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Modelling and numerical analysis of nonlocal conservation laws with applications to sedimentation and traffic flow problems.

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por

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Dedicatoria

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RESUMEN

Esta tesis tiene como objetivo el modelamiento, análisis y aproximación numérica mediante métodos de volumenes finitos, de leyes de balance hiperbólicas unidimensionales en espacio, con función de flujo no local, que son motivadas por aplicaciones en sedimentación y tráfico vehicular. En particular, estamos interesados en estudiar el buen planteamiento y en diseñar esquemas numéricos eficientes para calcular soluciones aproximadas de nuevos modelos que se proponen en el marco de las aplicaciones objetivo.

Primero se pretende modelar un proceso de sedimentación por lotes en una columna cerrada, para ello consideramos un problema de valor inicial y de frontera (IBVP) para una ley de conservación no local en el que el término no local viene dado por la convolución entre una función kernel y la velocidad de sedimentación, se asume que este operador no local tiene en cuenta la presencia de los términos frontera. De este primer modelo propuesto estudiamos el buen planteamiento y adaptamos a una versión no local un esquema numérico tipo Hilliges-Weidlich (HW). Especificamente, se demuestra que la unicidad de soluciones débiles de entropía para el modelo no local depende Lipschitz continuamente de los datos iniciales y de frontera; así mismo, a través del esquema numérico se proveen estimaciones de compacidad, junto con una desigualdad discreta de entropía, que demuestran la existencia de soluciones débiles y la convergencia de la sucesión de soluciones aproximadas hacia una solución débil de entropía del problema no local. Comparamos el esquema HW con los esquemas basados en el flujo de Lax-Friedrichs mediante ejemplos numéricos. También se presenta un esquema HW de segundo orden basado en métodos tipo MUSCL.

En segunda instancia, modelamos la dinámica del tráfico en una carretera con condiciones heterogéneas, a través de una ley de conservación cuyo flujo no local contiene un término de obstaculización y tiene una única discontinuidad espacial. El término no local refleja que los conductores adaptan su velocidad con respecto a lo que pasa en frente de ellos. Estas hipótesis conducen a una función de flujo en la que la velocidad depende de una convolución downstream entre la densidad de vehículos y una función kernel. Aproximamos el problema a través del esquema HW propuesto antes y proporcionamos algunas estimaciones uniformes sobre la sucesión de soluciones aproximadas lo cual nos permite probar existencia de una solución débil de entropía. También establecemos estabilidad $\mathbf{L^1}$ y por tanto unicidad de las soluciones de entropía.

Posteriormente, introducimos un modelo que describe la dinámica del tráfico vehicular en una carretera con rampas de entradas y salida, para lo cual consideramos una ley de balance no local en la que el término fuente describe de manera independiente el flujo de entrada y salida a través de las rampas. El término fuente depende de un término de convolución que describe el hecho de que los conductores

sobre la rampa de entrada pueden ver lo que pasa detrás y en frente de ellos en la carretera principal. La existencia de las soluciones débiles de entropía es probada aproximando las soluciones numéricas por medio del esquema HW junto con un operador splitting que tiene en cuenta el término de reacción y proporcionando estimaciones $\mathbf{L}^\infty$ y $\mathbf{BV}$ para la sucesión de soluciones aproximadas. La unicidad de la solución débil de entropía es probada a través de la dependencia Lipschitz contínua de la solución sobre el dato inicial, la razón de entrada y la razón de salida de las rampas. También estudiamos numéricamente el modelo límite cuando el soporte de la función kernel tiende a cero y presentamos algunas simulaciones numéricas que ilustran la dinámica del modelo estudiado. Además, motivados por problemas de optimización y control en tráfico vehicular, estudiamos la dependencia de las soluciones para el modelo de tráfico vehicular con rampas introducido antes, sobre el kernel de convolución dado en el término fuente. Obtenemos una estimación de la dependencia de la solución con respecto al kernel del término fuente, el dato inicial, la razón de entrada y la razón de salida de las rampas. La estabilidad es obtenida de la condición de entropía a través de la técnica de duplicación de variables. También proporcionamos algunas simulaciones numéricas que ilustran la dependencia anterior para algunos funcionales de costo.

Finalmente, con el fin de modelar el tráfico en una carretera con dos carriles y dos vías en el cual los conductores tienen un carril preferencial (el carril de la derecha) y el otro carril es usado solo para adelantar, proponemos un sistema de leyes de balance no local. En este modelo la parte convectiva describe la dinámica intra-carril de los vehículos, por esta razón las funciones de flujo consideran términos locales y no locales, a saber, la función velocidad en cada carril depende localmente de la densidad de los vehículos de la clase preferente y de una forma no local de la densidad de vehículos de la otra clase que viene en dirección opuesta sobre el mismo carril haciendo adelantamiento; a su vez, los términos fuente describien el acoplamiento inter-carril entre los dos carriles del modelo, por lo que consideramos los criterios de adelantamiento y retorno dependientes de un promedio ponderado de la densidad del tráfico downstream de la clase preferencial y de un promedio ponderado de la densidad del tráfico downstream de las clases viajando en dirección opuesta. Aproximamos las soluciones del problema usando el esquema HW desarrollado en esta tesis y por medio de estimaciones de compacidad probamos la existencia de soluciones débiles. También mostramos algunas simulaciones numéricas que describen el comportamiento de las soluciones en diferentes situaciones.

Palabras Claves: Leyes de conservación no local, leyes de balance no local, problema de valor inicial y de frontera, término de convolución, funciones kernel, solución débil de entropía, esquema numérico tipo HW, función de flujo discontinuo, modelos macroscópicos de tráfico vehicular, modelos de tráfico Lighthill-

Whitham-Richards, rampas de entrada y salida, buen planteamiento, modelo de tráfico multicarril, modelo de tráfico de dos vías y dos carriles.

ABSTRACT

This thesis aims at the modeling, analysis and numerical approximation by means of finite volume methods, of spatially one-dimensional hyperbolic balance laws, with nonlocal flux function, motivated by applications in sedimentation and vehicular traffic. Particularly, we are interested in studying well-posedness and to design efficient numerical schemes to compute approximate solutions of new proposed models in the frame of the target applications.

First we intend to model a batch sedimentation process in a closed column, for it we consider an initial boundary values problem (IBVP) for a nonlocal conservation law in which the nonlocal term is given by the convolution between a kernel function and the velocity of sedimentation. This nonlocal operator is assumed to be aware of boundary terms. For this first proposed model we study well-posedness and adapt a nonlocal version of a Hilliges-Weidlich (HW)-type numerical scheme. Specifically, it is proved that the uniqueness of entropy weak solutions to nonlocal model follows from the Lipschitz continuous dependence of the solution on initial and boundary data; likewise, by means of the numerical scheme we provide compactness estimates along with a discrete entropy solution, which show the existence of weak solutions and the convergence of the sequence of approximate solutions to an entropy weak solution of the nonlocal problem. We compare the HW-type scheme with schemes based on the Lax-Friedrichs flux through numerical examples. A second-order HW-type scheme based on MUSCL methods is also presented.

Second, we model the traffic dynamics on a road with rough conditions, by means of a conservation law whose nonlocal flux has a hidrance term and a single spatial discontinuity. The nonlocal term reflects that drivers adapt their velocity with respect to what happens in front of them. These assumptions lead to a expression flux in which the velocity depends on a convolution between density of vehicles and a function kernel. We approximate the problem by means of the HW-type numerical scheme proposed above and provide some uniform estimates on the sequence of approximate solutions what allow us to prove existence of a entropy weak solution. We also provide $\mathbf{L^1}$ stability and therefore uniqueness of entropy solutions.

Subsequently, we introduce a model that describes the vehicular traffic dynamics on a road with on- and off ramps, for which we consider a nonlocal balance law in which the source term independently describes the inflow and outflow via on-ramp and off-ramps. The source term depends on a downstream convolution term that describes that drivers on the on-ramp can see what happens behind and in front of them on the main road. Existence of entropy weak solutions is proved approximating the numerical solutions by means of the HW-type scheme

along with an operator splitting to account the reaction term and providing $\mathbf{L}^{\infty}$ and **BV** estimates to sequence of approximate solutions. Uniqueness of entropy weak solution is proved by means $\mathbf{L}^{\mathbf{1}}-$Lipschitz continuous dependence of solution on initial datum, on-ramp rate and off-ramp rate. We also study numerically the limit model as support of the kernel function tends to zero and also are presented some numerical simulations illustrating the dynamic of the studied model.

Then, motivated by optimization and control problems, we study the dependence of solutions to the vehicular traffic model with ramps introduced above on the convolution kernel given in the source term. We obtain an estimate of the dependence of the solution with respect to the kernel function in the source term, the initial datum, on-ramp rate and off-ramp rate. Stability is obtained from the entropy condition through doubling of variable technique. We also provide some numerical simulations illustrating the dependencies above for some cost functionals.

Finally, in order to model vehicular traffic flow on a two-lane and two-way road where drivers have a preferred lane, the lane on their right, and the left one is used only for overtaking slower vehicles, we propose a system of nonlocal balance laws. In this model the convective part describes the intra-lane dynamics, for this reason the flux functions consider local and nonlocal terms, namely, the velocity function in each lane depends locally on the density of vehicles of the preferential class and on a nonlocal form on the density of vehicles of the another class coming in opposite direction on the same lane overtaking; in turn, the source terms describe the inter-lane coupling between the two lanes, so that we consider the overtake and return criteria dependent on a weighted mean of the downstream traffic density of preferred class and a weighted mean of downstream traffic density of the classes traveling in opposite direction. We approximate the solutions of the problem by means of the HW-type numerical scheme developed in this thesis and prove existence of weak solutions by means of compactness estimates. We also show some numerical simulations that describe the behavior of the solutions in different situations.

Key Words: Nonlocal conservation laws, nonlocal balance laws, initial boundary values problem, convolution term, kernel functions, entropy weak solution, HW type numerical scheme, discontinuous flux function, macroscopic vehicular traffic models, Lighthill-Whitham-Richards traffic model, on- and off-ramps, well-posedness, multilane traffic model, two way and two lane traffic model.

Contents

Chapter 1

Introduction

Nonlocal balance laws are given by expressions of the type

$$\partial_t \mathbf{u} + \mathrm{div}_{\mathbf{x}} \mathbf{F}(t, \mathbf{x}, \mathbf{u}, W) = \mathbf{R}(t, \mathbf{x}, \mathbf{u}, W), \qquad t > 0,\ \mathbf{x} \in \mathbb{R}^d,\ d \geq 1, \tag{1.0.1}$$

where the variable $W = W(t, \mathbf{x})[\mathbf{u}]$ depends on a integral evaluation of $\mathbf{u}$. Such equations are intended to model macroscopically the action of nonlocal interactions occurring at the microscopic level and for this reason nonlocal balance laws are being developed to model various phenomena, such as the dynamics of crowd [33, 32, 34], vehicular traffic [11, 14, 26, 43, 44, 51], supply chains [7], granular materials [4] and sedimentation phenomena [13]. Challenges in these applications are related to the theoretical and numerical treatment of nonlocal terms in the presence of discontinuous solutions. In fact, discontinuous solutions may arise despite the expected regularizing effect of the convolution product. Consequently, one must consider generalized solutions or in the sense of distributions; furthermore, these solutions are not necessarily unique, and therefore some extra conditions, called *entropy conditions* [63], need to be imposed to guarantee stability and uniqueness of solutions to the Cauchy problem for the conservation law under consideration. Thus, nonlocal terms make the classical techniques developed for hyperbolic systems of conservation laws inapplicable and call for novel analytical and numerical methods. Moreover, integral terms strongly impact the cost of numerical simulations, especially in the case of multidimensional problems that possibly involve space-dependent convolution kernels [70, 71], which motivates the design of efficient approximation schemes. Until recently, numerical discretizations had been based on first-order Lax-Friedrichs-type methods [2, 6, 14] that usually exhibit high diffusion properties, or second-order central schemes [64] and discontinuous Galerkin schemes [52], which are more accurate but harder to implement and require the computation of a larger number of integral terms. The main objective of this thesis is to propose, study well-posedness and develop efficient numerical scheme, for hyperbolic conservation and balance laws with nonlocal flux function

that arise in applications related to sedimentation and vehicular traffic.
Motivated by the above discussion and taking into account our main goal, we first start studying an initial-boundary value problem (IBVP) for a scalar conservation law with nonlocal flux function, on a open bounded interval $]a,b[\subset \mathbb{R}$, which is a simplified version of the model introduced in [13, 50] and describes, for instance, a batch sedimentation process in a closed column. The main motivation for studying this nonlocal model is to propose an approach for a rigorous treatment of boundary conditions in the case of a spatially one-dimensional nonlocal problems due to the lack of literature on the subject so far. Regarding vehicular traffic applications, we propose and study several models which extend the classical LWR (Lighthill - Whitham [66] and Richards [72]) traffic model in several directions in order to simulate more realistic features of traffic on roads, e.g., traffic flow with abruptly changing road conditions, which we do by means of a nonlocal conservation law with discontinuous flux function; traffic flow in a road with on- and off-ramps, which we do by means of a nonlocal balance law, where the key feature is in the right hand side due to the presence of a nonlocal source; traffic flow in a two way and two lanes road, which we introduce through a nonlocal system of balance laws including nonlocal criteria for changing lane maneuvers in the source terms.

Let us introduce the problems to work in this thesis, and then give a description to solve each.

Chapter 2 is concerned with an IBVP for a scalar conservation law with nonlocal flux function, on a open bounded interval $]a,b[\subset \mathbb{R}$ which reads

$$\begin{aligned}
&\partial_t \rho + \partial_x\big(f(\rho)V(t,x)\big) = 0, && (t,x) \in \mathbb{R}^+ \times]a,b[,\\
&\rho(x,0) = \rho_0(x), && x \in]a,b[,\\
&\rho(a,t) = \rho_a(t), \quad \rho(b,t) = \rho_b(t), && t \in \mathbb{R}^+,
\end{aligned}$$

where

$$\begin{aligned}
f(\rho) &:= \rho g(\rho),\\
V(t,x) &:= \big(\omega * v(\rho)\big)(t,x) = \frac{1}{W(x)} \int_a^b v\big(\rho(t,y)\big)\omega(y-x)\,\mathrm{d}y
\end{aligned}$$

with $W(x) := \int_a^b \omega(y-x)\,\mathrm{d}y$ for a suitable convolution kernel ω and g a hindrance function. This model is in part motivated by a model of layered sedimentation observable in a batch process in a closed column and is a simplified version of models proposed in [13, 50]. The layered sedimentation phenomenon occurs because homogeneous suspensions of small solid particles dispersed in a viscous fluid do not always sediment in a smooth continuous fashion as described, for instance,

in the well-known model of sedimentation arising from Kynch theory [65]; instead, layers of different concentrations (staircasing) are often observed after settling has proceeded after a finite time. This effect is particularly well documented in the paper by Siano [78].
The aim of Chapter 2 is to propose an approach for a rigorous treatment of boundary conditions in the case of spatially one-dimensional nonlocal problems, through the development of new numerical schemes that are more accurate and less diffusive than, for instance, Lax-Friedrichs type-schemes. For this purpose, we adapt a numerical scheme, which is based on one given in [15, 55] but including a nonlocal term which is assumed to be aware the boundary conditions, following [36, 50]. Observe that in the proposed model we take the average of velocities instead of average of concentration. The proposed numerical method and the way of computing the convolutions are the main novelties of Chapter 2. In order to study well-posedness of model, we first give a definition of entropy weak solution and impose an appropriate Courant-Levy-Friedrichs (CFL) condition, then we prove maximum principle and Bounded Variations (**BV**) bounds in space and time, which allows us to apply the Helly's Compactness Theorem in order to prove convergence and existence of solutions to model. We also prove uniqueness of entropy weak solution through Lipschitz continuous dependence on initial and boundary data of solutions, which is obtained from results for a local IBVP.
The contents of this chapter gave rise to the following submitted preprint [19]:

- R. Bürger, H. D. Contreras and L. M. Villada. A Hilliges-Weidlich-type scheme for a one-dimensional scalar conservation law with nonlocal flux.

The second problem, which motivates Chapter 3, is related to a nonlocal conservation law with discontinuous flux which reads

$$\begin{cases} \partial_t \rho + \partial_x f(t,x,\rho) = 0, & (t,x) \in (0,\infty) \times \mathbb{R}, \\ \rho(0,x) = \rho_0(x), & x \in \mathbb{R}, \end{cases}$$

with

$$f(t,x,\rho) = H(-x)\ \rho\, g(\rho)\, v_l(\omega_\eta * \rho) + H(x)\ \rho\, g(\rho)\, v_r(\omega_\eta * \rho),$$

where $H(x)$ is the Heaviside function and the flux $f(t,x,\rho)$ has a discontinuity at $x = 0$ if the velocity functions $v_l(\rho)$ and $v_r(\rho)$ are different. The hindrance function g is assumed nonnegative and such that $g'(\rho) \leq 0$ and $g(\rho_{\max}) = 0$ and the convolution term is defined as

$$(\omega_\eta * \rho)(t,x) = \int_x^{x+\eta} \rho(t,y)\omega_\eta(y-x)\mathrm{d}y, \quad \eta > 0.$$

In the vehicular traffic context this model describes traffic flow on a road with rough conditions, and ρ represents the density of vehicles on the road, ω_η is a non-increasing kernel function whose support η is proportional to the look-ahead distance of drivers, that are supposed to adapt their velocity with respect to the mean downstream traffic density. Although there are many works on conservation laws with discontinuous flux, see e.g. [9, 15, 16, 17, 58, 76, 83], due to the lack of works on conservation laws with nonlocal and discontinuous flux, the main novelty of proposed model is precisely the inclusion of the nonlocal term in this type of problems. The purpose of this chapter is to establish the well-posedness of the proposed model. First, we propose to use a suitable notion of entropy solution for this problem and then existence of a entropy solution is established by proving convergence of a HW-type numerical scheme based on the proposed one in Chapter 2. Afterwards, we prove uniqueness ($\mathbf{L^1}$ stability) of the entropy solution and finally, the performance of scheme is demonstrated by numerical examples.
The contents of this chapter gave rise to the following submitted preprint [22]:

- F. A. Chiarello, H. D. Contreras and L. M. Villada. Existence of entropy weak solutions for 1D nonlocal traffic models with space-discontinuous flux.

Chapter 4 deals with the nonlocal balance law,

$$\rho_t + (\rho v(\rho * \omega_\eta))_x = S_{\text{on}}(t, x, \rho, \rho * \omega_{\eta,\delta}) - S_{\text{off}}(t, x, \rho), \quad x \in \mathbb{R},$$

which models vehicular traffic flow on a road with presence of on- and off-ramps and where the non-negative functions $S_{\text{on}}(t, x, \rho, \rho * \omega_{\eta,\delta})$ and $S_{\text{off}}(t, x, \rho)$ are the source terms that describe the inflow and output flow on a main road, via on- and off-ramps, respectively. The algebraic expressions of these terms will be specified in the development of chapter. The main novelty of this model is the presence of a nonlocal source in the right hand side, that describes the fact that drivers on the on-ramp can see what happens behind and in front of them on the main road. The purpose of this chapter is to study the well-posedness of proposed model, i.e., to establish existence and uniqueness of entropy weak solution. Keeping this in mind, we first introduce the definition of weak and entropy weak solution, then the existence of a weak entropy solution is proved by constructing a converging sequence of finite volume approximate solutions, defined using the HW-type numerical scheme along with an operator splitting to account the reaction term. We derive the $\mathbf{L^1}$ Lipschitz continuous dependence of solution of the proposed model on the initial data and on parameters of the source terms.
The contents of this chapter corresponds to the following paper [23]:

- F. A. Chiarello, H. D. Contreras and L. M. Villada. Nonlocal reaction traffic flow model with on-off ramps. Networks and Heterogeneous Media, volume 17, number 2, 2022.

In Chapter 5 we study the dependence of solutions to the model introduced in Chapter 4, on the convolution kernel given in the source term S_{on}. We get an estimate of the dependence of the solution with respect to the initial datum, the on-ramp rate, the off-ramp rate and the kernel function $\omega_{\eta,\delta}$ through doubling of variable technique. Then, we study numerically an optimization problem for traffic flow in which we want to identify optimal values for η and δ such that less congestion is generated when vehicles enter the main road through the on-ramp. The contents of this chapter corresponds to the following preprint [24], which was submitted to the HYP 2022 Proceedings,

- F. A. Chiarello, H. D. Contreras and L. M. Villada. Stability estimates for nonlocal balance laws arising in traffic modelling.

In Chapter 6 the following system of balance laws is introduced

$$\begin{aligned}
\partial_t\rho_1 + \partial_x(\rho_1 v_1(\rho_1 + (\rho_{\max} - \rho_1)\chi_\varepsilon(\tilde\rho_2 * \omega_\eta))) &= -S_{\text{O}}(\rho_1,\rho_2,\tilde\rho_1,\tilde\rho_2) + S_{\text{R}}(\rho_1,\rho_2,\tilde\rho_1,\tilde\rho_2),\\
\partial_t\rho_2 + \partial_x(\rho_2 v_2(\rho_2 + (\rho_{\max} - \rho_2)\chi_\varepsilon(\tilde\rho_1 * \omega_\eta))) &= S_{\text{O}}(\rho_1,\rho_2,\tilde\rho_1,\tilde\rho_2) - S_{\text{R}}(\rho_1,\rho_2,\tilde\rho_1,\tilde\rho_2),\\
\partial_t\tilde\rho_1 - \partial_x(\tilde\rho_1 v_1(\tilde\rho_1 + (\rho_{\max} - \tilde\rho_1)\chi_\varepsilon(\rho_2 * \hat\omega_\eta))) &= -\tilde S_{\text{O}}(\tilde\rho_1,\tilde\rho_2,\rho_1,\rho_2) + \tilde S_{\text{R}}(\tilde\rho_1,\tilde\rho_2,\rho_1,\rho_2),\\
\partial_t\tilde\rho_2 - \partial_x(\tilde\rho_2 v_2(\tilde\rho_2 + (\rho_{\max} - \tilde\rho_2)\chi_\varepsilon(\rho_1 * \hat\omega_\eta))) &= \tilde S_{\text{O}}(\tilde\rho_1,\tilde\rho_2,\rho_1,\rho_2) - \tilde S_{\text{R}}(\tilde\rho_1,\tilde\rho_2,\rho_1,\rho_2).
\end{aligned}$$

This system models vehicular traffic flow on a two lane (labeled lane 1 and lane 2) and two way road where drivers have a preferred lane, the lane on their right, and the left one is used only for overtaking slower vehicles. We denote by ρ_1 and ρ_2 the density of classes of cars traveling from the left to right on lane 1 and lane 2, respectively; by $\tilde\rho_1$ and $\tilde\rho_2$ the density of classes of cars traveling from the right to left on lane 2 and lane 1, respectively. In each equation, the convective part describes the intralane dynamics, while the right hand side models the interplay between Lane 1 and Lane 2. The main novelties of model lie on the one hand, in that to our knowledge, this is the first nonlocal macroscopic traffic model for describing the dynamics of this type of road which are common in many Latin American cities, and on the other hand in the way in which the velocity function is written that depends on densities of vehicles traveling in opposite directions on a same lane as well as the criteria that allows overtaking and returning. The goal of Chapter 5 consists in establishing existence of weak solutions on the proposed nonlocal balance law system. To carry out this purpose, like in Chapter 4, we introduce the definition of weak solution, then by constructing a converging sequence of finite volume approximate solutions, defined using the HW-type numerical scheme along with an operator splitting to account the reaction term, we prove compactness estimates in order to apply Helly's Theorem, this latter allows us to get convergence and thus existence of weak solutions. The contents of this chapter corresponds to the following submitted preprint [37]:

- H.D. Contreras, P. Goatin and L.M. Villada. Two way nonlocal traffic model.

Meetings, Conferences and Research stays

During my studies in doctoral program, the following meetings and conferences have been attended,

- XLVIII Semana de la Matemática.
  Departament of Mathematics, Pontificia Universidad Católica de Valparaíso. Valparaíso (Chile), October 11-14, 2022.
  Tittle of talk: Well-posedness of nonlocal conservation laws with continuous and discontinuous fluxes.

- XXX Congreso de Matemática Capricornio COMCA 2022 (Online).
  Departament of Mathematics, Arturo Prat University, Iquique (Chile), August 3-5, 2022.
  Tittle of talk: Existence of entropy weak solutions for 1D nonlocal traffic models with space-discontinuous flux.

- HYP 2022 XVIII International Conference On Hyperbolic Problems: Theory, Numerics, Applications.
  Málaga (Spain). June 20-24, 2022.
  Tittle of talk: Nonlocal reaction traffic flow model with on-off ramps.

- XVII Encuentro Internacional de Matemáticas (EIMAT) 2021.
  Departament of Mathematics, Universidad del Atlántico, Barranquilla (Colombia). November 16-19, 2021.
  Tittle of talk: Nonlocal reaction traffic flow.

- XXXIII Jornada Matemática de la Zona Sur 2021 (Online).
  Universidad de la Frontera, Temuco (Chile). April 20-24, 2021.
  Talk's title: A Hilliges-Weidlich type scheme for a 1D scalar conservation law IBVP with nonlocal flux.

Poster presentations

- A numerical scheme for a nonlocal conservation law modelling sedimentation. COMCA 2019 XXVIII Congreso de Matemáticas Capricornio. Universidad de la Serena, La serena (Chile). July 31 – August 2, 2019.

During my studies in doctoral program I did the following research stays

Research stays

- Institut National de Recherche en Informatique et en Automatique (INRIA) October 21th 2021 - January 5th 2022. Supervisor researcher: Paola Goatin.

- Institut National de Recherche en Informatique et en Automatique (INRIA) June 28th 2022 - July 12th 2022. Supervisor researcher: Paola Goatin

Chapter 2

A Hilliges-Weidlich-type scheme for a one-dimensional scalar conservation law with nonlocal flux

2.1 Introduction

2.1.1 Scope

The aim of this chapter is to propose an approach for a rigorous treatment of boundary conditions in the case of a spatially one-dimensional nonlocal problems, through development of new numerical schemes that are more accurate and less diffusive in comparison, for instance that Lax-Friedrichs-type numerical schemes. The strategies that we employ are inspired by the results obtained in [13, 15, 50]. Particularly, we propose to study a simplification of the problem studied in [13], we adopt the treatment of the boundary conditions proposed in [50] and we present a numerical scheme based on local one studied in [15, 55]. Our proposed scheme takes advantage of the form in which the flow is written, namely density ρ times a local decreasing factor $g(\rho)$ times a nonlocal convolution term $V(x,t) = (\omega * v(\rho))$, where v is a given velocity function and ω is a convolution kernel such that the governing conservation law becomes

$$\partial_t \rho + \partial_x \big(\rho g(\rho) V(x,t)\big) = 0. \tag{2.1.1}$$

In the case of a standard (local) conservation law, captured by setting $V = \text{const.}$, the above-mentioned approach results in a monotone scheme [15], so it is possible to invoke standard arguments to prove its convergence to an entropy solution. This idea is extended herein to the nonlocal equation (2.1.1), although we emphasize that the resulting scheme is *not* monotone.

2.1.2 Related work.

There are many works about existence and uniqueness results for nonlocal equations, see e.g. [6, 39, 42, 50] for the scalar case in one space dimension. In these papers a first-order Lax-Friedrichs (LxF)-type numerical scheme is used to approximate the problem and to prove the existence of solutions and the nonlocal term is considered as a convolution between a kernel function and the unknown (mean downstream density approach). LxF-type schemes are the most common approach used to solve nonlocal conservation laws because they are easy to implement and due to their monotonicity, they make it possible to numerically analyze nonlocal flux problems. Their well-known main disadvantage is, however, their large amount of numerical diffusion that smears out sharp features of the exact solution. To reduce this phenomenon, Friedrich et al. [44] proposed a Godunov-type numerical scheme where the nonlocal term is considered as a convolution between a kernel function and the velocity of unknown (mean downstream velocity approach). We adopt this idea about the convolution to propose and develop our model and computations. A well-known early analysis of initial-boundary value problems (IBVP) for conservation laws is due to Bardos et al. [10], existence, uniqueness and continuous dependence of the solution on initial data in the case of zero boundary data are prove. These results were extended to more general but smooth boundary data by Colombo and Rossi [35]. Rossi [75] studied an IBVP for a general scalar balance law in one space dimension. Under rather general assumptions on the flux and source functions, the author proves the well-posedness of the problem and stability of its solutions with respect to variations in the flux and the source terms. For both results, the initial and boundary data are required to be bounded functions with bounded total variation. In [39] a global well-posedness result for a class of weak entropy solutions of bounded variations of scalar conservation laws with nonlocal flux on bounded domains is established under suitable regularity assumptions on the flux function. The nonlocal operator is the standard convolution product. The existence of solutions is obtained by proving the convergence of an adapted LxF algorithm. Lipschitz continuous dependence from initial and boundary data is derived applying Kružkov's doubling of variables technique. In [50] Goatin and Rossi study the same problem as Filippis and Goatin [39], but with a different approach, namely following the treatment of the boundary conditions proposed by Colombo and Rossi in [36] where a particular multi-dimensional system of conservation laws in bounded domains with zero boundary conditions was considered. More specifically, the nonlocal operator in the flux function is not a mere convolution product, but it is assumed to be 'aware' of boundaries and by introducing an adapted LxF algorithm, various estimates on the approximate solutions that allow to prove the existence of solutions to the original IBVP are introduced. Uniqueness was derived from the Lipschitz continuous dependence on initial and boundary data, which is proved exploiting results available for the local problem.

2.1.3 Outline of Chapter 2.

This chapter is organized as follows: In Section 2.2 we present the considered class of nonlocal conservation laws and the assumptions needed on studied problem as well as the main result of this chapter, whose proof is postponed to end of Section 2.3. Lipschitz continuous dependence of solutions to the studied problem on initial and boundary data is proved in Section 2.3. Afterwards, in Section 2.4 we introduce the numerical scheme and derive some of its important properties such as the maximum principle, **BV**, and L^1 Lipschitz continuity in time estimates. These imply convergence of the scheme proposed, which in turn covers the existence part of the well-posedness of the governing model. Throughout the chapter we address the new scheme as a "HW scheme" according to the proponent of the original idea (Hilliges and Weidlich [55]), and in Section 2.5 we provide a second-order version of a HW-type numerical scheme. Finally, In Section 2.6 we present some numerical examples, analysing the L^1-error of the approximate solutions of studied problem computed with different schemes. Appendix A collects some estimates necessary throughout the chapter.

2.2 Initial-boundary value problem

We consider a particular initial-boundary value problem which is a version of a nonlocal model of sedimentation proposed in [13]. Our model has the following structure:

$$\begin{aligned} &\partial_t \rho + \partial_x \big(f(\rho) V(x,t) \big) = 0, \quad (x,t) \in]a,b[\times \mathbb{R}^+, \\ &\rho(x,0) = \rho_0(x), \quad x \in]a,b[, \\ &\rho(a,t) = \rho_a(t), \quad \rho(b,t) = \rho_b(t), \quad t \in \mathbb{R}^+, \end{aligned} \tag{2.2.1}$$

where

$$f(\rho) := \rho g(\rho), \tag{2.2.2}$$

$$V(x,t) := \big(\omega * v(\rho)\big)(x,t) = \frac{1}{W(x)} \int_a^b v\big(\rho(y,t)\big) \omega(y-x)\, \mathrm{d}y \tag{2.2.3}$$

with $W(x) := \int_a^b \omega(y-x)\, \mathrm{d}y$ for a suitable convolution kernel ω.

Remark 2.1. *The particular combination of local and nonlocal evaluations of ρ present in (2.2.2), (2.2.3) can be motivated by following the discussion of [13, Sect. 1.2] for a model of sedimentation. Namely, if we assume that the nonlocal model describes the volume fraction of solids $\rho \in [0,1]$ within a solid-fluid two-phase flow system, then the solid and fluid conservation equations in differential form are $\partial_t \rho + \partial_x(\rho v_{\mathrm{s}}) = 0$ and $\partial_t(1-\rho) + \partial_x((1-\rho)v_{\mathrm{f}}) = 0$, where v_{s} and v_{f} are the solid and fluid phase velocities and x is the vertical spatial coordinate. One then defines the volume average velocity of the mixture $q := \rho v_{\mathrm{s}} + (1-\rho)v_{\mathrm{f}}$ and the solid-fluid relative velocity $v_{\mathrm{r}} = v_{\mathrm{s}} - v_{\mathrm{f}}$. Now for the particular case of batch settling in a closed column, we have $q = 0$ for all x and t, and then $\rho v_{\mathrm{s}} = \rho(1-\rho)v_{\mathrm{r}}$, so that the unique PDE to be solved is*

$$\partial_t \rho + \partial_x \big(\rho(1-\rho) v_{\mathrm{r}} \big) = 0, \tag{2.2.4}$$

where v_{r} is specified by some constitutive function. This scenario corresponds to (2.2.1)–(2.2.3) *if we choose $g(\rho) = 1 - \rho$ and assume that v_{r} is given through the nonlocal convolution*

$$v_{\mathrm{r}} = v_{\mathrm{r}}(x,t) = \big(\omega * v(\rho)\big)(x,t), \tag{2.2.5}$$

where $\rho \mapsto v(\rho)$ is a given, in general nonlinear function. In other words, the local and nonlocal evaluations of ρ in (2.2.1) *arise from the combination of properly defined volume fractions in mixture theory with the constitutive assumption* (2.2.5)*. The standard local evaluation $v_{\mathrm{r}}(x,t) = v(\rho(x,t))$ corresponds to the well-known kinematic sedimentation model, while utilizing $v(\omega * \rho)(x,t)$ instead of $(\omega * v(\rho))(x,t)$ is the model alternative explored in [13].*

Assumptions 2.2.1. *The initial-boundary value problem* (2.2.1) *is studied under the following assumptions:*

(i) The initial datum satisfies $\rho_0 \in BV(I;\mathbb{R}^+)$, where $I :=]a,b[\subseteq \mathbb{R}^+$.

(ii) The function g satisfies

$$g \in C^2([0,1];\mathbb{R}_0^+),\ \ g'(\rho) \le 0 \textit{ for } \rho \in [0,1], \textit{and } g(1) = 0.$$

(iii) The function v satisfies

$$v \in C^2([0,1];\mathbb{R}^+), v'(\rho) \le 0 \textit{ for } \rho \in [0,1], \textit{and } 0 = v(1) \le v(\rho) \le v(0) = 1.$$

(iv) The convolution kernel ω satisfies

$$\omega \in (C^2 \cap W^{2,1} \cap W^{2,\infty})(\mathbb{R};\mathbb{R}) \textit{ such that } \int_{\mathbb{R}} \omega(y)\,\mathrm{d}y = 1$$

and there exists $K_\omega > 0$ such that for all $x \in I$, $W(x) = \int_a^b \omega(y-x)\mathrm{d}y \ge K_\omega$.

In what follows in this chapter, we denote $\|\cdot\|_{L^\infty([0,1])} := \|\cdot\|_\infty$. The weak entropy solution of problem (2.2.1) is defined, as in [39, 50], in the following sense:

Definition 2.2. *A function $\rho \in (L^1 \cap L^\infty \cap BV)(]a,b[\times\mathbb{R}^+;\mathbb{R})$ is an entropy weak solution to problem* (2.2.1) *if for all $\varphi \in C^1_{\mathrm{c}}(\mathbb{R}^2;\mathbb{R}^+)$ and $k \in \mathbb{R}$,*

$$\begin{aligned}
&\int_0^\infty \int_a^b \Big(|\rho - k|\varphi_t + \mathrm{sgn}(\rho - k)\big(f(\rho) - f(k)\big)V\varphi_x - \mathrm{sgn}(\rho - k) f(k) V_x \varphi \Big)\,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_a^b \big|\rho_0(x) - k\big|\varphi(x,0)\,\mathrm{d}x \\
&+ \int_0^\infty \mathrm{sgn}(\rho_a - \kappa)\big(f(\rho(a+,t)) - f(\kappa)\big)V(a,t)\varphi(a,t)\,\mathrm{d}t \\
&- \int_0^\infty \mathrm{sgn}(\rho_b - \kappa)\big(f(\kappa) - f(\rho(b-,t))\big)V(b,t)\varphi(b,t)\,\mathrm{d}t \ge 0.
\end{aligned}$$

Definition 2.3. *A function* $\rho \in L^\infty(]a,b[\times\mathbb{R}^+;[0,1])$ *is an entropy weak solution to problem* (2.2.1) *if, for all* $\varphi \in C^1_c(\mathbb{R}^2,\mathbb{R}^+)$ *and* $k \in \mathbb{R}$,

$$\int_0^\infty \int_a^b \Big((\rho-k)^\pm \partial_t\varphi(x,t) + \operatorname{sgn}^\pm(\rho-k)\big(f(\rho)-f(k)\big)V(x,t)\partial_x\varphi(x,t) \\ - \operatorname{sgn}^\pm(\rho-k)f(k)\varphi(x,t)\partial_x V(x,t)\Big)\,\mathrm{d}x\,\mathrm{d}t + \int_a^b \big(\rho_0(x)-k\big)^\pm \varphi(x,0)\,\mathrm{d}x \\ + L\bigg(\int_0^\infty \big(\rho_a(t)-k\big)^\pm \varphi(a,t)\,\mathrm{d}t + \int_0^\infty \big(\rho_b(t)-k\big)^\pm \varphi(b,t)\,\mathrm{d}t\bigg) \geq 0,$$

where

$$L := \|v\|_\infty(\|g\|_\infty + \|g'\|_\infty). \tag{2.2.6}$$

Here we have used the notation $s^+ := \max\{s,0\}$, $s^- := \max\{-s,0\}$, and

$$\operatorname{sgn}^+(s) := \begin{cases} 1 & \text{if } s>0, \\ 0 & \text{if } s \leq 0, \end{cases} \qquad \operatorname{sgn}^-(s) := \begin{cases} 0 & \text{if } s \geq 0, \\ -1 & \text{if } s<0. \end{cases}$$

Definition 2.2 will be useful in the existence proof, while Definition 2.3 will be used in the uniqueness proof. We need to remark that in the frame of functions in L^∞, Definition 2.3 implies Definition 2.2, for more details see [74]. In the rest of chapter, we will denote $\mathcal{I}(r,s) = [\min\{r,s\}, \max\{r,s\}]$, for any $r,s \in \mathbb{R}$.

Our main result concerning the new model is given by the following theorem, which states the well-posedness of the problem.

Theorem 2.4 (Well-posedness). *Let* $\rho_0 \in BV(I;\mathbb{R}^+)$, $\rho_a, \rho_b \in BV(\mathbb{R}^+;[0,1])$ *and Assumptions 2.2.1 be in effect. Then, for all* $T>0$, *the Cauchy problem* (2.2.1) *admits a unique entropy weak solution* $\rho \in (L^1 \cap L^\infty \cap BV)(I \times [0,T];\mathbb{R}^+)$ *in the sense of the Definitions 2.2 and 2.3. Moreover, the following estimates hold: for any* $t \in [0,T]$,

$$0 \leq \rho(x,t) \leq 1 \quad \text{for all } x \in I, \tag{2.2.7}$$

$$\big\|\rho(\cdot,t)\big\|_{L^1(I)} \leq \mathcal{R}_1, \tag{2.2.8}$$

$$\operatorname{TV}\big(\rho(\cdot,t);I\big) \leq \mathrm{e}^{t\mathcal{T}_1}\Big(\operatorname{TV}(\rho_0;I) + \operatorname{TV}(\rho_a;]0,T[) + \big|\rho_0(a+)-\rho_a(0+)\big| \\ + \operatorname{TV}(\rho_b;]0,T[) + \big|\rho_0(b-)-\rho_b(0+)\big|\Big) + \frac{\mathcal{T}_2(t)}{\mathcal{T}_1(t)}(\mathrm{e}^{t\mathcal{T}_1}-1), \tag{2.2.9}$$

and for $\tau > 0$,

$$\big\|\rho(\cdot,t)-\rho(\cdot,t-\tau)\big\|_{L^1(I)} \leq \tau\Big(\mathcal{C}_t(t) + L\big(\operatorname{TV}(\rho_a;]t-\tau,t[) + \operatorname{TV}(\rho_a;]t-\tau,t[)\big)\Big)$$

where L *is defined by* (2.2.6) *and*

$$\mathcal{R}_1(t) := c\|\rho_0\|_{L^1(I)} + L\big(\|\rho_a\|_{L^1([0,t])} + \|\rho_b\|_{L^1([0,t])}\big),$$

$$\begin{aligned}
\mathcal{T}_1 &:= \mathcal{L}\big(\|g\|_\infty + \|g'\|_\infty\big), \\
\mathcal{L} &:= 2K_\omega^{-1}\|v\|_\infty\|\omega'\|_{L^1(\mathbb{R})}, && (2.2.10)\\
\mathcal{T}_2 &:= (\mathcal{W}\mathcal{R}_1(t) + 2\mathcal{L})\|g\|_\infty, \\
\mathcal{W} &:= 2K_\omega^{-1}\|v\|_\infty\|\omega''\|_{L^1(\mathbb{R})} + 4K_\omega^{-2}\|\omega'\|^2_{L^1(I)}\|v\|_\infty, && (2.2.11)\\
\mathcal{C}_t(t) &:= \|v\|_\infty\big(\|g'\|_\infty + \|g\|_\infty\big)\mathcal{C}_x(t) + \|g\|_\infty\mathcal{L}\mathcal{R}_1(t).
\end{aligned}$$

The proof consists of two parts: existence and uniqueness of entropy solutions. While uniqueness follows from the Lipschitz continuous dependence of weak entropy solutions on initial and boundary data, existence is based on a construction of a converging sequence of approximate solutions defined by a numerical scheme.

2.3 Uniqueness of entropy solutions

One part of the proof of Theorem 2.4 is to show uniqueness of weak entropy solutions for the model (2.2.1). Therefore, we prove the Lipschitz continuous dependence of weak entropy solutions with respect to initial and boundary data. Here, we follow [50]. We define $V(x,t)$ by (2.2.3) and analogously $U(x,t)$ by replacing ρ in (2.2.3) by another function σ. Furthermore, we let $r(x,t,u) := ug(u)V(x,t)$ and $h(x,t,u) := ug(u)U(x,t)$. Observe that by the definition of V and U,

$$V(x,t) \le \|v\|_\infty, \quad U(x,t) \le \|v\|_\infty,$$

furthermore, we have the following estimates derived in Appendix A:

$$\big|\partial_x V(x,t)\big| \le 2K_\omega^{-1}\|v\|_\infty\|\omega'\|_{L^1(I)} =: \mathcal{P}_1, \tag{2.3.1}$$

$$\begin{aligned}
\big|\partial^2_{xx} V\big| &\le K_\omega^{-2}\|v\|_\infty\|\omega''\|_{L^1(I)}\|\omega\|_{L^1(I)} + \mathcal{P}_1 K_\omega^{-2}\|\omega\|_{L^1(I)} \\
&\quad + \mathcal{P}_1 K_\omega^{-1} + K_\omega^{-1}\|v\|_\infty\|\omega''\|_{L^1(I)} =: \mathcal{P}_2,
\end{aligned} \tag{2.3.2}$$

$$\big|V(x,t) - U(x,t)\big| \le \mathcal{P}_3\int_a^b \big|\rho(y,t) - \sigma(y,t)\big|\,\mathrm{d}y, \quad \mathcal{P}_3 := K_\omega^{-1}\|\omega\|_{L^\infty(\mathbb{R})}\|v'\|_\infty, \tag{2.3.3}$$

$$\begin{aligned}
\big|\partial_x V(x,t) - \partial_x U(x,t)\big| &\le \mathcal{M}\int_a^b \big|\rho(y,t) - \sigma(y,t)\big|\mathrm{d}y, \\
\mathcal{M} &:= K_\omega^{-2}\|\omega'\|_{L^1(\mathbb{R})}\|v'\|_\infty\|\omega\|_{L^\infty(\mathbb{R})} + K_\omega^{-1}\|v'\|_\infty\|\omega'\|_{L^\infty(\mathbb{R})}.
\end{aligned} \tag{2.3.4}$$

In order to obtain the desired estimate, we first consider the local initial-boundary value problem

$$\begin{aligned}
&\partial_t\phi + \partial_x r(x,t,\phi) = 0, \quad (x,t) \in I\times]0,T[, \\
&\phi(x,0) = \sigma_0(x), \quad x \in I; \quad \phi(a,t) = \sigma_a(t), \quad \phi(b,t) = \sigma_b(t), \quad t \in]0,T[.
\end{aligned} \tag{2.3.5}$$

By Assumptions 2.2.1, $r \in C^2(I \times [0,T] \times \mathbb{R};\mathbb{R})$ and $\partial_\phi r \in L^\infty(I \times [0,T] \times \mathbb{R};\mathbb{R})$ and by estimation (2.3.1), $\partial^2_{x\phi} r(x,t,\phi) < \infty$. Thus, we may apply Theorem 2.4 of [75] to deduce that problem (2.3.5) admits a unique solution in $(L^\infty \cap BV)(I\times]0,T[;\mathbb{R})$ which satisfies, for all $t\in[0,T[$, $0\le \phi(x,t)\le 1$ for all $x \in I$ and

$$\begin{aligned}\mathrm{TV}(\phi(t)) \le \mathrm{e}^{t\mathcal{C}_2(t)}\big(&\,\mathrm{TV}(\sigma_0)+\mathrm{TV}(\sigma_a;]0,t[)+\left|\sigma_0(a+)-\sigma_a(0+)\right|\\ &+\mathrm{TV}(\sigma_b;]0,t[)+\left|\sigma_0(b-)-\sigma_b(0+)\right|+\mathcal{K}t\big),\end{aligned}\tag{2.3.6}$$

where

$$\mathcal{C}_2(t) := \mathcal{P}_1\big(\|g\|_\infty+\|g'\|_\infty\big),\quad \mathcal{K} := 2\big((\mathcal{P}_1+\mathcal{P}_2)\|g\|_\infty+\mathcal{P}_1\|g'\|_\infty\big).$$

Assume that ρ is a solution to the IBVP

$$\begin{aligned}&\partial_t\rho+\partial_x r(x,t,\rho)=0,\quad (x,t)\in I\times]0,T[,\\ &\rho(x,0)=\rho_0(x),\quad x\in I;\quad \rho(a,t)=\rho_a(t),\quad \rho(b,t)=\rho_b(t),\quad t\in]0,T[\end{aligned}$$

and that σ is a solution to the analogous IBVP

$$\begin{aligned}&\partial_t\sigma+\partial_x h(x,t,\sigma)=0,\quad (x,t)\in I\times]0,T[,\\ &\sigma(x,0)=\sigma_0(x),\quad x\in I;\quad \sigma(a,t)=\sigma_a(t),\quad \sigma(b,t)=\sigma_b(t),\quad t\in]0,T[.\end{aligned}$$

Therefore, for $t>0$ we compute

$$\left\|\rho(\cdot,t)-\sigma(\cdot,t)\right\|_{L^1(I)} \le \left\|\rho(\cdot,t)-\phi(\cdot,t)\right\|_{L^1(I)}+\left\|\phi(\cdot,t)-\sigma(\cdot,t)\right\|_{L^1(I)},\tag{2.3.7}$$

where the first term on the right-hand side of (2.3.7) evaluates the distance between solutions to IBVPs with the same flux function, but different initial and boundary data. Then, we can apply Proposition 3.7 of [75] to get

$$\begin{aligned}&\left\|\rho(\cdot,t)-\phi(\cdot,t)\right\|_{L^1(I)}\\ &\quad\le \|\rho_0-\sigma_0\|_{L^1(I)}+L\big(\|\rho_a-\sigma_a\|_{L^1([0,t])}+\|\rho_b-\sigma_b\|_{L^1([0,t])}\big) =: A(t).\end{aligned}$$

Now, the second term on the right-hand side of (2.3.7) evaluates the distance between solutions to IBVPs with different flux functions, but same initial and boundary data. Therefore, we apply Theorem 2.6 of [75] to obtain

$$\begin{aligned}&\left\|\phi(\cdot,t)-\sigma(\cdot,t)\right\|_{L^1(I)}\\ &\le \int_0^t\int_a^b \left\|\partial_x(r-h)(x,s,\cdot)\right\|_{L^\infty(\mathcal{U})}\,\mathrm{d}x\,\mathrm{d}s\\ &\quad+\int_0^t\left\|\partial_u(r-h)(\cdot,s,\cdot)\right\|_{L^\infty(I\times\mathcal{U}))}\min\big\{\mathrm{TV}(\sigma(\cdot,s)),\mathrm{TV}(\phi(\cdot,s))\big\}\,\mathrm{d}s\\ &\quad+2\int_0^t\left\|(r-h)(a,s,\cdot)\right\|_{L^\infty(\mathcal{U})}\mathrm{d}s+2\int_0^t\left\|(r-h)(b,s,\cdot)\right\|_{L^\infty(\mathcal{U})}\mathrm{d}s,\end{aligned}\tag{2.3.8}$$

where

$$\mathcal{U} := [-\max\{\|\pi(s)\|_{L^\infty(I)},\|\sigma(s)\|_{L^\infty(I)}\},\max\{\|\pi(s)\|_{L^\infty(I)},\|\sigma(s)\|_{L^\infty(I)}\}]$$

$= [-1, 1].$

Next, we estimate all terms appearing in (2.3.8). First of all, by Theorem 2.4,

$$\begin{aligned}\mathrm{TV}\big(\sigma(\cdot,t)\big) \leq \mathrm{e}^{t\mathcal{T}_1(t)}\Big(&\mathrm{TV}(\sigma_0;I) + \mathrm{TV}\big(\sigma_a;(0,t)\big) + \big|\sigma_0(a+) - \sigma_a(0+)\big| \\ &+ \mathrm{TV}\big(\sigma_b;(0,t)\big) + \big|\sigma_0(b-) - \sigma_b(0+)\big|\Big) + \frac{\mathcal{T}_2(t)}{\mathcal{T}_1(t)}(\mathrm{e}^{t\mathcal{T}_1(t)} - 1)\end{aligned} \tag{2.3.9}$$

with

$$\mathcal{T}_1(t) := \mathcal{L}\big(\|g\|_\infty + \|g'\|_\infty\big), \quad \mathcal{T}_2(t) := \big(\mathcal{W}\mathcal{S}_1(t) + 2\mathcal{L}\big)\|g\|_\infty,$$
$$\mathcal{S}_1(t) := \|\sigma_0\|_{L^1(I)} + L\big(\|\sigma_a\|_{L^1([0,t])} + \|\sigma_b\|_{L^1([0,t])}\big).$$

Thus, by (2.3.6) and (2.3.9), we get

$$\begin{aligned}&\min\big\{\mathrm{TV}\big(\sigma(\cdot,s)\big), \mathrm{TV}\big(\phi(\cdot,s)\big)\big\} \\ &\leq \mathrm{e}^{t\mathcal{T}_3(t)}\Big(\mathrm{TV}(\sigma_0;I) + \mathrm{TV}\big(\sigma_a;(0,t)\big) + \big|\sigma_0(a+) - \sigma_a(0+)\big| \\ &\quad + \mathrm{TV}\big(\sigma_b;(0,t)\big) + \big|\sigma_0(b-) - \sigma_b(0+)\big|\Big) \\ &\quad + \min\big\{\mathcal{K}t\mathrm{e}^{\mathcal{C}_2(t)t}, (\mathcal{T}_2(t)/\mathcal{T}_1(t))(\mathrm{e}^{t\mathcal{T}_1} - 1)\big\} =: \mathcal{T}_4(t).\end{aligned} \tag{2.3.10}$$

To handle the first term on the right-hand side of (2.3.8), we use the estimate

$$\big|\partial_x(r-h)(x,t,u)\big| = \big|ug(u)\partial_x(V-U)\big| \leq C|u|\mathcal{M}\int_a^b \big|\rho(y,t) - \sigma(y,t)\big|\,\mathrm{d}y,$$

which implies

$$\big\|\partial_x(r-h)(x,s,\cdot)\big\|_{L^\infty(\mathcal{U})} \leq C\mathcal{M}\int_a^b \big|\rho(y,t) - \sigma(y,t)\big|\,\mathrm{d}y. \tag{2.3.11}$$

Next, in view of $\partial_u(r-h)(x,t,u) = \partial_u(ug(u))(V-U)$ we get

$$\begin{aligned}\big\|\partial_u(r-h)(\cdot,s,\cdot)\big\|_{L^\infty(I\times\mathcal{U})} &\leq \big\|\partial_u(ug(u))\big\|_\infty \mathcal{P}_3\int_a^b \big|\rho(y,s) - \sigma(y,s)\big|\,\mathrm{d}y \\ &\leq \big(\|g\|_\infty + \|g'\|_\infty\big)\mathcal{P}_3\int_a^b \big|\rho(y,s) - \sigma(y,s)\big|\,\mathrm{d}y.\end{aligned} \tag{2.3.12}$$

The third integral on the right-hand side of (2.3.8) is estimated by considering that

$$\big|(r-h)(a,t,u)\big| = \big|ug(u)(V-U)\big| \leq C|u|\mathcal{P}_3\int_a^b \big|\rho(y,t) - \sigma(y,t)\big|\,\mathrm{d}y,$$

hence

$$\big\|(r-h)(a,s,\cdot)\big\|_{L^\infty([-1,1])} \leq C\mathcal{P}_3\int_a^b \big|\rho(y,t) - \sigma(y,t)\big|\,\mathrm{d}y; \tag{2.3.13}$$

the fourth integral is treated similarly. Finally, combining (2.3.10) to (2.3.13) we get

$$\big\|\phi(\cdot,t)-\sigma(\cdot,t)\big\|_{L^1(I)} \leq B(t)\int_0^t\int_a^b \big|\rho(y,t)-\sigma(y,t)\big|\,\mathrm{d}y\,\mathrm{d}s, \tag{2.3.14}$$

where we define

$$B(t)=C\mathcal{M}+\mathcal{P}_3\Big(\big(\|g\|_\infty+\|g'\|_\infty\big)\mathcal{T}_4(t)+4C\Big). \tag{2.3.15}$$

Inserting $A(t)$ and (2.3.14) into (2.3.7) yields

$$\|\rho(\cdot,t)-\sigma(\cdot,t)\|_{L^1(I)} \leq A(t)+B(t)\int_0^t \big\|\rho(\cdot,s)-\sigma(\cdot,s)\big\|_{L^1(I)}\mathrm{d}s,$$

so by an application of Gronwall's lemma we arrive at the estimate

$$\begin{aligned}\big\|\rho(\cdot,t)-\sigma(\cdot,t)\big\|_{L^1(I)} &\leq A(t)+\int_0^t A(s)B(s)\exp\bigg(\int_s^t B(\tau)\,\mathrm{d}\tau\bigg)\,\mathrm{d}s\\ &\leq A(t)+B(t)\int_0^t A(s)\mathrm{e}^{B(t)(t-s)}\,\mathrm{d}s \leq A(t)\big(1+B(t)t\mathrm{e}^{B(t)t}\big).\end{aligned}$$

Consequently, we have proven the following lemma.

Lemma 2.5 (Lipschitz continuous dependence on initial and boundary data)**.** *If Assumptions* (2.2.1) *are in effect and ρ and σ are two entropy solutions to* (2.2.1) *with initial data $\rho_0,\ \sigma_0 \in \mathbf{BV}(I;\mathbb{R}^+)$ and $\rho_a,\rho_b,\sigma_a,\sigma_b \in \mathbf{BV}(]0,T[;[0,1])$, then the estimate*

$$\begin{aligned}&\big\|\rho(\cdot,T)-\sigma(\cdot,T)\big\|_{L^1(I)}\\ &\leq \Big(\|\rho_0-\sigma_0\|+L\big(\|\rho_a-\sigma_a\|_{L^1([0,T])}+\|\rho_b-\sigma_b\|_{L^1([0,T])}\big)\Big)\\ &\quad\times\big(1+B(T)T\mathrm{e}^{B(T)T}\big)\end{aligned} \tag{2.3.16}$$

holds for any $T>0$, where $B(T)$ is defined in (2.3.15)*.*

2.4 Existence of solutions

The proof of existence of solutions consists of several steps that are developed in this section. We construct a sequence of approximate solutions to (2.2.1) and derive the compactness estimates necessary to prove its convergence by Helly's theorem. We then show that the limit function is a weak entropy solution to the IBVP (2.2.1).

2.4.1 Numerical scheme

Fix $T>0$, we take a space step $\Delta x=(b-a)/M$ with $M\in\mathbb{N}$ and a time step Δt that is subject to a CFL condition specified later, and we set $\lambda=\Delta t/\Delta x$. We denote the center of the cells by $x_j := a+(j-1/2)\Delta x$ for $j = 1,\dots,M$, and $x_{j+1/2} = a+j\Delta x$, $j = 0,\dots,M$ are the cells interfaces. Moreover, we set $N_T = \lfloor T\Delta t \rfloor$ and, for $n = 0,\dots,N_T$ let $t^n = n\Delta t$ be the time mesh. The initial datum and the boundary data are approximated as

$$\rho_j^0 := \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} \rho_0(x)\,\mathrm{d}x, \quad j = 1,\dots,M;$$
$$\rho_a^n := \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} \rho_a(t)\,\mathrm{d}t, \quad \rho_b^n := \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} \rho_b(t)\,\mathrm{d}t \quad n = 0,\dots,N_T-1,$$

furthermore, we set $\rho_0^n := \rho_a^n$ and $\rho_{M+1}^n := \rho_b^n$. For $n = 0,\dots,N_T-1$, we set $\omega^k := \omega((k-1/2)\Delta x)$ for $k \in \mathbb{Z}$ and define

$$W_{j+1/2} := \Delta x \sum_{k=1}^{M} \omega^{k-j}, \quad V_{j+1/2}^n := \frac{\Delta x}{W_{j+1/2}} \sum_{k=1}^{M} \omega^{k-j} v(\rho_k^n) \quad \text{for } j = 0,\dots,M.$$

We define a piecewise constant approximate solution $\rho_\Delta(x,t)$ to (2.2.1) as

$$\rho_\Delta(x,t) = \rho_j^n \quad \text{for } t \in [t^n, t^{n+1}[,\ x \in [x_{j-1/2}, x_{j+1/2}[, \tag{2.4.1}$$

where $n = 0,\dots,N_T-1$, $j = 1,\dots,M$, through the numerical scheme

$$\rho_j^{n+1} = \rho_j^n - \lambda\big(F_{j+1/2}^n(\rho_j^n, \rho_{j+1}^n) - F_{j-1/2}^n(\rho_{j-1}^n, \rho_j^n)\big), \quad j = 1,\dots,M, \tag{2.4.2}$$

where a nonlocal version of the monotone numerical flux proposed in [55] and also used in [15] is employed, namely

$$F_{j+1/2}^n(u,w) = ug(w)V_{j+1/2}^n. \tag{2.4.3}$$

Next, we study the properties of the numerical scheme (2.4.2)-(2.4.3). Particularly, we going to prove that the sequence of approximate solutions $\rho_\Delta(x,t)$ satisfies the assumptions of Helly´s compactness theorem.

2.4.2 Uniform bounds of numerical solutions

Lemma 2.6 (Maximum principle). *If Assumptions 2.2.1 and the CFL condition*

$$\lambda\|v\|_\infty\big(\|g\|_\infty + \|g'\|_\infty\big) \le 1 \tag{2.4.4}$$

hold, then if $\rho_0(x) \in [0,1]$ for $x \in I$, the approximate solution satisfies

$$0 \le \rho_j^n \le 1 \quad \text{for all } j = 1,\dots,M \text{ and } n = 1,\dots,N_T.$$

Proof. We assume that $0 \le \rho_j^n \le 1$ for $j = 1,\dots,M$. From (2.4.2) we have

$$\rho_j^{n+1} = \rho_j^n - \lambda\big(\rho_j^n g(\rho_{j+1}^n)V_{j+1/2}^n - \rho_{j-1}^n g(\rho_j^n)V_{j-1/2}^n\big)$$

$$\leq \rho_j^n + \lambda \rho_{j-1}^n g(\rho_j^n) V_{j-1/2}^n \leq \rho_j^n + \lambda g(\rho_j^n) V_{j-1/2}^n =: G(\rho_j^n).$$

In view of $G'(\rho) = 1 + \lambda g'(\rho) V_{j-1/2}^n$, under the CFL condition (2.4.4), G is a non-decreasing function of ρ. Thus

$$\max_{\rho_j^n \in [0,1]} G(\rho_j^n) = G(1) = 1,$$

which implies that $\rho_j^{n+1} \leq 1$. Returning to (2.4.2), we obtain that

$$\rho_j^{n+1} \geq \rho_j^n - \lambda \rho_j^n g(\rho_{j+1}^n) V_{j+1/2}^n = \big(1 - \lambda g(\rho_{j+1}^n) V_{j+1/2}^n\big)\rho_j^n, \quad j = 1, \dots, M.$$

Consequently, if (2.4.4) is in effect, then $\rho_j^{n+1} \geq 0$. □

Lemma 2.7 (L^1 bound)**.** *Let Assumptions 2.2.1 and the CFL condition* (2.4.4) *hold. If* $\rho_0 \in L^\infty(I; [0,1])$ *and* $\rho_a, \rho_b \in L^\infty(\mathbb{R}^+; [0,1])$*, then, for all* $t > 0$*,* ρ_Δ *satisfies*

$$\big\|\rho_\Delta(\cdot, t)\big\|_{L^1(I)} \leq \|\rho_0\|_{L^1(I)} + L\big(\|\rho_a\|_{L^1([0,t])} + \|\rho_b\|_{L^1([0,t])}\big) =: \mathcal{C}_1(t), \tag{2.4.5}$$

where L *is defined in* (2.2.6)*.*

Proof. Lemma 2.6 (for $n = 0, \dots, N$) and the assumption $g(1) = 0$ imply

$$\begin{aligned} &\big\|\rho_\Delta(\cdot, t^{n+1})\big\|_{L^1(I)} \\ &= \Delta x\big(\rho_1^n + \cdots + \rho_M^n\big) + \Delta t\big(\rho_a^n g(\rho_1^n) V_{1/2} - \rho_M^n g(\rho_b^n) V_{M+1/2}^n\big) \\ &= \|\rho_\Delta(\cdot, t^n)\|_{L^1(I)} + \Delta t\big(\rho_a^n g(\rho_1^n) V_{1/2}^n + \rho_M^n (g(1) - g(\rho_b^n)) V_{M+1/2}^n\big) \\ &= \|\rho_\Delta(\cdot, t^n)\|_{L^1(I)} + \Delta t \rho_a^n g(\rho_1^n) V_{1/2}^n + \Delta t \rho_M^n g'(\zeta_j^n)(1 - \rho_b^n) V_{M+1/2}^n, \end{aligned}$$

where $\zeta_j^n \in \mathcal{I}(\rho_b^n, 1)$. Now, using item (ii) of Assumptions 2.2.1 and the nonnegativity of ρ_a^n and ρ_b^n we have

$$\|\boldsymbol{\rho}^{n+1}\|_{L^1(I)} \leq \|\rho^n\|_{L^1(I)} + \Delta t \|v\|_\infty \big(\|g\|_\infty + \|g'\|\big)(\rho_a^n + \rho_b^n).$$

An iterative argument yields the desired estimate (2.4.5). □

Lemma 2.8 (**BV** estimate in space)**.** *Let Assumptions 2.2.1 hold,* $\rho_0 \in BV(I; \mathbb{R}^+)$*,* $\rho_a, \rho_b \in BV(\mathbb{R}^+, [0,1])$ *and let* ρ_Δ *be given by* (2.4.2)*. If the CFL condition* (2.4.4) *holds, then for all* $n = 1, \dots, N_T$ *the discrete space* BV *estimate*

$$\sum_{j=0}^{M} \big|\rho_{j+1}^n - \rho_j^n\big| \leq \mathcal{C}_x(t^n) \tag{2.4.6}$$

is satisfied, where we define the time-dependent bound

$$\begin{aligned} \mathcal{C}_x(t^n) := \mathrm{e}^{\mathcal{K}_1 t^n} &\left(\sum_{j=0}^{M} \big|\rho_{j+1}^0 - \rho_j^0\big| + \sum_{m=1}^{n} \big|\rho_a^m - \rho_a^{m-1}\big| + \sum_{m=1}^{n} \big|\rho_b^m - \rho_b^{m-1}\big| \right) \\ &+ \mathcal{K}_1^{-1} \mathcal{K}_2 \big(\mathrm{e}^{\mathcal{K}_1 t^n} - 1\big) \end{aligned} \tag{2.4.7}$$

and $\mathcal{K}_1$ *and* $\mathcal{K}_2$ *are defined as*

$$\mathcal{K}_1 := \mathcal{L}\big(\|g'\|_\infty + \|g\|_\infty\big), \quad \mathcal{K}_2 := \big(\mathcal{W}\mathcal{C}_1(t) + 2\mathcal{L}\big)\|g\|_\infty. \tag{2.4.8}$$

Proof. Subtracting two versions of (2.4.2) from each other, we obtain

$$\rho_{j+1}^{n+1} - \rho_j^{n+1} = \mathcal{A}_j^n - \lambda \mathcal{B}_j^n,$$

where

$$\begin{aligned}\mathcal{A}_j^n &:= \rho_{j+1}^n - \rho_j^n - \lambda\big(\rho_{j+1}^n g(\rho_{j+2}^n)V_{j+3/2}^n - \rho_j^n g(\rho_{j+1}^n)V_{j+1/2}^n - \rho_j^n g(\rho_{j+1}^n)V_{j+3/2}^n \\ &\quad + \rho_{j-1}^n g(\rho_j^n)V_{j+1/2}^n\big),\\ \mathcal{B}_j^n &:= \rho_j^n g(\rho_{j+1}^n)V_{j+3/2}^n - \rho_j^n g(\rho_{j+1}^n)V_{j+1/2}^n + \rho_{j-1}^n g(\rho_j^n)V_{j-1/2}^n - \rho_{j-1}^n g(\rho_j^n)V_{j+1/2}^n.\end{aligned}$$

A straightforward computation reveals that $\mathcal{A}_j^n$ can be written in the form

$$\begin{aligned}\mathcal{A}_j^n &= \Big(1 - \lambda\big(g(\rho_{j+1}^n)V_{j+3/2}^n - \rho_j^n g'(\xi_{j+1/2}^n)V_{j+1/2}^n\big)\Big)(\rho_{j+1}^n - \rho_j^n)\\ &\quad - \lambda\rho_{j+1}^n g'(\xi_{j+3/2}^n)V_{j+3/2}^n(\rho_{j+2}^n - \rho_{j+1}^n) + \lambda g(\rho_j^n)V_{j+1/2}^n(\rho_j^n - \rho_{j-1}^n),\end{aligned} \tag{2.4.9}$$

where $\xi_{j+3/2}^n \in \mathcal{I}(\rho_{j+1}^n, \rho_{j+2}^n)$. By the CFL condition (2.4.4), the first term in the right-hand side of (2.4.9) is positive, thus summing over $j \in \{1, \dots, M-1\}$ yields

$$\begin{aligned}\sum_{j=1}^{M-1} \big|\mathcal{A}_j^n\big| &\le \sum_{j=1}^{M-1} \big|\rho_{j+1}^n - \rho_j^n\big| + \lambda g(\rho_1^n)V_{3/2}^n|\rho_1^n - \rho_a^n| - \lambda g(\rho_M^n)V_{M+1/2}^n|\rho_M^n - \rho_{M-1}^n|\\ &\quad - \lambda\rho_M^n g'(\xi_{M+1/2}^n)V_{M+1/2}^n|\rho_b^n - \rho_M^n| + \lambda\rho_1^n g'(\xi_{3/2}^n)V_{3/2}^n|\rho_2^n - \rho_1^n|.\end{aligned} \tag{2.4.10}$$

On the other hand,

$$\begin{aligned}\mathcal{B}_j^n &= -\rho_j^n\big(V_{j+1/2}^n - V_{j+3/2}^n\big)g'(\xi_{j+1/2}^n)(\rho_{j+1}^n - \rho_j^n)\\ &\quad + g(\rho_j^n)\big(V_{j+1/2}^n - V_{j-1/2}^n\big)(\rho_j^n - \rho_{j-1}^n) + \rho_j^n g(\rho_j^n)\big(V_{j+3/2}^n - 2V_{j+1/2}^n + V_{j-1/2}^n\big).\end{aligned}$$

Taking absolute values and summing over $j \in \{1, \dots, M-1\}$ we have

$$\begin{aligned}\lambda\sum_{j=1}^{M-1}\big|\mathcal{B}_j^n\big| &\le -\lambda\sum_{j=1}^{M-1}\rho_j^n g'(\xi_{j+1/2}^n)\big|V_{j+3/2}^n - V_{j+1/2}^n\big|\big|\rho_{j+1}^n - \rho_j^n\big|\\ &\quad + \lambda\sum_{j=1}^{M-1} g(\rho_j^n)\big|V_{j-1/2}^n - V_{j+1/2}^n\big|\big|\rho_j^n - \rho_{j-1}^n\big|\\ &\quad + \lambda\sum_{j=1}^{M-1}\rho_j^n g(\rho_j^n)\big|V_{j+3/2}^n - 2V_{j+1/2}^n + V_{j-1/2}^n\big|\\ &= -\lambda\sum_{j=1}^{M-1}\big(\rho_j^n g'(\xi_{j+1/2}^n) - g(\rho_{j+1}^n)\big)\big|V_{j+3/2}^n - V_{j+1/2}^n\big|\big|\rho_{j+1}^n - \rho_j^n\big|\\ &\quad + \lambda\sum_{j=1}^{M-1}\rho_j^n g(\rho_j^n)\big|V_{j+3/2}^n - 2V_{j+1/2}^n + V_{j-1/2}^n\big|\end{aligned}$$

$$+ \lambda g(\rho_1^n)|V_{3/2}^n - V_{1/2}^n||\rho_1^n - \rho_a^n|$$
$$- \lambda g(\rho_M^n)|V_{M+1/2}^n - V_{M-1/2}^n||\rho_M^n - \rho_{M-1}^n|.$$

By using the following estimations (which are proved in Appendix A)

$$\left|V_{j+3/2}^n - V_{j+1/2}^n\right| \le \mathcal{L}\Delta x, \quad \left|V_{j+3/2}^n - 2V_{j+1/2}^n + V_{j-1/2}^n\right| \le \Delta x^2 \mathcal{W}, \tag{2.4.11}$$

we obtain

$$\begin{aligned}
\lambda \sum_{j=1}^{M-1} \left|\mathcal{B}_j^n\right| \le & -\lambda \mathcal{L}\Delta x \sum_{j=1}^{M-1} \left(\rho_j^n g'(\xi_{j+1/2}^n) - g(\rho_{j+1}^n)\right)\left|\rho_{j+1}^n - \rho_j^n\right| \\
& + \lambda \Delta x^2 \mathcal{W} \sum_{j=1}^{M-1} \rho_j^n g(\rho_j^n) + \lambda g(\rho_1^n)|V_{3/2}^n - V_{1/2}^n||\rho_1^n - \rho_a^n| \\
& - \lambda g(\rho_M^n)|V_{M+1/2}^n - V_{M-1/2}^n||\rho_M^n - \rho_{M-1}^n| \\
\le & -\mathcal{L}\Delta t \sum_{j=1}^{M-1} \left(\rho_j^n g'(\xi_{j+1/2}^n) - g(\rho_{j+1}^n)\right)\left|\rho_{j+1}^n - \rho_j^n\right| \\
& + \Delta t \mathcal{W} \|g\|_\infty \|\rho\|_{L^1(I)} + \mathcal{L}\Delta t g(\rho_1^n)|\rho_1^n - \rho_a^n|.
\end{aligned} \tag{2.4.12}$$

We now deal with the boundary terms, first for the left boundary term. By the definition of the scheme (2.4.2),

$$\begin{aligned}
\rho_1^{n+1} - \rho_a^{n+1} &= \rho_1^n - \rho_a^n + \rho_a^n - \rho_a^{n+1} - \lambda\Big(\big(\rho_1^n g(\rho_2^n) - \rho_1^n g(\rho_1^n)\big)V_{3/2}^n \\
&\quad + \big(\rho_1^n g(\rho_1^n) - \rho_a^n g(\rho_1^n)\big)V_{3/2}^n + \rho_a^n g(\rho_1^n)(V_{3/2}^n - V_{1/2}^n)\Big) \\
&= \rho_1^n - \rho_a^n + \rho_a^n - \rho_a^{n+1} - \lambda\Big(\rho_1^n g'(\xi_{3/2}^n)V_{3/2}^n(\rho_2^n - \rho_1^n) \\
&\quad + (\rho_1^n - \rho_a^n)g(\rho_1^n)V_{3/2}^n + \rho_a^n g(\rho_1^n)(V_{3/2}^n - V_{1/2}^n)\Big) \\
&= \rho_a^n - \rho_a^{n+1} + \big(1 - \lambda g(\rho_1^n)V_{3/2}^n\big)(\rho_1^n - \rho_a^n) - \lambda \rho_1^n g'(\xi_{3/2}^n)(\rho_2^n - \rho_1^n) \\
&\quad - \lambda \rho_a^n g(\rho_1^n)\big(V_{3/2}^n - V_{1/2}^n\big).
\end{aligned}$$

Taking absolute values, invoking (2.4.4) and using (2.4.11) we obtain

$$\begin{aligned}
\left|\rho_1^{n+1} - \rho_a^{n+1}\right| \le & \left|\rho_a^n - \rho_a^{n+1}\right| + \big(1 - \lambda g(\rho_1^n)V_{3/2}^n\big)\left|\rho_1^n - \rho_a^n\right| \\
& - \lambda \rho_1^n g'(\xi_{3/2}^n)V_{3/2}^n\left|\rho_2^n - \rho_1^n\right| + \mathcal{L}\Delta t \rho_a^n g(\rho_1^n).
\end{aligned} \tag{2.4.13}$$

An analogous discussion of the other boundary term yields

$$\begin{aligned}
\left|\rho_b^{n+1} - \rho_M^{n+1}\right| \le & \left|\rho_b^n - \rho_b^{n+1}\right| + (1 + \lambda \rho_M^n g'(\xi_{M+1/2}^n)V_{M+1/2}^n)\left|\rho_M^n - \rho_b^n\right| \\
& + \lambda g(\rho_M^n)V_{M+1/2}^n\left|\rho_M^n - \rho_{M-1}^n\right| + \Delta t \rho_{M-1}^n g(\rho_M^n)\mathcal{L}.
\end{aligned} \tag{2.4.14}$$

Finally, collecting the estimates (2.4.10), (2.4.12), (2.4.13) and (2.4.14) we arrive at

$$\begin{aligned}
&\sum_{j=0}^{M} \left|\rho_{j+1}^{n+1} - \rho_j^{n+1}\right| \\
&\leq \left|\rho_a^n - \rho_a^{n+1}\right| + \sum_{j=0}^{M} \left|\rho_{j+1}^n - \rho_j^n\right| - \mathcal{L}\Delta t \sum_{j=1}^{M-1} \left(\rho_j^n g'(\xi_{j+1/2}^n) - g(\rho_{j+1}^n)\right)\left|\rho_{j+1}^n - \rho_j^n\right| \\
&\quad + \Delta t \mathcal{W} \|g\|_\infty \|\rho\|_{L^1(I)} + \Delta t \mathcal{L} g(\rho_1^n)|\rho_1^n - \rho_a^n| + |\rho_b^n - \rho_b^{n+1}| \\
&\quad + \Delta t \mathcal{L}(\rho_a^n g(\rho_1^n) + \rho_{M-1}^n g(\rho_M^n)) \\
&\leq \left|\rho_a^n - \rho_a^{n+1}\right| + \Big(1 + \Delta t \mathcal{L}(\|g'\|_\infty + \|g\|_\infty)\Big) \sum_{j=0}^{M} |\rho_{j+1}^n - \rho_j^n| \\
&\quad + \Delta t \mathcal{W} \|g\|_\infty \|\rho\|_{L^1(I)} + |\rho_b^n - \rho_b^{n+1}| + 2\Delta t \mathcal{L} \|g\|_\infty \\
&= \left|\rho_a^n - \rho_a^{n+1}\right| + (1 + \Delta t \mathcal{K}_1) \sum_{j=0}^{M} |\rho_{j+1}^n - \rho_j^n| + |\rho_b^n - \rho_b^{n+1}| + \Delta t \mathcal{K}_2,
\end{aligned}$$

with $\mathcal{L}_1$, $\mathcal{W}$, $\mathcal{K}_1$ and $\mathcal{K}_2$ defined as in (2.2.10), (2.2.11), and (2.4.8). The previous estimate implies (2.4.6)-(2.4.7) by standard arguments.

□

Lemma 2.9 (BV estimate in space and time). *Let $\rho_0 \in BV(I; \mathbb{R}^+)$ and ρ_a, $\rho_b \in BV(\mathbb{R}^+; [0,1])$. If Assumptions 2.2.1 and the CFL condition (2.4.4) hold, then for all $n = 0, \ldots, N_T$, the estimate*

$$\sum_{m=0}^{n-1} \sum_{j=0}^{M} \Delta t \left|\rho_{j+1}^m - \rho_j^m\right| + \sum_{m=0}^{n-1} \sum_{j=0}^{M+1} \Delta x \left|\rho_j^{m+1} - \rho_j^m\right| \leq \mathcal{C}_{xt}(t^n) \tag{2.4.15}$$

holds, where

$$\mathcal{C}_{xt}(t^n) = t^n \mathcal{C}_x(t^n) + \mathcal{C}_t(t^n) + \Delta x\big(\mathrm{TV}(\rho_a; [0,T]) + \mathrm{TV}(\rho_b; [0,T])\big).$$

Proof. By Lemma 2.8 we have

$$\sum_{m=0}^{n-1} \sum_{j=0}^{M} \Delta t \left|\rho_{j+1}^m - \rho_j^m\right| \quad \leq \quad n \Delta t \mathcal{C}_x(n\Delta t). \tag{2.4.16}$$

By the definition of numerical scheme (2.4.2), for $m \in \{0, \ldots, n-1\}$ and $j \in \{1, \ldots, M\}$ we get

$$\begin{aligned}
\left|\rho_j^{m+1} - \rho_j^m\right| = &\left|\lambda \rho_j^m g'(\xi_{j+1/2}^m) V_{j+1/2}^m (\rho_{j+1}^m - \rho_j^m) + \lambda \rho_j^m g(\rho_j^m)(V_{j+1/2}^m - V_{j-1/2}^m)\right. \\
&\left. + \lambda g(\rho_j^m) V_{j-1/2}^m (\rho_j^m - \rho_j^m)\right| \\
\leq &\ \lambda \|g'\|_\infty \|v\|_\infty \left|\rho_{j+1}^m - \rho_j^m\right| + \lambda \|g\|_\infty \mathcal{L} \Delta x \rho_j^m
\end{aligned}$$

$$+ \lambda\|g\|_\infty\|v\|_\infty\left|\rho_j^m - \rho_{j-1}^m\right|.$$

Multiplying the last inequality by Δx and summing for j from 1 to M we get

$$\sum_{j=1}^{M}\Delta x\left|\rho_j^{m+1}-\rho_j^m\right|$$
$$\leq \Delta t\|v\|_\infty\big(\|g'\|_\infty + \|g\|_\infty\big)\sum_{j=0}^{M}\left|\rho_{j+1}^m - \rho_j^m\right| + \Delta t\|g\|_\infty\mathcal{L}\|\rho^m\|_{L^1(I)}.$$

Lemmas 2.7 and 2.8 now imply that

$$\sum_{j=1}^{M}\Delta x\left|\rho_j^{m+1}-\rho_j^m\right| \leq \Delta t\|v\|_\infty\big(\|g'\|_\infty + \|g\|_\infty\big)\mathcal{C}_x(m\Delta t)$$
$$+ \Delta t\|g\|_\infty\mathcal{L}\mathcal{C}_1(m\Delta t) = \Delta t\mathcal{C}_t(m\Delta t),$$

where we define

$$\mathcal{C}_t(\tau) := \|v\|_{L^\infty([0,1])}\big(\|g'\|_\infty + \|g\|_\infty\big)\mathcal{C}_x(\tau) + \|g\|_\infty\mathcal{L}\mathcal{C}_1(\tau).$$

In particular,

$$\begin{aligned}\sum_{j=0}^{M+1}\Delta x\left|\rho_j^{m+1}-\rho_j^m\right| &= \Delta x\left|\rho_a^{m+1}-\rho_a^m\right| + \Delta x\left|\rho_b^{m+1}-\rho_b^m\right| + \sum_{j=1}^{M}\Delta x\left|\rho_j^{m+1}-\rho_j^m\right| \\ &\leq \Delta x\left|\rho_a^{m+1}-\rho_a^m\right| + \Delta x\left|\rho_b^{m+1}-\rho_b^m\right| + \Delta t\mathcal{C}_t(m\Delta t),\end{aligned} \tag{2.4.17}$$

which, summed over $m = 0, \ldots, n-1$, yields

$$\sum_{m=0}^{n-1}\sum_{j=0}^{M+1}\Delta x\left|\rho_j^{m+1}-\rho_j^m\right| \leq \Delta x\sum_{m=0}^{n-1}\big(\left|\rho_a^{m+1}-\rho_a^m\right| + \left|\rho_b^{m+1}-\rho_b^m\right|\big) + n\Delta t\mathcal{C}_t(n\Delta t). \tag{2.4.18}$$

Summing (2.4.16) and (2.4.18) we get the desired estimate (2.4.15). □

2.4.3 Convergence analysis

Lemmas 2.6 and 2.9 allow us to apply Helly's compactness theorem that ensures the existence of a subsequence of ρ_Δ, still denoted by ρ_Δ, that converges in L^1 to a function $\rho \in L^\infty(I \times [0,T])$, for all $T > 0$. Now we need to prove that this limit function is indeed an entropy weak solution to (2.2.1) in the sense of Definition 2.3. First we will show that the approximate solutions obtained by the scheme (2.4.2) satisfy a discrete entropy inequality. To this end, for $j = 1, \ldots, M$, $n = 0, \ldots, N_T - 1$, and $k \in \mathbb{R}$, we define

$$H_j^n(u,w,z) := w - \lambda\big(F_{j+1/2}^n(w,z) - F_{j-1/2}^n(u,w)\big),$$

$$\begin{aligned} G^{n,k}_{j+1/2}(u,w) &:= F^n_{j+1/2}(u\vee k, w\vee k) - F^n_{j+1/2}(k,k),\\ L^{n,k}_{j+1/2}(u,w) &:= F^n_{j+1/2}(k,k) - F^n_{j+1/2}(u\wedge k, w\wedge k),\end{aligned}$$

where $w\wedge z := \min\{w,z\}$, $w\vee z := \max\{w,z\}$, and $F^n_{j+1/2}(u,w)$ is defined as in (2.4.3). Observe that, due to the definition of the scheme,

$$\rho^{n+1}_j = H^n_j\big(\rho^n_{j-1},\rho^n_j,\rho^n_{j+1}\big),$$

and we also recall the equivalence $(s-k)^+ = s\vee k - k$ and $(s-k)^- = k - s\wedge k$.

Lemma 2.10 (Discrete entropy inequalities)**.** *If Assumptions 2.2.1 and the CFL condition* (2.4.4) *are in effect, then the approximate solution ρ_Δ in* (2.4.1) *satisfies the discrete entropy inequalities*

$$\begin{aligned} &\big(\rho^{n+1}_j - k\big)^+ - (\rho^n_j - k)^+ + \lambda\big(G^{n,k}_{j+1/2}(\rho^n_j,\rho^n_{j+1}) - G^{n,k}_{j-1/2}(\rho^n_{j-1},\rho^n_j)\big)\\ &+ \lambda\,\mathrm{sgn}^+\big(\rho^{n+1}_j - k\big) f(k)\big(V^n_{j+1/2} - V^n_{j-1/2}\big) \le 0 \quad \text{and} \qquad (2.4.19)\\ &\big(\rho^{n+1}_j - k\big)^- - \big(\rho^n_j - k\big)^- + \lambda\big(L^{n,k}_{j+1/2}(\rho^n_j,\rho^n_{j+1}) - L^{n,k}_{j-1/2}(\rho^n_{j-1},\rho^n_j)\big)\\ &+ \lambda\,\mathrm{sgn}^-(\rho^{n+1}_j - k) f(k)\big(V^n_{j+1/2} - V^n_{j-1/2}\big) \le 0 \qquad (2.4.20)\end{aligned}$$

for $j = 1,\dots,M,\ n = 0,\dots,N_T-1$ and $k\in\mathbb{R}$.

Proof. By the CFL condition (2.4.4), the map $(u,w,z)\mapsto H^n_j(u,w,z)$ satisfies

$$\begin{aligned} \partial_u H^n_j(u,w,z) &= \lambda g(w) V^n_{j-1/2} \ge 0,\\ \partial_w H^n_j(u,w,z) &= 1 - \lambda\big(g(z)V^n_{j+1/2} - u g'(w) V^n_{j-1/2}\big) \ge 0, \qquad (2.4.21)\\ \partial_z H^n_j(u,w,z) &= -\lambda w g'(z) V^n_{j+1/2} \ge 0.\end{aligned}$$

Notice that

$$H^n_j(k,k,k) = k - \lambda f(k)\big(V^n_{j+1/2} - V^n_{j-1/2}\big).$$

The monotonicity properties (2.4.21) imply that

$$\begin{aligned} &H^n_j(\rho^n_{j-1}\vee k, \rho^n_j\vee k, \rho^n_{j+1}\vee k) - H^n_j(k,k,k)\\ &\ge H^n_j(\rho^n_{j-1},\rho^n_j,\rho^n_{j+1}) \vee H^n_j(k,k,k) - H^n_j(k,k,k)\\ &= \big(H^n_j(\rho^n_{j-1},\rho^n_j,\rho^n_{j+1}) - H^n_j(k,k,k)\big)^+ = \big(\rho^{n+1}_j - k + \lambda f(k)\big(V^n_{j+1/2} - V^n_{j-1/2}\big)\big)^+,\end{aligned}$$

moreover, we also have

$$\begin{aligned} &H^n_j\big(\rho^n_{j-1}\vee k, \rho^n_j\vee k, \rho^n_{j+1}\vee k\big) - H^n_j(k,k,k)\\ &= \big(\rho^n_j\vee k\big) - \lambda\big(F^n_{j+1/2}\big(\rho^n_j\vee k, \rho^n_{j+1}\vee k\big) - F^n_{j-1/2}\big(\rho^n_{j-1}\vee k, \rho^n_j\vee k\big)\big)\\ &\quad - \big(k - \lambda\big(F^n_{j+1/2}(k,k) - F^n_{j-1/2}(k,k)\big)\big)\end{aligned}$$

$$
\begin{aligned}
&= \big(\rho_j^n \vee k\big) - k - \lambda\big(F_{j+1/2}^n\big(\rho_j^n \vee k, \rho_{j+1}^n \vee k\big) - F_{j-1/2}^n\big(\rho_{j-1}^n \vee k, \rho_j^n \vee k\big) \\
&\quad - F_{j+1/2}^n(k,k) + F_{j-1/2}^n(k,k)\big) \\
&= (\rho_j^n - k)^+ - \lambda\big(G_{j+1/2}^{n,k}(\rho_j^n, \rho_{j+1}^n) - G_{j-1/2}^{n,k}(\rho_{j-1}^n, \rho_j^n)\big),
\end{aligned}
$$

hence

$$
\begin{aligned}
&\big(\rho_j^n - k\big)^+ - \lambda\big(G_{j+1/2}^{n,k}\big(\rho_j^n, \rho_{j+1}^n\big) - G_{j-1/2}^{n,k}\big(\rho_{j-1}^n, \rho_j^n\big)\big) \\
&\geq \big(\rho_j^{n+1} - k + \lambda f(k)\big(V_{j+1/2}^n - V_{j-1/2}^n\big)\big)^+ \\
&= \operatorname{sgn}^+\big(\rho_j^{n+1} - k + \lambda f(k)\big(V_{j+1/2}^n - V_{j-1/2}^n\big)\big)\big(\rho_j^{n+1} - k + \lambda f(k)\big(V_{j+1/2}^n - V_{j-1/2}^n\big)\big) \\
&\geq \big(\rho_j^{n+1} - k\big) + \lambda \operatorname{sgn}^+\big(\rho_j^{n+1} - k\big) f(k)\big(V_{j+1/2}^n - V_{j-1/2}^n\big),
\end{aligned}
$$

which proves (2.4.19), while (2.4.20) is proven in an entirely analogous way. □

Remark 2.11. *Lemma 2.10 and its proof mimic standard arguments known for monotone schemes of local conservation laws [38], although the scheme is* not *monotone in the proper sense since the argument* (2.4.21) *suppresses the presence of* ρ_{j-1}^n*,* ρ_j^n *and* ρ_{j+1}^n *within* $V_{j-1/2}^n$ *and* $V_{j+1/2}^n$*.*

Lemma 2.12. *Let* $\rho_0 \in BV(]a,b[;\mathbb{R}^+)$*,* $\rho_a, \rho_b \in BV(\mathbb{R}^+;\mathbb{R}^+)$*, and Assumptions 2.2.1 and the CFL condition* (2.4.4) *be in effect. Then the piecewise constant approximate solutions* ρ_Δ *in* (2.4.1) *resulting from the HW scheme* (2.4.2) *converge, as* $\Delta x \to 0$*, towards an entropy weak solution of initial boundary value problem* (2.2.1)*.*

Proof. Adding and subtracting $\lambda G_{j+1/2}^{n,k}(\rho_j^n, \rho_j^n)$ we may rewrite (2.4.19) as

$$
\begin{aligned}
0 &\geq \big(\rho_j^{n+1} - k\big)^+ - \big(\rho_j^n - k\big)^+ + \lambda\big(G_{j+1/2}^{n,k}(\rho_j^n, \rho_{j+1}^n) - G_{j+1/2}^{n,k}(\rho_j^n, \rho_j^n)\big) \\
&\quad + \lambda\big(G_{j+1/2}^{n,k}(\rho_j^n, \rho_j^n) - G_{j-1/2}^{n,k}(\rho_{j-1}^n, \rho_j^n)\big) \\
&\quad + \lambda \operatorname{sgn}^+\big(\rho_j^{n+1} - k\big) f(k)\big(V_{j+1/2}^n - V_{j-1/2}^n\big).
\end{aligned}
$$

Let $\varphi \in C_c^1([0,T] \times I; \mathbb{R}^+)$ for some $T > 0$. Multiplying the inequality above by $\Delta x \varphi(x_j, t^n)$ and summing over $j = 1, \ldots, M$ and $n \in \mathbb{N}$ yields

$$
0 \geq T_1 + T_2 + T_3 + T_4,
$$

where we define the terms

$$
\begin{aligned}
T_1 &:= \Delta x \sum_{n=0}^{\infty} \sum_{j=1}^{M} \big((\rho_j^{n+1} - k)^+ - (\rho_j^n - k)^+\big)\varphi(x_j, t^n), \\
T_2 &:= \Delta t \sum_{n=0}^{\infty} \sum_{j=1}^{M} \big(G_{j+1/2}^{n,k}(\rho_j^n, \rho_{j+1}^n) - G_{j+1/2}^{n,k}(\rho_j^n, \rho_j^n)\big)\varphi(x_j, t^n), \\
T_3 &:= -\Delta t \sum_{n=0}^{\infty} \sum_{j=1}^{M} \big(G_{j-1/2}^{n,k}(\rho_{j-1}^n, \rho_j^n) - G_{j+1/2}^{n,k}(\rho_j^n, \rho_j^n)\big)\varphi(x_j, t^n),
\end{aligned}
$$

$$T_4 := \Delta t \sum_{n=0}^{\infty}\sum_{j=1}^{M} \operatorname{sgn}^+\big(\rho_j^{n+1}-k\big) f(k)\big(V_{j+1/2}^n - V_{j-1/2}^n\big)\varphi(x_j,t^n).$$

Summing by parts, we obtain

$$\begin{aligned} T_1 &= \Delta x \sum_{n=1}^{\infty}\sum_{j=1}^{M}\big(\rho_j^n - k\big)^+\varphi(x_j,t^{n-1}) - \Delta x \sum_{n=0}^{\infty}\sum_{j=1}^{M}\big(\rho_j^n - k\big)^+\varphi(x_j,t^n) \\ &= -\Delta x \sum_{j=1}^{M}\big(\rho_j^0 - k\big)^+\varphi(x_j,0) - \Delta x\Delta t \sum_{n=1}^{\infty}\sum_{j=1}^{M}\big(\rho_j^n - k\big)^+\frac{\varphi(x_j,t^n)-\varphi(x_j,t^{n-1})}{\Delta t}, \end{aligned}$$

and by the dominated convergence theorem,

$$T_1 \overset{\Delta x\to 0^+}{\longrightarrow} -\int_a^b \big(\rho_0(x)-k\big)^+\varphi(x_j,0)\,\mathrm{d}x - \int_0^\infty\int_a^b \big(\rho(x,t)-k\big)^+\partial_t\varphi(x,t)\,\mathrm{d}x\,\mathrm{d}t.$$

Again by the dominated convergence theorem,

$$\begin{aligned} T_4 &= \Delta t\Delta x \sum_{n=0}^{\infty}\sum_{j=1}^{M}\operatorname{sgn}^+\big(\rho_j^{n+1}-k\big) f(k)\frac{V_{j+1/2}^n - V_{j-1/2}^n}{\Delta x}\varphi(x_j,t^n) \\ &\overset{\Delta x\to 0^+}{\longrightarrow} \int_0^\infty\int_a^b \operatorname{sgn}^+\big(\rho(x,t)-k\big) f(k)(\partial_x V)\varphi(x,t)\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

Concerning T_2 and T_3, we get

$$\begin{aligned} T_2 + T_3 &= \Delta t \sum_{n=0}^{\infty}\sum_{j=1}^{M}\big(G_{j+1/2}^{n,k}\big(\rho_j^n,\rho_{j+1}^n\big) - G_{j+1/2}^{n,k}\big(\rho_j^n,\rho_j^n\big)\big)\varphi(x_j,t^n) \\ &\quad - \Delta t \sum_{n=0}^{\infty}\sum_{j=0}^{M-1}\big(G_{j+1/2}^{n,k}\big(\rho_j^n,\rho_{j+1}^n\big) - G_{j+3/2}^{n,k}\big(\rho_{j+1}^n,\rho_{j+1}^n\big)\big)\varphi(x_{j+1},t^n) \\ &= T_{20} + T_{30} = T_{23}, \end{aligned}$$

where we define

$$\begin{aligned} T_{20} &:= \Delta t \sum_{n=0}^{\infty}\sum_{j=1}^{M-1}\Big(\big(G_{j+1/2}^{n,k}(\rho_j^n,\rho_{j+1}^n) - G_{j+1/2}^{n,k}(\rho_j^n,\rho_j^n)\big)\varphi(x_j,t^n) \\ &\qquad - \big(G_{j+1/2}^{n,k}(\rho_j^n,\rho_{j+1}^n) - G_{j+3/2}^{n,k}(\rho_{j+1}^n,\rho_{j+1}^n)\big)\varphi(x_{j+1},t^n)\Big), \\ T_{30} &:= \Delta t \sum_{n=0}^{\infty}\Big(\big(G_{M+1/2}^{n,k}(\rho_M^n,\rho_b^n) - G_{M+1/2}^{n,k}(\rho_M^n,\rho_M^n)\big)\varphi(x_M,t^n) \\ &\qquad - \big(G_{1/2}^{n,k}(\rho_a^n,\rho_1^n) - G_{3/2}^{n,k}(\rho_1^n,\rho_1^n)\big)\varphi(x_1,t^n)\Big). \end{aligned}$$

Now we define

$$
\begin{aligned}
S := -\Delta x \Delta t \sum_{n=0}^{+\infty} \sum_{j=1}^{M} G^{n,k}_{j+1/2}(\rho^n_j, \rho^n_j) \frac{\varphi(x_{j+1}, t^n) - \varphi(x_j, t^n)}{\Delta x} \\
- L \Delta t \sum_{n=0}^{+\infty} \big((\rho^n_a - k)^+ \varphi(a, t^n) + (\rho^n_b - k)^+ \varphi(b, t^n) \big).
\end{aligned}
\tag{2.4.22}
$$

Since

$$
\begin{aligned}
G^{n,k}_{j+1/2}\big(\rho^n_j, \rho^n_j\big) &= F^n_{j+1/2}\big(\rho^n_j \vee k, \rho^n_j \vee k\big) - F^n_{j+1/2}(k,k) \\
&= \big(f(\rho^n_j \vee k) - f(k)\big) V^n_{j+1/2} = \operatorname{sgn}^+(\rho^n_j - k)\big(f(\rho^n_j) - f(k)\big) V^n_{j+1/2},
\end{aligned}
$$

it follows that

$$
\begin{aligned}
S \overset{\Delta x \to 0^+}{\longrightarrow} & - \int_0^{+\infty} \int_a^b \operatorname{sgn}^+(\rho(x,t) - k)\big(f(\rho^n_j) - f(k)\big) V \partial_x \varphi(x,t) \, \mathrm{d}x \, \mathrm{d}t \\
& - L \left(\int_0^{+\infty} (\rho_a(t) - k)^+ \varphi(a,t) \, \mathrm{d}t + \int_0^{+\infty} (\rho_b(t) - k)^+ \varphi(b,t) \, \mathrm{d}t \right).
\end{aligned}
$$

Let us rewrite S (2.4.22) as follows

$$
\begin{aligned}
S = & -\Delta t \sum_{n=0}^{+\infty} \sum_{j=1}^{M} G^{n,k}_{j+1/2}(\rho^n_j, \rho^n_j)\big(\varphi(x_{j+1}, t^n) - \varphi(x_j, t^n)\big) \\
& - L \Delta t \sum_{n=0}^{+\infty} \big((\rho^n_a - k)^+ \varphi(a, t^n) + (\rho^n_b - k)^+ \varphi(b, t^n) \big) \\
= & -\Delta t \sum_{n=0}^{+\infty} \left(\sum_{j=1}^{M} G^{n,k}_{j+1/2}(\rho^n_j, \rho^n_j) \varphi(x_{j+1}, t^n) \right. \\
& \left. \qquad - \sum_{j=0}^{M-1} G^{n,k}_{j+3/2}(\rho^n_{j+1}, \rho^n_{j+1}) \varphi(x_{j+1}, t^n) \right) \\
& - L \Delta t \sum_{n=0}^{+\infty} \big((\rho^n_a - k)^+ \varphi(a, t^n) + (\rho^n_b - k)^+ \varphi(b, t^n) \big) = S_{20} + S_{30},
\end{aligned}
$$

where we define

$$
\begin{aligned}
S_{20} &:= -\Delta t \sum_{n=0}^{\infty} \sum_{j=1}^{M-1} \big\{ G^{n,k}_{j+1/2}\big(\rho^n_j, \rho^n_j\big) - G^{n,k}_{j+3/2}\big(\rho^n_{j+1}, \rho^n_{j+1}\big) \big\} \varphi(x_{j+1}, t^n), \\
S_{30} &:= -\Delta t \sum_{n=0}^{+\infty} \big(G^{n,k}_{M+1/2}(\rho^n_M, \rho^n_M) \varphi(x_{M+1}, t^n) - G^{n,k}_{3/2}(\rho^n_1, \rho^n_1) \varphi(x_1, t^n) \big)
\end{aligned}
$$

$$- L\Delta t \sum_{n=0}^{+\infty} \big((\rho_a^n - k)^+ \varphi(a, t^n) + (\rho_b^n - k)^+ \varphi(b, t^n) \big).$$

Adding and subtracting $G^{n,k}_{j+1/2}(\rho_j^n, \rho_j^n)$, we may rewrite S_{20} as

$$\begin{aligned} S_{20} = &-\Delta t \sum_{n=0}^{\infty} \sum_{j=1}^{M-1} \big(G^{n,k}_{j+1/2}(\rho_j^n, \rho_{j+1}^n) - G^{n,k}_{j+1/2}(\rho_j^n, \rho_j^n) \big) \varphi(x_{j+1}, t^n) \\ &- \Delta t \sum_{n=0}^{\infty} \sum_{j=1}^{M-1} \big(G^{n,k}_{j+1/2}(\rho_j^n, \rho_{j+1}^n) - G^{n,k}_{j+3/2}(\rho_{j+1}^n, \rho_{j+1}^n) \big) \varphi(x_{j+1}, t^n). \end{aligned}$$

We evaluate now the distance between T_{20} and S_{20}:

$$\begin{aligned} &\big| T_{20} - S_{20} \big| \\ &\leq \Delta t \sum_{n=0}^{\infty} \sum_{j=1}^{M-1} \big| G^{n,k}_{j+1/2}(\rho_j^n, \rho_{j+1}^n) - G^{n,k}_{j+1/2}(\rho_j^n, \rho_j^n) \big| \big| \varphi(x_{j+1}, t^n) - \varphi(x_j, t^n) \big|. \end{aligned}$$

Since

$$\begin{aligned} &\big| G^{n,k}_{j+1/2}(\rho_j^n, \rho_{j+1}^n) - G^{n,k}_{j+1/2}(\rho_j^n, \rho_j^n) \big| \\ &= \big| F^n_{j+1/2}(\rho_j^n \vee k, \rho_{j+1}^n \vee k) - F^n_{j+1/2}(\rho_j^n \vee k, \rho_j^n \vee k) \big| \\ &= \big| (\rho_j^n \vee k) \big(g(\rho_{j+1}^n \vee k) - g(\rho_j^n \vee k) \big) V^n_{j+1/2} \big| \\ &= \big| (\rho_j^n \vee k) g'(\eta^n_{j+1/2}) \big((\rho_{j+1}^n \vee k) - (\rho_j^n \vee k) \big) V^n_{j+1/2} \big| \\ &\leq \|v\|_\infty \|g'\|_\infty \big| (\rho_j^n \vee k) \big((\rho_{j+1}^n \vee k) - (\rho_j^n \vee k) \big) \big| \\ &\leq \|v\|_\infty \|g'\|_\infty \big| \rho_{j+1}^n - \rho_j^n \big| \leq L \big| \rho_{j+1}^n - \rho_j^n \big|, \end{aligned}$$

in light of the uniform BV estimate (2.4.6) we deduce that

$$\begin{aligned} \big| T_{20} - S_{20} \big| &\leq L \Delta x \Delta t \|\partial_x \varphi\|_\infty \sum_{n=0}^{\infty} \sum_{j=1}^{M-1} \big| \rho_{j+1}^n - \rho_j^n \big| \\ &\leq L \Delta x T \|\partial_x \varphi\|_\infty \max_{0 \leq n \leq T/\Delta t} \mathrm{TV}\big(\rho_\Delta(\cdot, t^n) \big) = \mathcal{O}(\Delta x). \end{aligned} \tag{2.4.23}$$

Furthermore, we obtain

$$\begin{aligned} S_{30} - T_{30} = &-\Delta t \sum_{n=0}^{\infty} \big(G^{n,k}_{M+1/2}(\rho_M^n, \rho_M^n) \varphi(x_{M+1}, t^n) - G^{n,k}_{3/2}(\rho_1^n, \rho_1^n) \varphi(x_1, t^n) \big) \\ &- L\Delta t \sum_{n=0}^{\infty} \big((\rho_a^n - k)^+ \varphi(a, t^n) + (\rho_b^n - k)^+ \varphi(b, t^n) \big) \\ &- \Delta t \sum_{n=0}^{\infty} \Big(\big(G^{n,k}_{M+1/2}(\rho_M^n, \rho_b^n) - G^{n,k}_{M+1/2}(\rho_M^n, \rho_M^n) \big) \varphi(x_M, t^n) \end{aligned}$$

$$- \big(G^{n,k}_{1/2}(\rho^n_a, \rho^n_1) - G^{n,k}_{3/2}(\rho^n_1, \rho^n_1)\big)\varphi(x_1, t^n)\Big),$$

which we can write as $S_{30} - T_{30} = R_1 + R_2 + R_3$ with

$$\begin{aligned}
R_1 &:= \Delta t \sum_{n=0}^{\infty} \big(G^{n,k}_{1/2}(\rho^n_a, \rho^n_1)\varphi(x_1, t^n) - L(\rho^n_a - k)^+ \varphi(a, t^n)\big),\\
R_2 &:= -\Delta t \sum_{n=0}^{\infty} \big(L(\rho^n_b - k)^+ \varphi(b, t^n) + G^{n,k}_{M+1/2}(\rho^n_M, \rho^n_b)\varphi(x_M, t^n)\big),\\
R_3 &:= -\Delta t \sum_{n=0}^{\infty} G^{n,k}_{M+1/2}(\rho^n_M, \rho^n_M)\big(\varphi(x_{M+1}, t^n) - \varphi(x_M, t^n)\big).
\end{aligned}$$

Observe that

$$\begin{aligned}
\partial_u F^n_{j+1/2}(u, z) &= \partial_u\big(ug(z)V^n_{j+1/2}\big) = g(z)V^n_{j+1/2} \geq 0,\\
\partial_z F^n_{j+1/2}(u, z) &= \partial_z\big(ug(z)V^n_{j+1/2}\big) = ug'(z)V^n_{j+1/2} \leq 0,
\end{aligned}$$

meaning that the numerical flux is increasing with respect to the first variable and the decreasing with respect to the second one. Consequently,

$$\begin{aligned}
G^{n,k}_{j+1/2}(u, z) &= F^n_{j+1/2}(u \vee k, z \vee k) - F^n_{j+1/2}(k, k)\\
&\geq F^n_{j+1/2}(k, z \vee k) - F^n_{j+1/2}(k, k) = (kg(z \vee k) - kg(k))V^n_{j+1/2}\\
&= kg'(\nu^n_{j+1/2})((z \vee k) - k)V^n_{j+1/2} \geq -\|v\|_\infty \|g'\|_\infty (z - k)^+\\
&\geq -L(z - k)^+,\\
G^{n,k}_{j+1/2}(u, z) &= F^n_{j+1/2}(u \vee k, z \vee k) - F^n_{j+1/2}(k, k)\\
&\leq F^n_{j+1/2}(u \vee k, k) - F^n_{j+1/2}(k, k) = (u \vee k)g(k)V^n_{j+1/2} - kg(k)V^n_{j+1/2}\\
&= \big((u \vee k) - k\big)g(k)V^n_{j+1/2} \leq \|v\|_\infty \|g\|_\infty (u - k)^+ \leq L(u - k)^+,
\end{aligned}$$

hence

$$\begin{aligned}
R_1 &= \Delta t \sum_{n=0}^{\infty} G^{n,k}_{1/2}(\rho^n_a, \rho^n_1)\big(\varphi(x_1, t^n) - \varphi(a, t^n)\big)\\
&\quad + \Delta t \sum_{n=0}^{\infty} \big(G^{n,k}_{1/2}(\rho^n_a, \rho^n_1) - L(\rho^n_a - k)^+\big)\varphi(a, t^n)\\
&\leq LT\Delta x \|\partial_x \varphi\|_\infty \sup_{0 \leq n \leq T/\Delta t} (\rho^n_a - k)^+ + L\Delta t \sum_{n=0}^{\infty} \big((\rho^n_a - k)^+ - (\rho^n_a - k)^+\big)\varphi(a, t^n)\\
&\leq T\Delta x \|\partial_x \varphi\|_{L^\infty} \|\rho_a\|_{L^\infty([0,t])} = \mathcal{O}(\Delta x),\\
R_2 &= -\Delta t \sum_{n=0}^{\infty} \big(L(\rho^n_b - k)^+ + G^{n,k}_{M+1/2}(\rho^n_M, \rho^n_b)\big)\varphi(b, t^n)
\end{aligned}$$

$$
\begin{aligned}
&- \Delta t \sum_{n=0}^{\infty} G^{n,k}_{M+1/2}(\rho^n_M, \rho^n_b)\big(\varphi(x_M, t^n) - \varphi(b, t^n)\big)\\
&\leq -L\Delta t \sum_{n=0}^{\infty} \big((\rho^n_b - k)^+ - (\rho^n_b - k)^+\big)\varphi(b, t^n)\\
&\quad + LT\Delta x \|\partial_x \varphi\|_{L^\infty} \sup_{0\leq n \leq T/\Delta t} (\rho^n_b - k)^+\\
&\leq LT\Delta x \|\partial_x \varphi\|_{L^\infty} \|\rho_b\|_{L^\infty([0,t])} = \mathcal{O}(\Delta x),\\
R_3 &\leq \Delta t \left| \sum_{n=0}^{\infty} G^{n,k}_{M+1/2}(\rho^n_M, \rho^n_M)\big(\varphi(x_{M+1}, t^n) - \varphi(x_M, t^n)\big) \right|\\
&\leq \Delta t \Delta x \|\partial_x \varphi\|_{L^\infty} \sum_{n=0}^{+\infty} \big|G^{n,k}_{M+1/2}(\rho^n_M, \rho^n_M)\big|.
\end{aligned}
$$

Taking into account that

$$
\begin{aligned}
G^{n,k}_{M+1/2}(\rho^n_M, \rho^n_M) &= F^n_{M+1/2}(\rho^n_M \vee k, \rho^n_M \vee k) - F^n_{M+1/2}(k, k)\\
&= (\rho^n_M \vee k) g(\rho^n_M \vee k) V^n_{M+1/2} - k g(k) V^n_{M+1/2}\\
&= \big((\rho^n_M \vee k)(g(\rho^n_M \vee k) - g(k)) + (\rho^n_M \vee k - k) g(k)\big) V^n_{M+1/2}\\
&= \big((\rho^n_M \vee k) g'(\eta^n_{j+1/2})(\rho^n_M \vee k - k) + (\rho^n_M \vee k - k) g(k)\big) V^n_{M+1/2}\\
&= \big((\rho^n_M \vee k) g'(\eta^n_{j+1/2}) + g(k)\big)(\rho^n_M \vee k - k) V^n_{M+1/2},
\end{aligned}
$$

we get

$$
\begin{aligned}
R_3 &= \Delta t \Delta x \|\partial_x \varphi\|_{L^\infty} \sum_{n=0}^{+\infty} \big| \big((\rho^n_M \vee k) g'(\eta^n_{j+1/2}) + g(k)\big)(\rho^n_M \vee k - k) V^n_{M+1/2} \big|\\
&\leq \Delta t \Delta x \|\partial_x \varphi\|_{L^\infty} \sum_{n=0}^{+\infty} \big| \big(g'(\eta^n_{j+1/2}) + g(k)\big)(\rho^n_M \vee k - k) V^n_{M+1/2} \big|\\
&\leq L\Delta t \Delta x \|\partial_x \varphi\|_{L^\infty} \|v\|_\infty \sum_{n=0}^{\infty} \big|\rho^n_M \vee k - k\big|\\
&= L\Delta t \Delta x \|\partial_x \varphi\|_{L^\infty} \|v\|_\infty \sum_{n=0}^{\infty} (\rho^n_M - k)^+\\
&\leq LT\Delta x \|\partial_x \varphi\|_{L^\infty} \|v\|_\infty \sup_{0\leq n\leq T/\Delta t} \|\rho^n\|_\infty \leq LT\Delta x \|\partial_x \varphi\|_{L^\infty} \|v\|_\infty = \mathcal{O}(\Delta x),
\end{aligned}
$$

thanks to the maximum principle estimate. Hence, $S_{30} - T_{30} \leq \mathcal{O}(\Delta x)$, so that we finally get

$$
0 \geq T_1 + T_2 + T_3 + T_4 = T_1 + T_4 + T_{23} \pm S = T_1 + T_4 + S - \mathcal{O}(\Delta x).
$$

This concludes the proof. □

Proof of Theorem 2.4. The existence of solutions to problem (2.2.1) follows from the results of Section 2.4. The uniqueness is ensured by the Lipschitz continuous dependence of solutions to (2.2.1) on initial and boundary data, see Section 2.3. The estimates on the solution to (2.2.1) are obtained from the corresponding discrete estimates passing to the limit. In particular, the L^1 bound follows from (2.7), the Maximum principle from (2.6), the total variation bound from (2.8) and the Lipschitz continuity in time from (2.4.17), since $\Delta x = \Delta t/\lambda$ and taking $\lambda = 1/L$. □

2.5 A second-order scheme

The scheme (2.4.2), (2.4.3) is only first-order accurate. We propose here a second-order accuracy scheme, constructed using MUSCL-type variable extrapolation and Runge-Kutta temporal differencing. To implement it, we approximate $\rho(x, t^n)$ by a piecewise linear functions in each cell, i.e. $\hat{\rho}_j(x, t^n) = \rho_j^n + \sigma_j^n(x - x_j)$, where the slopes σ_j^n are calculated via the generalized minmod limiter, i.e.

$$\sigma_j^n = \frac{1}{\Delta x}\,\mathrm{minmod}\Big(\vartheta(\rho_j^n - \rho_{j-1}^n), \frac{1}{2}(\rho_{j+1}^n - \rho_{j-1}^n), \vartheta(\rho_{j+1}^n - \rho_j^n)\Big),$$

where $\vartheta \in [1,2]$ and

$$\mathrm{minmod}(a,b,c) := \begin{cases} \mathrm{sgn}(a)\min\{|a|,|b|,|c|\} & \text{if } \mathrm{sgn}(c) = \mathrm{sgn}(b) = \mathrm{sgn}(a), \\ 0 & \text{otherwise.} \end{cases}$$

This extrapolation enables one to define left and right values at the cell interfaces respectively by

$$\begin{aligned} \rho_{j+1/2}^{\mathrm{L}} &:= \hat{\rho}_j\left(x_j + \frac{\Delta x}{2}, t^n\right) = \rho_j^n + \sigma_j^n \frac{\Delta x}{2}, \\ \rho_{j-1/2}^{\mathrm{R}} &:= \hat{\rho}_j\left(x_j - \frac{\Delta x}{2}, t^n\right) = \rho_j^n - \sigma_j^n \frac{\Delta x}{2}. \end{aligned}$$

and

$$\begin{aligned} \hat{V}_{j+1/2}^n &= \frac{1}{\hat{W}_{j+1/2}} \int_a^b v(\hat{\rho}(y,t))\omega(y - x_{j+1/2})\,\mathrm{d}y \\ &= \frac{1}{\hat{W}_{j+1/2}} \sum_{k=1}^{M} \int_{x_{k-1/2}}^{x_{k+1/2}} v(\hat{\rho}_k(y,t))\omega(y - x_{j+1/2})\,\mathrm{d}y \\ &= \frac{\Delta x}{2\hat{W}_{j+1/2}} \sum_{k=1}^{M} \int_{-1}^{1} v\left(\hat{\rho}_k(\frac{\Delta x}{2}y + x_k, t^n)\right) \omega\left(\frac{\Delta x}{2}y + (k - j + 1/2)\Delta x\right) \mathrm{d}y \\ &= \frac{\Delta x}{2\hat{W}_{j+1/2}} \sum_{k=1}^{M} \sum_{e=1}^{N_G} p_e v\left(\hat{\rho}_k(\frac{\Delta x}{2}y_e + x_k, t^n)\right) \omega\left(\frac{\Delta x}{2}y_e + (k - j + 1/2)\Delta x\right) \end{aligned}$$

$$= \frac{\Delta x}{2\hat{W}_{j+1/2}} \sum_{k=1}^{M} \sum_{e=1}^{N_G} p_e v \left(\rho_k^n + \frac{\Delta x}{2} y_e \sigma_k^n \right) \omega \left(\frac{\Delta x}{2} y_e + (k - j + 1/2)\Delta x \right)$$

where $\hat{W}_{j+1/2} = \int_a^b \omega(y - x_{j+1/2}) \mathrm{d}y$ is computed in exact form, and y_e are the Gauss-Lobatto-Quadrature points. The MUSCL version of the numerical flux reads

$$F_{j+1/2}^n := \rho_{j+1/2}^{\mathrm{L}} g(\rho_{j+1/2}^{\mathrm{R}}) \hat{V}_{j+1/2}^n.$$

To achieve formal second-order accuracy also in time, we use second-order Runge-Kutta (RK) time stepping. More precisely, if we write our scheme with first-order Euler time differences and second-order spatial differences formally as

$$\rho_j^{n+1} = \rho_j^n - \lambda L_j(\rho^n) := \rho_j^n - \lambda\big(F_{j+1/2}^n - F_{j-1/2}^n\big), \tag{2.5.1}$$

then the RK version takes the two-step form

$$\rho_j^1 = \rho_j^n - \lambda L_j(\rho^n); \quad \rho_j^{n+1} = \frac{1}{2}(\rho_j^n + \rho_j^1) - \frac{\lambda}{2} L_j(\rho_j^1). \tag{2.5.2}$$

2.6 Numerical examples

In this section we solve (2.2.1) by using the numerical scheme (2.4.2) on the x-interval $I = [0, 1]$ with suitable boundary conditions and $t \in [0, T]$, with T specified later. In the numerical examples we consider the equation (2.2.1) with $g(\rho) = 1-\rho$ and $v(\rho) = (1-\rho)^4$, where we recall that $f(\rho) = \rho g(\rho)$ and $V(x, t) = (\omega * v(\rho))(x, t)$, where the kernel function $\omega(x)$ is specified later in each case. For numerical experiments the interval I is subdivided into M subintervals of length $\Delta x = 1/M$, and the time step is computed taking account the CFL condition (2.4.4), and for each numerical experiment, we specify the inital and boundary conditions.

2.6.1 Example 1

In this example we compare numerical approximations obtained by scheme (2.4.2) with those generated by an adapted LxF-type scheme proposed by Goatin and Rossi in [50], starting from the initial and boundary conditions

$$\rho_0(x) = 0.2 \quad \text{for } x \in I; \quad \rho_a(t) = 0.1, \quad \rho_b(t) = 0.5 \quad \text{for } t > 0.$$

Here we employ the symmetric kernel function

$$\omega(y) = \omega_1(y) := \frac{3}{4\eta} \left(1 - \frac{y^2}{\eta^2} \right) \chi_{[-\eta,\eta]}(y) \tag{2.6.1}$$

with $\eta = 0.05$. In Figure 2.6.1 we display the numerical approximations for $M = 800$ at simulated times $T = 2$ and $T = 8$ and compare them with the reference solution which is computed with the LxF scheme with $M_{\text{ref}} = 12800$. We observe better accuracy for the proposed scheme. This property also becomes apparent in Table 2.6.1 where we show the corresponding approximate L^1 errors for discretizations $M = 100 \times 2^l$, $l = 0, 1, \dots, 4$ and respective experimental orders of convergence (E.O.C.). We observe that the approximate L^1 errors decrease as the grid is refined and E.O.C. assumes values close to one, in agreement with the formal first order of accuracy of the scheme.

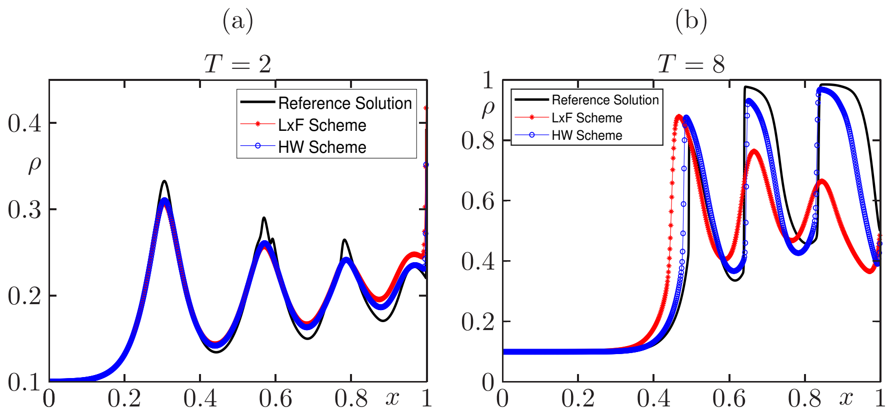

Figure 2.6.1: Example 1: numerical approximations obtained with HW and LxF numerical flux with $M = 800$ and symmetric kernel $\omega_1(y)$ with $\eta = 0.05$ at simulated times (a) $T = 2$, (b) $T = 8$.

Table 2.6.1: Example 1: approximate L^1-error $e_M(u)$ and E.O.C. for the LxF and HW numerical fluxes with $\Delta x = 1/M$ and symmetric kernel (2.6.1) with $\eta = 0.05$ at simulated times $T = 2$ and $T = 8$.

	$T=2$				$T=8$			
	LxF		HW		LxF		HW	
M	$e_M(\rho_\Delta)$	E.O.C.	$e_M(\rho_\Delta)$	E.O.C.				
100	1.71e-01	—	1.02e-01	—	6.34e-01	—	5.28e-01	—
200	1.11e-01	0.63	5.42e-02	0.92	5.96e-01	0.89	5.80e-01	-0.14
400	4.64e-02	1.25	2.74e-02	0.98	4.89e-01	0.28	3.69e-01	0.65
800	2.02e-02	1.20	1.39e-02	0.98	2.86e-01	0.78	1.19e-01	1.63
1600	9.58e-03	1.07	6.84e-03	1.03	1.11e-01	1.37	4.18e-02	1.51

2.6.2 Example 2

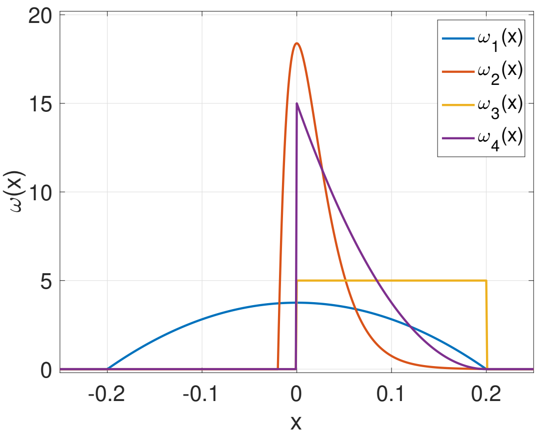

Figure 2.6.2: Example 2: kernel functions $\omega(x) = \omega_i(x)$, $i = 1, \dots, 4$, given by (2.6.1), (2.6.2) and (2.6.3) with $\eta = 0.2$.

We now compare the dynamics in the solution of model (2.2.1) by using various kernel functions. We consider the symmetric kernel ω_1 (2.6.1) as in Example 1 along with a non-symmetric kernel

$$\omega_2(y) := \frac{20}{\eta}\left(\frac{5y}{\eta} + \frac{1}{2}\right) \exp\left(-\frac{10y}{\eta} - 1\right) \chi_{[-\frac{\eta}{10},\eta]}(y) \tag{2.6.2}$$

and the anisotropic discontinuous kernels

$$\omega_3(y) := \frac{1}{\eta}\chi_{[0,\eta]}(y), \quad \omega_4(y) := \frac{3}{\eta^3}(\eta - y)^2 \chi_{[0,\eta]}(y), \tag{2.6.3}$$

where in all cases we choose $\eta = 0.2$. In Figure 2.6.2 we display the different kernel functions. The initial and boundary conditions are given by

$$\rho_0(x) = 0.1 \quad \text{for } x \in I; \quad \rho_a(t) = 0.1, \quad \rho_b(t) = 1 \quad \text{for } t > 0.$$

In Figure 2.6.3 we display numerical approximations with $\Delta x = 1/400$ at times $T = 10$ and $T = 100$. We can evidence that the dynamics of the solution is different for each kernel functions; by using ω_1 we observe that numerical solution goes faster to a stationary state solution than for other kernel functions, in which this stationary state is not observed for enough large simulation time. Regarding the kernel ω_2 we can see the formation of some oscillations. Now, for the kernel ω_3 we can see the formation of some layers on the numerical solution and that the period of these layers are proportional to η. Finally, for ω_4 we can observe a numerical solution more smooth than in the previous solutions.

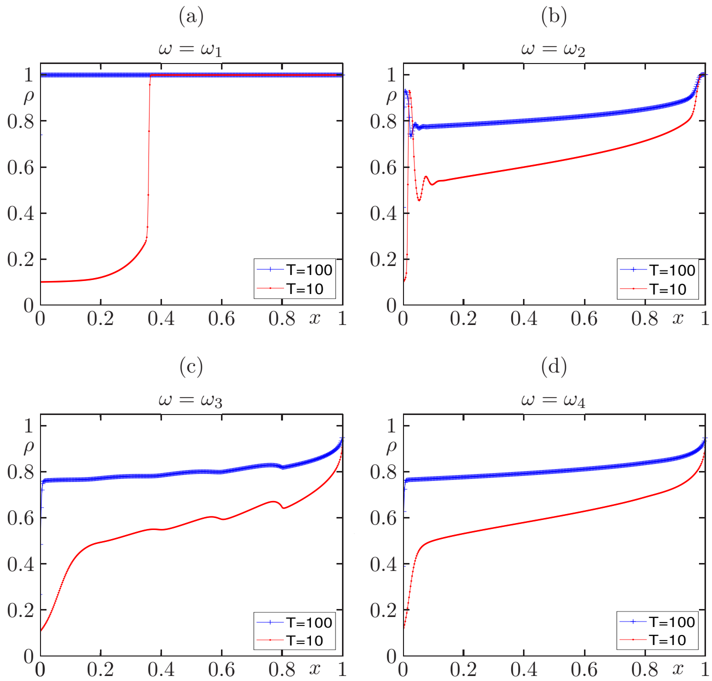

Figure 2.6.3: Example 2: dynamics of the model (2.2.1) for various kernel functions (ω_1, ω_2, ω_3 or ω_4). Numerical solutions with $M = 800$ at simulated times $T = 10$ and $T = 100$.

2.6.3 Example 3

The aim of the present example is to investigate the behavior of numerical solutions considering the kernel functions ω_1, ω_2, and ω_3 as well as for two different values of the parameter η, namely $\eta = 0.1$ and $\eta = 0.025$. The initial and boundary conditions are

$$\rho_0(x) = 0.5 \quad \text{for } x \in I; \quad \rho_a(t) = 0, \quad \rho_b(t) = 1 \quad \text{for } t > 0,$$

which leaves zero flow conditions at boundary, i.e. $f_{1/2} = f_{M+1/2} = 0$. Numerical approximations are computed at simulated times $T = 2$, $T = 7$ and $T = 15$ with discretization $M = 400$. In Figure 2.6.4, first we observe that numerical solutions for the nonlocal problem (2.2.1) get closer to the solution of the local problem as η takes a smaller value, but ω_1 and ω_2 make it slower due to the presence of the oscillations that the numerical solutions present when we use these functions.

2.6.4 Example 4: Error test for second order scheme

We consider the problem (2.2.1), with a smooth initial datum

$$\rho_0(x) = 0.9 \exp\bigl(-70(x - 0.4)^2\bigr) \quad \text{for } x \in [0, 1], \tag{2.6.4}$$

with boundary conditions $\rho_a = \rho_b = 0$, and with the symmetric kernel function ω_1 with $\eta = 0.2$. In Fig. 2.6.5 we display the numerical approximations obtained with the second order scheme (2.5.1)–(2.5.2), computed with $M = 100$ at $T = 0.1$. The reference solution is computed with $M = 6400$. As expected, the numerical solutions obtained with second order version of the HW scheme capture the reference solution better than the first order one. In Table 2.6.2 we compute the L^1−error and E.O.C. We recover the correct order of accuracy for the second order HW scheme. Instead, we obtain just first order accuracy for HW scheme (2.4.2). We also can observe for scheme (2.4.2) that the L^1−error for each level of refinement is bigger than the error of the second order version scheme.

Table 2.6.2: Example 4: approximate L^1 errors $e_M(\rho)$ and E.O.C for the first and second order version of the HW scheme with $\Delta x = 1/M$, at $T = 0.1$.

	$T = 0.1$			
	HW first-order version		HW second-order version	
M	$e_M(\rho_\Delta)$	E.O.C.	$e_M(\rho_\Delta)$	E.O.C.
100	8.71e-03	-	3.20e-04	—
200	4.60e-03	9.19e-01	8.63e-05	1.89
400	2.38e-03	9.54e-01	2.24e-05	1.95
800	1.21e-03	9.77e-01	5.53e-06	2.02

2.7 Conclusions of Chapter 2

In this chapter we extend to its nonlocal version a numerical scheme presented in [15, 55] where we take advantages of the form in which the flow is written, $\rho v(\rho)V(x,t)$, where $v(\rho)$ is a positive and non-increasing function, and $V(x,t)$ is a positive function containing the nonlocal terms. We have proved a maximum principle, L^1-bound, and **BV** estimations, also, a discrete entropy inequality in order to prove the well-posedness of an 1-D and nonlocal IBVP. Likewise, using kernel functions of different shapes (symmetric, non-symmetric, isotropic) we have conducted numerical experiments aiming at assessing whether (2.2.1) can possibly explain the phenomenon of layering in sedimentation. In Figures 2.6.3 and 2.6.4 we can observe some fluctuations of the concentration ρ like staircasing.

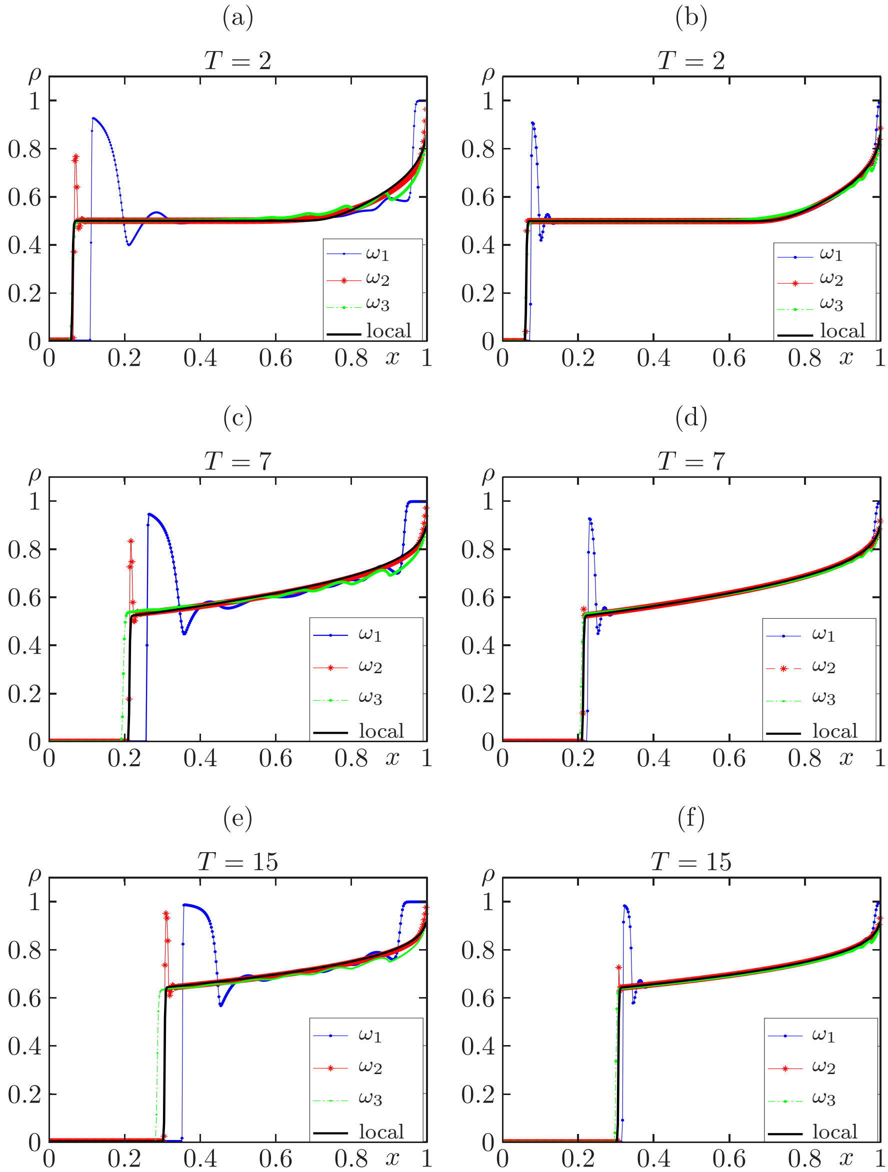

Figure 2.6.4: Example 3: numerical solutions of (2.2.1) for $M = 400$ at indicated simulated times with (left) $\eta = 0.1$ and (right) $\eta = 0.025$.

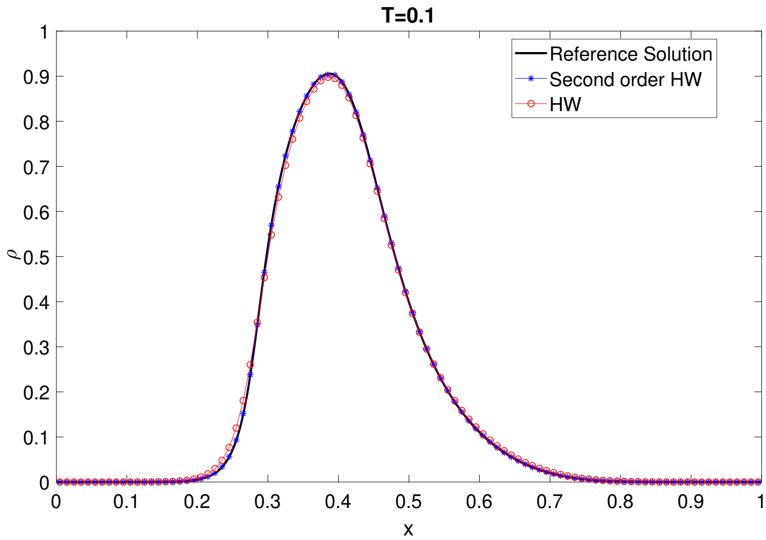

Figure 2.6.5: Example 4: comparison of the numerical solutions at $T = 0.1$ corresponding to initial condition (2.6.4), computed with $1/\Delta x = 100$ and kernel function ω_1.

Chapter 3

Existence of entropy weak solutions for 1D non-local traffic models with space-discontinuous flux

3.1 Introduction

We are interested in the analysis of the well-posedness and the numerical approximation of solutions of non-local conservation laws with a single spatial discontinuity in the flux

$$\begin{cases} \partial_t \rho + \partial_x f(t,x,\rho) = 0, & (t,x) \in (0,\infty) \times \mathbb{R}, \\ \rho(0,x) = \rho_0(x), & x \in \mathbb{R}, \end{cases} \tag{3.1.1}$$

with

$$f(t,x,\rho) = H(-x)\ \rho\, g(\rho)\, v_l(\omega_\eta * \rho) + H(x)\ \rho\, g(\rho)\, v_r(\omega_\eta * \rho),$$

where $H(x)$ is the Heaviside function, and the flux $f(t,x,\rho)$ has a discontinuity at $x = 0$ if the velocity functions $v_l(\rho)$ and $v_r(\rho)$ are different. The function g is assumed to be nonnegative and such that $g'(\rho) \leq 0$ and $g(\rho_{\max}) = 0$. We assume that the convolution term and the kernel function ω_η satisfy

$$(\omega_\eta * \rho)(t,x) = \int_x^{x+\eta} \rho(t,y)\omega_\eta(y-x)\mathrm{d}y, \quad \eta > 0 \tag{3.1.2}$$

$$\omega_\eta \in \mathbf{C}^2([0,\eta],\mathbb{R}^+), \quad \omega_\eta' \leq 0, \quad \omega_\eta(\eta) = 0, \tag{3.1.3}$$

and the following hypothesis hold on the velocity functions

$$v_s(\rho) = k_s\psi(\rho), \quad s = l,r, \quad \psi \in \mathbf{C}^2(\mathbb{R}), \text{ s.t. } \psi' \leq 0. \tag{3.1.4}$$

In the vehicular traffic context ρ represents the density of vehicles on the roads, ω_η is a non-increasing kernel function whose support η is proportional to the look-ahead distance of drivers, that are supposed to adapt their velocity with respect to the mean downstream traffic density. The equation in (3.1.1) is a non-local version of a generalized Lighthill-Whitham-Richards traffic model [66, 72, 46] with a discontinuous velocity field [31, 62].

Models of conservation laws with non-local flux describe several phenomena such as slow erosion of granular flow [4, 77], synchronization [3], sedimentation [13], crowd dynamics [34], navigation processes [5] and traffic flow [14, 61, 20, 42, 29]. In particular, non-local traffic models describe the behaviour of drivers that adapt their velocity with respect to what happens to the cars in front of them. See [20] for an overview about non-local traffic models and [25] for a continuous non-local model describing the behavior of drivers on two stretches of a road with different velocities and capacities. There are many results relating to existence, uniqueness, stability and numerical approximation of weak entropy solutions of *local* conservation laws with a spatially discontinuous flux [31, 62, 1, 8, 15, 18, 45, 47, 48, 58, 59, 60]. Conversely, in the nonlocal case, traveling wave for a traffic flow model with rough road conditions were studied in [73], with the following velocity functions:

$$v_s(\rho) = k_s(1-\rho), \quad s = l, r \quad g(\rho) = 1.$$

But it is worth pointing out that in the latter case with $k_l > k_r$ and $g(\rho) = 1$, the non-local model does not satisfy the maximum principle, as it is showed in [21, 30]. On the contrary, model (3.1.1) satisfies the maximum principle and this makes it more realistic in the sense of traffic flow dynamics.

In this sense, the aim of this chapter is twofold:

- we prove the well-posedness of the non-local space-discontinuous traffic model (3.1.1) for a general non-increasing speed function ψ, approximating the problem through a monotone numerical scheme and proving standard compactness estimates;
- we numerically study the limit model as the support of the kernel function tends to 0^+.

Following [59], we recall the following definitions of solution.

Definition 3.1. *We say that a function $\rho \in (\mathbf{L}^1 \cap \mathbf{L}^\infty)([0,T] \times \mathbb{R}; [0, \rho_{\max}])$ is a weak solution of the initial value problem (3.1.1) if for any test function $\varphi \in \mathbf{C}_c^1([0,T[\times\mathbb{R};\mathbb{R})$,*

$$\int_0^T \int_{\mathbb{R}} (\rho \partial_t \varphi + f(t,x,\rho) \partial_x \varphi(t,x)) \, \mathrm{d}x \mathrm{d}t + \int_{\mathbb{R}} \rho_0(x) \varphi(0,x) \mathrm{d}x = 0.$$

Definition 3.2. *A function $\rho \in (\mathbf{L}^1 \cap \mathbf{L}^\infty)([0,T] \times \mathbb{R}; [0, \rho_{\max}])$ is an entropy weak solution of (3.1.1) if for all $c \in [0, \rho_{\max}]$, and any test function $\varphi \in \mathbf{C}_c^1([0,T[\times\mathbb{R};\mathbb{R}^+)$,*

$$\int_0^T \int_{\mathbb{R}} |\rho - c| \, \varphi_t + \operatorname{sgn}(\rho - c) \left(f(t,x,\rho) - f(t,x,c) \right) \varphi_x \, \mathrm{d}x \, \mathrm{d}t$$
$$- \int_0^T \int_{\mathbb{R}^*} \operatorname{sgn}(\rho - c) \, f(t,x,c)_x \varphi \, \mathrm{d}x \, \mathrm{d}t + \int_{\mathbb{R}} |\rho_0(x) - c| \, \varphi(0,x) \mathrm{d}x$$

$$+\int_0^T |(k_r - k_l)c\, g(c)\, \psi(\rho * \omega_\eta)|\, \varphi(t,0)\mathrm{d}t \geq 0.$$

This chapter is organized as follows. In Section 3.2, we introduce the numerical scheme that we use to discretize our problem. After that, in Section 3.3 we prove the existence and uniqueness of weak entropy solutions with $\mathbf{L}^\infty$ and $\mathbf{BV}$ bounds. Finally, in Section 3.4, we show some numerical tests illustrating the behaviour of solutions and investigating the limit model as the support of the kernel $\eta \to 0^+$.

3.2 Numerical scheme

We introduce a uniform space mesh of width Δx and a time step Δt, subject to a CFL condition, to be detailed later on. The spatial domain is discretized into uniform cells $I_j = [x_{j-1/2}, x_{j+1/2})$, where $x_{j+1/2} = x_j + \Delta x/2$ are the cell interfaces, and $x_j = j\Delta x$ the cell centers, in particular $x = 0$ where the flux function changes, falls at the midpoint of the cell $I_0 = [x_{-1/2}, x_{1/2})$. We take Δx such that $\eta = N\Delta x$ for some $N \in \mathbb{N}$. Let $t^n = n\Delta t$ be the time mesh and $\lambda = \Delta t/\Delta x$. We aim to construct a finite volume approximate solution ρ_Δ such that $\rho_\Delta(t,x) = \rho_j^n$ for $(t,x) \in [t^n, t^{n+1}[\times[x_{j-1/2}, x_{j+1/2})$. To this end, we approximate the initial datum ρ_0 with the cell averages

$$\rho_j^0 = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} \rho_0(x)\mathrm{d}x,$$

we denote $\omega_k := \dfrac{1}{\Delta x}\displaystyle\int_{k\Delta x}^{(k+1)\Delta x} \omega(y)\mathrm{d}y$ for $k = 0, \ldots, N-1$ and set the convolution term

$$R(x_{j+1/2}, t^n) = (\omega_\eta * \rho_\Delta)(x_{j+1/2}, t^n) \approx \Delta x \sum_{k=0}^{N-1} \omega_k \rho_{j+k+1}^n.$$

In this way we can define the following finite volume scheme $\forall j \in \mathbb{Z}$

$$\rho_j^{n+1} = \rho_j^n - \lambda\left(F(x_{j+1/2}, \rho_j^n, \rho_{j+1}^n, R_{j+1/2}^n) - F(x_{j-1/2}, \rho_{j-1}^n, \rho_j^n, R_{j-1/2}^n)\right), \tag{3.2.1}$$

where F is defined by

$$F(x_{j+1/2}, \rho_j, \rho_{j+1}, R_{j+1/2}) = \begin{cases} \rho_j g(\rho_{j+1})\, v_l(R_{j+1/2}) \text{ if } x_{j+1/2} < 0, \\ \rho_j g(\rho_{j+1})\, v_r(R_{j+1/2}) \text{ if } x_{j+1/2} > 0. \end{cases} \tag{3.2.2}$$

This proposed scheme is a non-local version of one which was proposed in [15] and it takes advantage of the form in which the flow is written, namely density ρ times a local decreasing factor $g(\rho)$ times $v(\omega * \rho)$, where v is a given velocity function and ω is a convolution kernel.

3.3 Well-posedness of (3.1.1)

In this Section, we prove some properties of the finite volume scheme (3.2.1)-(3.2.2).

Lemma 3.3. *Let hypotheses* (3.1.4) *hold. Given an initial datum such that* $0 \le \rho_j^0 \le \rho_{\max}$ *for* $j \in \mathbb{Z}$*, the finite volume scheme* (3.2.1)-(3.2.2) *is such that*

$$0 \le \rho_j^{n+1} \le \rho_{\max}, \qquad j \in \mathbb{Z},$$

under the CFL condition

$$\Delta t \le \min_{s=l,r} \left\{ \frac{\Delta x}{\rho_{\max} k_s \left\| g' \right\|_{\mathbf{L}^\infty} \left\| \psi \right\|_{\mathbf{L}^\infty}}, \frac{\Delta x}{k_s \left\| g \right\|_{\mathbf{L}^\infty} \left\| \psi \right\|_{\mathbf{L}^\infty}} \right\}. \tag{3.3.1}$$

Proof. By induction, assume that $0 \le \rho_j^n \le \rho_{\max}$ for all $j \in \mathbb{Z}$. Let us consider $j \ne 0$ and set $v(\rho) := k_s \psi(\rho)$ for $s = l, r$. In this case, we can observe that

$$\begin{aligned} \rho_j^{n+1} &= \rho_j^n - \lambda \left(\rho_j^n g(\rho_{j+1}^n) v(R_{j+1/2}^n) - \rho_{j-1}^n g(\rho_j^n) v(R_{j-1/2}^n) \right) \\ &\le \rho_j^n + \lambda \rho_{\max} \left\| g' \right\|_{\mathbf{L}^\infty} k_s \left\| \psi \right\|_{\mathbf{L}^\infty} (\rho_{\max} - \rho_j^n). \end{aligned}$$

Under the CFL condition (3.3.1), we conclude that $\rho_j^{n+1} \le \rho_{\max}$ for all $j \in \mathbb{Z}^*$. For $j = 0$, we obtain

$$\begin{aligned} \rho_0^{n+1} &= \rho_0^n - \lambda \left(\rho_0^n g(\rho_1^n) v_r(R_{1/2}^n) - \rho_{-1}^n g(\rho_0^n) v_l(R_{-1/2}^n) \right) \\ &\le \rho_0^n + \lambda \left(\rho_{\max} g(\rho_0^n) v_l(R_{-1/2}^n) \right) \\ &\le \rho_0^n + \lambda k_l \rho_{\max} \left\| g' \right\|_{\mathbf{L}^\infty} \left\| \psi \right\|_{\mathbf{L}^\infty} (\rho_{\max} - \rho_0^n) \le \rho_{\max}. \end{aligned}$$

To prove the positivity $\rho_j^{n+1} \ge 0$, we observe that

$$\begin{aligned} \rho_j^{n+1} &= \rho_j^n - \lambda \left(\rho_j^n g(\rho_{j+1}^n) v(R_{j+1/2}^n) - \rho_{j-1}^n g(\rho_j^n) v(R_{j-1/2}^n) \right) \\ &\ge \rho_j^n \left(1 - \lambda g(\rho_{j+1}^n) v(R_{j+1/2}^n) \right) \\ &\ge 0. \end{aligned}$$

This concludes the proof. □

Lemma 3.4 ($\mathbf{L}^1$ norm)**.** *Let hypotheses* (3.1.4) *hold. If* $\rho_0 \in \mathbf{L}^1(\mathbb{R}; \mathbb{R}^+)$ *then under the CFL condition* (3.3.1)*, the approximate solution* ρ_Δ *constructed through the finite volume scheme* (3.2.1)-(3.2.2) *satisfies*

$$\left\| \rho_\Delta(t, \cdot) \right\|_{\mathbf{L}^1} = \left\| \rho_0 \right\|_{\mathbf{L}^1} \quad \textit{for all } t > 0. \tag{3.3.2}$$

Proof. By induction, suppose that (3.3.2) holds for $t^n = n\Delta t$. Thanks to the positivity and the conservative form of the numerical scheme (3.2.1) we have

$$\left\| \rho^{n+1} \right\|_{\mathbf{L}^1} = \Delta x \sum_{j \in \mathbb{Z}} \rho_j^{n+1} = \left\| \rho^n \right\|_{\mathbf{L}^1}.$$

□

We now prove the $\mathbf{L}^1$-continuity in time by following the idea introduced in [58]. For the sake of simplicity we use the following notation throughout the proof, let us define

$$v^n_{j+1/2} := \begin{cases} v_l(R^n_{j+1/2}), & \text{if } j < 0, \\ v_r(R^n_{j+1/2}), & \text{if } j \geq 0. \end{cases}$$

Lemma 3.5. *Set $N_T = \lfloor T/\Delta t \rfloor$. Let $\rho_0 \in \mathbf{BV}(\mathbb{R}; [0, \rho_{\max}])$ with $\|\rho_0\|_{\mathbf{L}^1} < +\infty$. Assume that the following CFL condition holds*

$$\Delta t \leq \min_{s=l,r} \left\{ \frac{\Delta x}{\rho_{\max} k_s \|\psi\|_{\mathbf{L}^\infty} (\|g\|_{\mathbf{L}^\infty} + \|g'\|_{\mathbf{L}^\infty}) + \Delta x \rho_{\max} \omega_\eta(0) k_s \|\psi'\|_{\mathbf{L}^\infty} \|g\|_{\mathbf{L}^\infty}} \right\}, \tag{3.3.3}$$

then, for $n = 0, \ldots, N_T - 1$

$$\Delta x \sum_{j \in \mathbb{Z}} |\rho^{n+1}_j - \rho^n_j| \leq C(T), \tag{3.3.4}$$

where

$$\begin{aligned} C(T) \;=\; & e^{(2T\rho_{\max}\|g\|_{\mathbf{L}^\infty}\|v'\|_{\mathbf{L}^\infty})} \Delta t \bigg(3 \max_{s=l,r} \big\{ \big(\|\psi\|_{\mathbf{L}^\infty} \|g\|_{\mathbf{L}^\infty} + \rho_{\max} \left\|\psi'\right\|_{\mathbf{L}^\infty} \|\omega\|_{\mathbf{L}^1} \|g\|_{\mathbf{L}^\infty} \\ & + \rho_{\max} \|\psi\|_\infty \left\|g'\right\|\big) k_s \big\} \, TV(\rho_0) + \Delta t \rho_{\max} |k_r - k_l| \, \|\psi\|_{\mathbf{L}^\infty} \|g\|_{\mathbf{L}^\infty} \bigg). \end{aligned}$$

Proof. First, we fix $j \in \mathbb{Z}$, by (3.2.1) we have

$$\begin{aligned} \rho^{n+1}_j - \rho^n_j \;=\; & \rho^n_j - \rho^{n-1}_j - \lambda \Big(\rho^n_j g(\rho^n_{j+1}) v^n_{j+1/2} \pm \rho^{n-1}_j g(\rho^{n-1}_{j+1}) v^n_{j+1/2} \\ & - \rho^{n-1}_j g(\rho^{n-1}_{j+1}) v^{n-1}_{j+1/2} - \rho^n_{j-1} g(\rho^n_j) v^n_{j-1/2} \pm \rho^{n-1}_{j-1} g(\rho^{n-1}_j) v^n_{j-1/2} \\ & + \rho^{n-1}_{j-1} g(\rho^{n-1}_j) v^{n-1}_{j-1/2} \Big) \\ =\; & \rho^n_j - \rho^{n-1}_j - \lambda \Big((\rho^n_j g(\rho^n_{j+1}) - \rho^{n-1}_j g(\rho^{n-1}_{j+1})) v^n_{j+1/2} \\ & + \rho^{n-1}_j g(\rho^{n-1}_{j+1}) (v^n_{j+1/2} - v^{n-1}_{j+1/2}) - (\rho^n_{j-1} g(\rho^n_j) - \rho^{n-1}_{j-1} g(\rho^{n-1}_j)) v^n_{j-1/2} \\ & - \rho^{n-1}_{j-1} g(\rho^{n-1}_j) (v^n_{j-1/2} - v^{n-1}_{j-1/2}) \Big), \end{aligned}$$

and using the mean-value theorem, we take $R^{n-1/2}_{j+1/2} \in (R^n_{j+1/2}, R^{n-1}_{j+1/2})$ such that

$$\begin{aligned} v^n_{j+1/2} - v^{n-1}_{j+1/2} \;=\; & v'(R^{n-1/2}_{j+1/2})(R^n_{j+1/2} - R^{n-1}_{j+1/2}) \\ =\; & v'(R^{n-1/2}_{j+1/2}) \Delta x \sum_{k=0}^{N-1} \omega_k (\rho^n_{j+k+1} - \rho^{n-1}_{j+k+1}). \end{aligned}$$

Next, we can write

$$
\begin{aligned}
&\rho_j^{n+1} - \rho_j^n \\
&= \Big(1 - \lambda\big(v_{j+1/2}^n g(\rho_{j+1}^{n-1}) - \rho_{j-1}^n g'(\xi_{j-1/2}^{n-1/2}) v_{j-1/2}^n - \Delta x \omega_0 \rho_{j-1}^{n-1} g(\rho_j^{n-1}) v'(R_{j-1/2}^{n-1/2})\big)\Big) \\
&\quad \times (\rho_j^n - \rho_j^{n-1}) \\
&\quad -\lambda \left(\rho_j^n g'(\xi_{j+1/2}^{n-1/2}) v_{j+1/2}^n - v'\left(R_{j+1/2}^{n-1/2}\right) \rho_j^{n-1} g(\rho_{j+1}^{n-1}) \Delta x \omega_0\right) (\rho_{j+1}^n - \rho_{j+1}^{n-1}) \\
&\quad -\lambda v'(R_{j+1/2}^{n-1/2}) \rho_j^{n-1} g(\rho_{j+1}^{n-1}) \Delta x \sum_{k=1}^{N-1} \omega_k (\rho_{j+k}^n - \rho_{j+k}^{n-1}) + \lambda v_{j-1/2}^n g(\rho_j^{n-1}) (\rho_{j-1}^n - \rho_{j-1}^{n-1}) \\
&\quad +\lambda \rho_{j-1}^{n-1} g(\rho_j^{n-1}) v'(R_{j-1/2}^{n-1/2}) \Delta x \sum_{k=1}^{N-1} \omega_k (\rho_{j+k}^n - \rho_{j+k}^{n-1}),
\end{aligned}
$$

and thanks to the CFL condition (3.3.3), we have

$$
1 - \lambda \left(v_{j+1/2}^n g(\rho_{j+1}^{n-1}) - \rho_{j-1}^n g'(\xi_{j-1/2}^{n-1/2}) v_{j-1/2}^n - \Delta x \omega_0 \rho_{j-1}^{n-1} g(\rho_j^{n-1}) v'(R_{j-1/2}^{n-1/2})\right) \geq 0.
$$

Then, taking the absolute value we obtain

$$
\begin{aligned}
&|\rho_j^{n+1} - \rho_j^n| \\
&\leq \left(1 - \lambda \left(v_{j+1/2}^n g(\rho_{j+1}^{n-1}) - \rho_{j-1}^n g'(\xi_{j-1/2}^{n-1/2}) v_{j-1/2}^n - \Delta x \omega_0 \rho_{j-1}^{n-1} g(\rho_j^{n-1}) v'(R_{j-1/2}^{n-1/2})\right)\right) \\
&\quad \times |\rho_j^n - \rho_j^{n-1}| \\
&\quad +\lambda \left(-\rho_j^n g'(\xi_{j+1/2}^{n-1/2}) v_{j+1/2}^n - v'\left(R_{j+1/2}^{n-1/2}\right) \rho_j^{n-1} g(\rho_{j+1}^{n-1}) \Delta x \omega_0\right) |\rho_{j+1}^n - \rho_{j+1}^{n-1}| \\
&\quad -\lambda v'(R_{j+1/2}^{n-1/2}) \rho_j^{n-1} g(\rho_{j+1}^{n-1}) \Delta x \sum_{k=1}^{N-1} \omega_k |\rho_{j+k}^n - \rho_{j+k}^{n-1}| + \lambda v_{j-1/2}^n g(\rho_j^{n-1}) |\rho_{j-1}^n - \rho_{j-1}^{n-1}| \\
&\quad -\lambda \rho_{j-1}^{n-1} g(\rho_j^{n-1}) v'(R_{j-1/2}^{n-1/2}) \Delta x \sum_{k=1}^{N-1} \omega_k |\rho_{j+k}^n - \rho_{j+k}^{n-1}|.
\end{aligned}
$$

Now, multiplying by Δx and summing over j, we get

$$
\begin{aligned}
\sum_{j \in \mathbb{Z}} \Delta x |\rho_j^{n+1} - \rho_j^n| \;\; &\leq \;\; \sum_{j \in \mathbb{Z}} \Delta x |\rho_j^n - \rho_j^{n-1}| \\
&\quad -2\lambda \sum_{j \in \mathbb{Z}} \Delta x \rho_{j-1}^{n-1} g(\rho_j^{n-1}) v'(R_{j-1/2}^{n-1/2}) \Delta x \sum_{k=1}^{N-1} \omega_k |\rho_{j+k}^n - \rho_{j+k}^{n-1}| \\
&\leq \;\; \sum_{j \in \mathbb{Z}} \Delta x |\rho_j^n - \rho_j^{n-1}| \\
&\quad +2\Delta t \rho_{\max} \|g\|_{\mathbf{L}^\infty} \|v'\|_{\mathbf{L}^\infty} \sum_{k=1}^{N-1} \omega_k \sum_{j \in \mathbb{Z}} \Delta x |\rho_{j+k}^n - \rho_{j+k}^{n-1}|
\end{aligned}
$$

$$\leq \left(1 + 2\Delta t \rho_{\max} \|g\|_{\mathbf{L}^\infty} \|v'\|_{\mathbf{L}^\infty} \sum_{k=0}^{N-1} \omega_k \right) \sum_{j\in\mathbb{Z}} \Delta x |\rho_j^n - \rho_j^{n-1}|.$$

Thus,

$$\sum_{j\in\mathbb{Z}} \Delta x |\rho_j^{n+1} - \rho_j^n| \leq \mathrm{e}^{(2n\Delta t \rho_{\max} \|g\|_{\mathbf{L}^\infty} \|v'\|_{\mathbf{L}^\infty})} \sum_{j\in\mathbb{Z}} \Delta x |\rho_j^1 - \rho_j^0|.$$

On the other hand,

$$\begin{aligned} \sum_{j\in\mathbb{Z}} \Delta x |\rho_j^1 - \rho_j^0| \quad \leq \quad & \Delta t \sum_{j<0} \left| \rho_j^0 g(\rho_{j+1}^0) v_l(R_{j+1/2}^0) - \rho_{j-1}^0 g(\rho_j^0) v_l(R_{j-1/2}^0) \right| \\ & + \Delta t \left| \rho_{-1}^0 g(\rho_0^0) v_l(R_{-1/2}^0) - \rho_0^0 g(\rho_1^0) v_r(R_{1/2}^0) \right| \\ & + \Delta t \sum_{j>0} \left| \rho_j^0 g(\rho_{j+1}^0) v_r(R_{j+1/2}^0) - \rho_{j-1}^0 g(\rho_j^0) v_r(R_{j-1/2}^0) \right|. \end{aligned}$$

The first term of the right-hand side can be estimated as

$$\begin{aligned} & \sum_{j<0} |\rho_j^0 g(\rho_{j+1}^0) v_l(R_{j+1/2}^0) - \rho_{j-1}^0 g(\rho_j^0) v_l(R_{j-1/2}^0)| \\ & \leq \|v_l\|_{\mathbf{L}^\infty} \|g\|_{\mathbf{L}^\infty} \sum_{j<0} |\rho_j^0 - \rho_{j-1}^0| + \rho_{\max} \left\|g'\right\|_{\mathbf{L}^\infty} \|v_l\|_{\mathbf{L}^\infty} \sum_{j<0} |\rho_{j+1}^0 - \rho_j^0| \\ & \quad + \rho_{\max} \|g\|_{\mathbf{L}^\infty} \left\|v_l'\right\|_{\mathbf{L}^\infty} \sum_{j<0} \left| R_{j+1/2}^n - R_{j-1/2}^n \right| \\ & \leq \|v_l\|_{\mathbf{L}^\infty} \|g\|_{\mathbf{L}^\infty} \sum_{j<0} |\rho_j^0 - \rho_{j-1}^0| + \rho_{\max} \left\|g'\right\|_{\mathbf{L}^\infty} \|v_l\|_{\mathbf{L}^\infty} \sum_{j<0} |\rho_{j+1}^0 - \rho_j^0| \\ & \quad + \rho_{\max} \|g\|_{\mathbf{L}^\infty} \left\|v_l'\right\|_{\mathbf{L}^\infty} \sum_{j<0} \sum_{k=0}^{N-1} \Delta x \omega_k |\rho_{j+k+1}^0 - \rho_{j+k}^0| \\ & \leq \left(\|v_l\|_{\mathbf{L}^\infty} \|g\|_{\mathbf{L}^\infty} + \rho_{\max} \|g\|_{\mathbf{L}^\infty} \left\|v_l'\right\|_{\mathbf{L}^\infty} \|\omega\|_{\mathbf{L}^1} + \rho_{\max} \left\|g'\right\|_{\mathbf{L}^\infty} \|v_l\|_{\mathbf{L}^\infty} \right) \mathrm{TV}(\rho_0). \end{aligned}$$

Analogously,

$$\begin{aligned} & \sum_{j>0} |\rho_j^0 g(\rho_{j+1}^0) v_r(R_{j+1/2}^0) - \rho_{j-1}^0 g(\rho_j^0) v_r(R_{j-1/2}^0)| \\ & \leq (\|v_r\|_{\mathbf{L}^\infty} \|g\|_{\mathbf{L}^\infty} + \rho_{\max} \left\|v_r'\right\|_{\mathbf{L}^\infty} \|g\|_{\mathbf{L}^\infty} \|\omega\|_{\mathbf{L}^1} + \rho_{\max} \left\|g'\right\|_{\mathbf{L}^\infty} \|v_r\|_{\mathbf{L}^\infty}) \mathrm{TV}(\rho_0), \end{aligned}$$

and by hypothesis (3.1.4)

$$\begin{aligned} & |\rho_{-1}^0 g(\rho_0^0) v_l(R_{-1/2}^0) - \rho_0^0 g(\rho_1^0) v_r(R_{1/2}^0)| \\ & \leq |\rho_{-1}^0 g(\rho_0^0) v_l(R_{-1/2}^0) \pm \rho_{-1}^0 g(\rho_0^0) v_r(R_{1/2}^0) - \rho_0^0 g(\rho_1^0) v_r(R_{1/2}^0)| \\ & \leq \rho_{\max} \|g\|_{\mathbf{L}^\infty} \left| v_l(R_{-1/2}^0) - v_r(R_{1/2}^0) \right| + v_r(R_{1/2}^0) |\rho_{-1}^0 g(\rho_0^0) - \rho_0^0 g(\rho_1^0)| \\ & \leq \rho_{\max} \|g\|_{\mathbf{L}^\infty} \left| v_l(R_{-1/2}^0) \pm v_r(R_{-1/2}^0) - v_r(R_{1/2}^0) \right| \end{aligned}$$

$$
\begin{aligned}
&+v_r(R^0_{1/2})\left|\rho^0_{-1}g(\rho^0_0)-\rho^0_0 g(\rho^0_1)\right| \\
&\leq \rho_{\max}\left\|g\right\|_{\mathbf{L}^\infty}\left\|\psi\right\|_{\mathbf{L}^\infty}|k_l-k_r|+\rho_{\max}\left\|g\right\|_{\mathbf{L}^\infty}\left\|v_r'\right\|\left\|\omega\right\|_{\mathbf{L}^1}\mathrm{TV}(\rho_0) \\
&\quad+\left(\left\|v_r\right\|_{\mathbf{L}^\infty}\left\|g\right\|_{\mathbf{L}^\infty}+\rho_{\max}\left\|v_r\right\|_{\mathbf{L}^\infty}\left\|g'\right\|_{\mathbf{L}^\infty}\right)\mathrm{TV}(\rho_0).
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\sum_{j\in\mathbb{Z}}\Delta x|\rho^1_j-\rho^0_j| \quad\leq\quad & 3\Delta t\max_{s=l,r}\Big\{\Big(\left\|\psi\right\|_{\mathbf{L}^\infty}\left\|g\right\|_{\mathbf{L}^\infty}+\rho_{\max}\left\|\psi'\right\|_{\mathbf{L}^\infty}\left\|\omega\right\|_{\mathbf{L}^1}\left\|g\right\|_{\mathbf{L}^\infty} \\
&+\rho_{\max}\left\|\psi\right\|_\infty\left\|g'\right\|\Big)k_s\Big\}\mathrm{TV}(\rho_0)+\Delta t\rho_{\max}|k_r-k_l|\left\|\psi\right\|_{\mathbf{L}^\infty}\left\|g\right\|_{\mathbf{L}^\infty}.
\end{aligned}
$$

This completes the proof. □

3.3.1 Spatial BV estimates.

Lemma 3.6. *Let $\rho_0 \in \mathbf{L}^\infty \cap \mathbf{BV}(\mathbb{R};[0,\rho_{\max}])$. Assume that the CFL condition* (3.3.1) *holds. For any interval $[a,b]\subset\mathbb{R}$ such that $0\notin[a,b]$, fix $q>0$ such that $2q<\min\{|a|,|b|\}$ and $q>\Delta x$. Then, for any $n=1,...,N_T-1$ the following estimate holds:*

$$
\sum_{j\in\mathbf{J}^b_a}\left|\rho^n_{j+1}-\rho^n_j\right|\leq e^{2\mathcal{K}T}\left(TV(\rho_0)+2\frac{C(T)}{q}+\mathcal{K}_2T\right),
$$

with $\mathbf{J}^b_a=\{j\in\mathbb{Z}: a\leq x_j\leq b\}$.

Proof. Let

$$
\mathcal{M}_\Delta=\left\{j\in\mathbb{Z}: x_{j-1/2}\in[a-q-\Delta x,a]\right\},\ \mathcal{N}_\Delta=\left\{j\in\mathbb{Z}: x_{j+1/2}\in[b,b+q+\Delta x]\right\}.
$$

By the assumptions on q, observe that there are at least 2 elements in each of the sets above, i.e. $|\mathcal{M}_\Delta|$, $|\mathcal{N}_\Delta|\geq 2$. Moreover, $|\mathcal{M}_\Delta|\Delta x\geq q$ and $|\mathcal{N}_\Delta|\Delta x\geq q$. By Lemma 3.5 there exists a constant $C(T)$ such that

$$
\Delta x\sum_{n=0}^{N_T-1}\sum_{j\in\mathbb{Z}}|\rho^{n+1}_j-\rho^n_j|\leq TC(T), \tag{3.3.5}
$$

with $C(T)$ as in Lemma 3.5, then when restricting the sum over j in the set $\mathcal{M}_\Delta$, respectively $\mathcal{N}_\Delta$, it follows that

$$
\Delta x\sum_{n=0}^{N_T-1}\sum_{j\in\mathcal{M}_\Delta}|\rho^{n+1}_j-\rho^n_j|\leq TC(T) \quad\text{and}\quad \Delta x\sum_{n=0}^{N_T-1}\sum_{j\in\mathcal{N}_\Delta}|\rho^{n+1}_j-\rho^n_j|\leq TC(T). \tag{3.3.6}
$$

Let us choose $j_a\in\mathcal{M}_\Delta$ and j_b with $j_b+1\in\mathcal{N}_\Delta$ such that

$$
\sum_{n=0}^{N_T-1}|\rho^{n+1}_{j_a}-\rho^n_{j_a}| \quad=\quad \min_{j\in\mathcal{M}_\Delta}\sum_{n=0}^{N_T-1}|\rho^{n+1}_j-\rho^n_j|,
$$

$$\sum_{n=0}^{N_T-1} |\rho_{j_{b+1}}^{n+1} - \rho_{j_{b+1}}^n| \quad = \quad \min_{j\in\mathcal{N}_\Delta} \sum_{n=0}^{N_T-1} |\rho_j^{n+1} - \rho_j^n|,$$

thus,

$$\begin{aligned}
\sum_{n=0}^{N_T-1} |\rho_{j_a}^{n+1} - \rho_{j_a}^n| \quad &\leq \quad \frac{C}{|\mathcal{M}_\Delta|\Delta x} \leq \frac{TC(T)}{q}, \\
\sum_{n=0}^{N_T-1} |\rho_{j_{b+1}}^{n+1} - \rho_{j_{b+1}}^n| \quad &\leq \quad \frac{C}{|\mathcal{N}_\Delta|\Delta x} \leq \frac{TC(T)}{q}.
\end{aligned}$$

Moreover, observe that

$$\sum_{j=j_a}^{j_b} \left|\rho_{j+1}^{n+1} - \rho_j^{n+1}\right| = \left|\rho_{j_a+1}^{n+1} - \rho_{j_a}^{n+1}\right| + \sum_{j=j_a+1}^{j_b-1} \left|\rho_{j+1}^{n+1} - \rho_j^{n+1}\right| + \left|\rho_{j_b+1}^{n+1} - \rho_{j_b}^{n+1}\right|. \tag{3.3.7}$$

Now, let us focus on the central sum on the right-hand side of (3.3.7). We write

$$\rho_{j+1}^{n+1} - \rho_j^{n+1} = \mathcal{A}_j^n - \lambda \mathcal{B}_j^n$$

with

$$\begin{aligned}
\mathcal{A}_j^n \quad &:= \quad (1 - \lambda v_{j+3/2}^n g(\rho_{j+2}^n) + \lambda \rho_{j-1}^n g'(\xi_{j+1/2}^n) v_{j+1/2}^n)(\rho_{j+1}^n - \rho_j^n) \\
&\quad\; +\lambda v_{j+1/2}^n g(\rho_{j+1}^n)(\rho_j^n - \rho_{j-1}^n) - \lambda \rho_{j+1}^n g'(\xi_{j+3/2}^n) v_{j+3/2}^n (\rho_{j+2}^n - \rho_{j+1}^n), \\
\mathcal{B}_j^n \quad &:= \quad \rho_j^n g(\rho_{j+1}^n)(v_{j+3/2}^n - v_{j+1/2}^n) - \rho_{j-1}^n g(\rho_j^n)(v_{j+1/2}^n - v_{j-1/2}^n).
\end{aligned}$$

Taking the absolute value and summing,

$$\begin{aligned}
\sum_{j=j_a+1}^{j_b-1} |\mathcal{A}_j^n| \quad \leq \quad & \sum_{j=j_a+1}^{j_b-1} |\rho_{j+1}^n - \rho_j^n| + \lambda v_{j_a+3/2}^n g(\rho_{j_a+2}^n) |\rho_{j_a+1}^n - \rho_{j_a}^n| \\
& -\lambda v_{j_b+1/2}^n g(\rho_{j_b+1}^n) |\rho_{j_b}^n - \rho_{j_b-1}^n| + \lambda g'(\xi_{j_a+3/2}^n) \rho_{j_a+1}^n v_{j_a+3/2}^n \left|\rho_{j_a+2}^n - \rho_{j_a+1}^n\right| \\
& -\lambda g'(\xi_{j_b+1/2}^n) \rho_{j_b-1}^n v_{j_b+1/2}^n \left|\rho_{j_b+1}^n - \rho_{j_b}^n\right|.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\mathcal{B}_j^n \quad &= \quad \rho_j^n g(\rho_{j+1}^n)(v_{j+3/2}^n - v_{j+1/2}^n) - \rho_{j-1}^n g(\rho_j^n)(v_{j+1/2}^n - v_{j-1/2}^n) \\
&= \quad \rho_j^n g(\rho_{j+1}^n) v'(\tilde{R}_{j+1}^n) \left(R_{j+3/2}^n - R_{j+1/2}^n\right) - \rho_{j-1}^n g(\rho_j^n) v'(\tilde{R}_j^n) \left(R_{j+1/2}^n - R_{j-1/2}^n\right) \\
&= \quad \rho_j^n g(\rho_{j+1}^n) v'(\tilde{R}_{j+1}^n) \left(R_{j+3/2}^n - R_{j+1/2}^n\right) \pm \rho_{j-1}^n g(\rho_j^n) v'(\tilde{R}_{j+1}^n) \left(R_{j+3/2}^n - R_{j+1/2}^n\right) \\
&\quad\; -\rho_{j-1}^n g(\rho_j^n) v'(\tilde{R}_j^n) \left(R_{j+1/2}^n - R_{j-1/2}^n\right) \\
&= \quad \rho_j^n g'(\xi_{j+1/2}^n)(\rho_{j+1}^n - \rho_j^n) v'(\tilde{R}_{j+1}^n) \left(R_{j+3/2}^n - R_{j+1/2}^n\right)
\end{aligned}$$

$$
\begin{aligned}
&+g(\rho_j^n)(\rho_j^n-\rho_{j-1}^n)v'(\tilde{R}_{j+1}^n)\left(R_{j+3/2}^n-R_{j+1/2}^n\right)\\
&+\rho_{j-1}^n g(\rho_j^n)v'(\tilde{R}_{j+1}^n)\left(R_{j+3/2}^n-2R_{j+1/2}^n+R_{j-1/2}^n\right)\\
&+\rho_{j-1}^n g(\rho_j^n)v''(\bar{R}_{j+1/2}^n)\left(R_{j+1/2}^n-R_{j-1/2}^n\right)\left(\tilde{R}_{j+1}^n-\tilde{R}_j^n\right),
\end{aligned}
$$

where $\tilde{R}_j^n \in \mathcal{I}(R_{j-1/2}^n, R_{j+1/2}^n)$ and $\bar{R}_{j+1/2}^n \in \mathcal{I}(\tilde{R}_j^n, \tilde{R}_{j+1}^n)$. Now, by the assumptions (3.1.4) on the kernel function and defining $\omega_N := 0$, we get

$$
\begin{aligned}
\left|R_{j+1/2}^n-R_{j-1/2}^n\right| &= \left|\Delta x\sum_{k=0}^{N-1}\omega_k(\rho_{j+k+1}^n-\rho_{j+k}^n)\right|\\
&= \Delta x\left|-\omega_0\rho_j^n+\sum_{k=1}^{N-1}(\omega_{k-1}-\omega_k)\rho_{j+k}^n+\omega_{N-1}\rho_{j+N}^n\right|\\
&\leq \Delta x\left(2\omega_\eta(0)\rho_{\max}+\left\|\omega'\right\|_{\mathbf{L}^\infty}\left\|\rho\right\|_{\mathbf{L}^1}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&\left|R_{j+3/2}^n-2R_{j+1/2}^n+R_{j-1/2}^n\right|\\
&=\left|\Delta x\left(\sum_{k=0}^{N-1}\omega_k\rho_{j+k+2}^n-2\sum_{k=0}^{N-1}\omega_k\rho_{j+k+1}^n+\sum_{k=0}^{N-1}\omega_k\rho_{j+k}^n\right)\right|\\
&=\left|\Delta x\left(\sum_{k=1}^{N-1}(\omega_{k-1}-2\omega_k+\omega_{k+1})\rho_{j+k+1}^n+\Delta x\rho_{j+1}^n\frac{\omega_1-\omega_0}{\Delta x}\right.\right.\\
&\qquad\left.\left.+\frac{1}{\Delta x}(\omega_{N-1}\underbrace{-\omega_N}_{:=0})\rho_{j+N+1}^n\Delta x+\omega_0(\rho_j^n-\rho_{j+1}^n)\right)\right|\\
&\leq(\Delta x)^2\left\|\omega''\right\|_{\mathbf{L}^\infty}\left\|\rho\right\|_{\mathbf{L}^1}+2(\Delta x)^2\rho_{\max}\left\|\omega'\right\|_{\mathbf{L}^\infty}\\
&\quad+\Delta x\omega_0|\rho_j-\rho_{j+1}|.
\end{aligned}
$$

Now, we compute

$$
\begin{aligned}
\left|\tilde{R}_{j+1}^n-\tilde{R}_j^n\right| &= \left|\theta R_{j+3/2}^n+(1-\theta)R_{j+1/2}^n-\mu R_{j+1/2}^n-(1-\mu)R_{j-1/2}^n\right|\\
&\leq 3\Delta x\left(2\omega_\eta(0)\rho_{\max}+\|\omega'\|_{\mathbf{L}^\infty}\|\rho\|_{\mathbf{L}^1}\right),
\end{aligned}
$$

for some $\theta,\mu\in[0,1]$. We end up with

$$
\begin{aligned}
\left|\mathcal{B}_j^n\right| \leq\ & \Delta x(2\omega_\eta(0)\rho_{\max}+\|\omega'\|_{\mathbf{L}^\infty}\|\rho\|_{\mathbf{L}^1})\|g\|_{\mathbf{L}^\infty}\left\|v'\right\|_{\mathbf{L}^\infty}\left|\rho_j^n-\rho_{j-1}^n\right|\\
&+(\Delta x)^2\|g\|_{\mathbf{L}^\infty}\left(\left\|\omega''\right\|_{\mathbf{L}^\infty}\|\rho\|_{\mathbf{L}^1}\left\|v'\right\|_{\mathbf{L}^\infty}+2\rho_{\max}\left\|\omega'\right\|_{\mathbf{L}^\infty}\left\|v'\right\|_{\mathbf{L}^\infty}\right)\left|\rho_{j-1}^n\right|\\
&+\Delta x\rho_{\max}\Bigg(\omega_\eta(0)\|g\|_{\mathbf{L}^\infty}\left\|v'\right\|_{\mathbf{L}^\infty}\\
&+\left\|g'\right\|_{\mathbf{L}^\infty}\left\|v'\right\|_{\mathbf{L}^\infty}(2\omega_\eta(0)\rho_{\max}+\left\|\omega_\eta'\right\|_{\mathbf{L}^\infty}\|\rho\|_{\mathbf{L}^1})\Bigg)\left|\rho_{j+1}^n-\rho_j^n\right|
\end{aligned}
$$

$$+3(\Delta x)^2 \left(\|\omega'\|_{\mathbf{L}^\infty}\|\rho\|_{\mathbf{L}^1} + 2\omega_\eta(0)\rho_{\max}\right)^2 \|g\|_{\mathbf{L}^\infty} \left\|v''\right\|_{\mathbf{L}^\infty} \left|\rho^n_{j-1}\right|.$$

Summing,

$$\begin{aligned}
\lambda \sum_{j=j_a+1}^{j_b-1} |\mathcal{B}^n_j| \quad \leq \quad & \Delta t\mathcal{K} \sum_{j=j_a+1}^{j_b-1} \left|\rho^n_j - \rho^n_{j-1}\right| + \Delta t\Delta x\mathcal{K}_1 \sum_{j=j_a+1}^{j_b-1} \left|\rho^n_{j-1}\right| \\
& + \Delta t\mathcal{K}_2 \sum_{j=j_a+1}^{j_b-1} \left|\rho^n_{j+1} - \rho^n_j\right|,
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{K} &= (2\omega_\eta(0)\rho_{\max} + \|\omega'\|_{\mathbf{L}^\infty}\|\rho\|_{\mathbf{L}^1}) \|g\|_{\mathbf{L}^\infty} \left\|v'\right\|_{\mathbf{L}^\infty}, \\
\mathcal{K}_1 &= \|g\|_{\mathbf{L}^\infty} \left(\left\|\omega''\right\|_{\mathbf{L}^\infty} \|\rho\|_{\mathbf{L}^1} \left\|v'\right\|_{\mathbf{L}^\infty} + 2\rho_{\max} \left\|\omega'\right\|_{\mathbf{L}^\infty} \left\|v'\right\|_{\mathbf{L}^\infty}\right) \\
&\quad + 3\left(\|\omega'\|_{\mathbf{L}^\infty}\|\rho\|_{\mathbf{L}^1} + 2\omega_\eta(0)\rho_{\max}\right)^2 \left\|v''\right\|_{\mathbf{L}^\infty} \|g\|_{\mathbf{L}^\infty}, \\
\mathcal{K}_2 &= \left\|v'\right\|_{\mathbf{L}^\infty} \rho_{\max} \left(\omega_\eta(0) \|g\|_{\mathbf{L}^\infty} + \left\|g'\right\|_{\mathbf{L}^\infty} (2\omega_\eta(0)\rho_{\max} + \|\omega_\eta\|_{\mathbf{L}^\infty} \|\rho\|_{\mathbf{L}^1})\right).
\end{aligned}$$

We are left with the boundary terms in (3.3.7), for $j = j_a$, we have

$$\begin{aligned}
\rho^{n+1}_{j_a+1} - \rho^{n+1}_{j_a} &= \rho^n_{j_a+1} - \lambda\left(\rho^n_{j_a+1} g(\rho^n_{j_a+2}) v^n_{j_a+3/2} - \rho^n_{j_a} g(\rho^n_{j_a+1}) v^n_{j_a+1/2}\right) - \rho^{n+1}_{j_a} \pm \rho^n_{j_a} \\
&= (1 - \lambda v^n_{j+3/2} g(\rho^n_{j_a+1}))(\rho^n_{j_a+1} - \rho^n_{j_a}) - \lambda\rho^n_{j_a} g(\rho^n_{j_a+1})(v^n_{j_a+3/2} - v^n_{j_a+1/2}) \\
&\quad - \lambda v_{j_a+3/2} \rho^n_{j_a+1} g'(\xi^n_{j_a+3/2}) \left(\rho^n_{j_a+2} - \rho^n_{j_a+1}\right) + \rho^n_{j_a} - \rho^{n+1}_{j_a}.
\end{aligned}$$

and similarly for $j = j_b$

$$\begin{aligned}
\rho^{n+1}_{j_b+1} - \rho^{n+1}_{j_b} &= \rho^{n+1}_{j_b+1} - \rho^n_{j_b} + \lambda\left(\rho^n_{j_b} g(\rho^n_{j_b+1}) v^n_{j_b+1/2} - \rho^n_{j_b-1} g(\rho^n_{j_b}) v^n_{j_b-1/2}\right) \pm \rho^n_{j_b+1} \\
&= \rho^{n+1}_{j_b+1} - \rho^n_{j_b+1} + (\rho^n_{j_b+1} - \rho^n_{j_b})(1 + \lambda v^n_{j_b+1/2} \rho^n_{j_b-1} g'(\xi^n_{j_b+1/2})) \\
&\quad + \lambda v^n_{j_b+1/2} g(\rho^n_{j_b+1})(\rho^n_{j_b} - \rho^n_{j_b-1}) + \lambda\rho^n_{j_b-1} g(\rho^n_{j_b})(v^n_{j_b+1/2} - v^n_{j_b-1/2}).
\end{aligned}$$

Next, collecting the terms, taking the absolute value and summing over j

$$\begin{aligned}
& \sum_{j=j_a}^{j_b} \left|\rho^{n+1}_{j+1} - \rho^{n+1}_j\right| \\
\leq \; & \left|\rho^{n+1}_{j_a+1} - \rho^{n+1}_{j_a}\right| + \sum_{j=j_a+1}^{j_b-1} \left(|\mathcal{A}^n_j| + \lambda\left|\mathcal{B}^n_j\right|\right) + \left|\rho^{n+1}_{j_b+1} - \rho^{n+1}_{j_b}\right| \\
\leq \; & (1 - \lambda v^n_{j_a+3/2} g(\rho^n_{j_a+2}))\left|\rho^n_{j_a+1} - \rho^n_{j_a}\right| + \lambda\rho^n_{j_a} g(\rho^n_{j_a+1})\left|v^n_{j_a+3/2} - v^n_{j_a+1/2}\right| \\
& + \left|\rho^n_{j_a} - \rho^{n+1}_{j_a}\right| - \lambda v^n_{j_a+3/2} \rho^n_{j_a+1} g'(\xi^n_{j_a+3/2})\left|\rho^n_{j_a+2} - \rho^n_{j_a+1}\right| + \sum_{j=j_a+1}^{j_b-1} |\rho^n_{j+1} - \rho^n_j| \\
& + \lambda v^n_{j_a+3/2} g(\rho^n_{j_a+2})|\rho^n_{j_a+1} - \rho^n_{j_a}| - \lambda v^n_{j_b+1/2} g(\rho^n_{j_b+1})|\rho^n_{j_b} - \rho^n_{j_b-1}|
\end{aligned}$$

$$
\begin{aligned}
&+\lambda g'(\xi^n_{j_a+3/2})\rho^n_{j_a+1}v^n_{j_a+3/2}\left|\rho^n_{j_a+2}-\rho^n_{j_a+1}\right| - \lambda g'(\xi^n_{j_b+1/2})\rho^n_{j_b-1}v^n_{j_b+1/2}\left|\rho^n_{j_b+1}-\rho^n_{j_b}\right| \\
&+\Delta t\mathcal{K}\sum_{j=j_a+1}^{j_b-1}\left|\rho^n_j-\rho^n_{j-1}\right| + \Delta t\Delta x\mathcal{K}_1\sum_{j=j_a+1}^{j_b-1}\left|\rho^n_{j-1}\right| + \Delta t\mathcal{K}_2\sum_{j=j_a+1}^{j_b-1}\left|\rho^n_{j+1}-\rho^n_j\right| \\
&+\left|\rho^{n+1}_{j_b+1}-\rho^n_{j_b+1}\right| + \left|\rho^n_{j_b+1}-\rho^n_{j_b}\right|(1+\lambda v^n_{j_b+1/2}\rho^n_{j_b-1}g'(\xi^n_{j_b+1/2})) \\
&+\lambda v^n_{j_b+1/2}g(\rho^n_{j_b})\left|\rho^n_{j_b}-\rho^n_{j_b-1}\right| + \lambda\rho^n_{j_b-1}g(\rho^n_{j_b})\left|v^n_{j_b+1/2}-v^n_{j_b-1/2}\right| \\
= \; &\left|\rho^n_{j_a}-\rho^{n+1}_{j_a}\right| + \sum_{j=j_a}^{j_b}\left|\rho^n_{j+1}-\rho^n_j\right| + \lambda\rho^n_{j_a}g(\rho^n_{j_a+1})\left|v^n_{j_a+3/2}-v^n_{j_a+1/2}\right| + \left|\rho^{n+1}_{j_b+1}-\rho^n_{j_b+1}\right| \\
&+\Delta t\mathcal{K}\sum_{j=j_a+1}^{j_b-1}\left|\rho^n_j-\rho^n_{j-1}\right| + \Delta t\Delta x\mathcal{K}_1\sum_{j=j_a+1}^{j_b-1}\left|\rho^n_{j-1}\right| + \Delta t\mathcal{K}_2\sum_{j=j_a+1}^{j_b-1}\left|\rho^n_{j+1}-\rho^n_j\right| \\
&+\lambda\rho^n_{j_b-1}g(\rho^n_{j_b})\left|v^n_{j_b+1/2}-v^n_{j_b-1/2}\right| \\
\leq \; &\left|\rho^n_{j_a}-\rho^{n+1}_{j_a}\right| + (1+2\Delta t\mathcal{K})\sum_{j=j_a}^{j_b}\left|\rho^n_{j+1}-\rho^n_j\right| + \left|\rho^{n+1}_{j_b+1}-\rho^n_{j_b+1}\right| + \Delta t\mathcal{K}_3,
\end{aligned}
$$

where $\mathcal{K}_3 = 2\rho_{\max}\mathcal{K} + \|\rho\|_{\mathbf{L}^1}\mathcal{K}_1$. By a standard iterative procedure we can deduce, for $1 \leq n < N_T - 1$,

$$
\sum_{j=j_a}^{j_b}\left|\rho^{n+1}_{j+1}-\rho^{n+1}_j\right| \leq e^{2\mathcal{K}T}\left(\sum_{j=j_a}^{j_b}\left|\rho^0_{j+1}-\rho^0_j\right| + 2\frac{C(T)}{q} + \mathcal{K}_3 T\right).
$$

This concludes the proof because $[a,b] \subseteq [x_{j_a}, x_{j_b+1}]$. □

3.3.2 Discrete entropy inequality.

Next we show that the approximate solution obtained by the scheme (3.2.1) satisfies a discrete entropy inequality. Let us define

$$
G_{j+1/2}(u) = ug(u)v^n_{j+1/2}, \qquad \mathcal{F}^c_{j+1/2}(u) := G_{j+1/2}(u\vee c) - G_{j+1/2}(u\wedge c)
$$

with $a \vee b = \max\{a,b\}$ and $a \wedge b = \min\{a,b\}$.

Lemma 3.7. *Let ρ^n_j for $j \in \mathbb{Z}$ and $n \in \mathbb{N}$ given by (3.2.1), and let the CFL condition (3.3.1) and the hypothesis (3.1.4) hold. Then we have*

$$
\begin{aligned}
&\left|\rho^{n+1}_j - c\right| - \left|\rho^n_j - c\right| + \lambda(\mathcal{F}^c_{j+1/2}(\rho^n_j) - \mathcal{F}^c_{j-1/2}(\rho^n_{j-1})) \\
&+\lambda\,\mathrm{sgn}\left(\rho^{n+1}_j - c\right)cg(c)(v^n_{j+1/2} - v^n_{j-1/2}) \leq 0,
\end{aligned} \tag{3.3.8}
$$

for all $j \in \mathbb{Z}$, $n \in \mathbb{N}$ and $c \in [0,\rho_{\max}]$.

Proof. For a complete proof see [44, Section 3.4]. □

3.3.3 Convergence to an entropy solution.

Theorem 3.8. *Let* $\rho_0 \in \mathbf{BV} \cap \mathbf{L}^\infty(\mathbb{R}, [0, \rho_{\max}])$*. Let* $\Delta x \to 0$ *with* $\lambda = \frac{\Delta t}{\Delta x}$ *constant and satisfying the CFL condition* (3.3.1)*. The sequence of approximate solution* ρ_Δ *constructed through finite volume scheme* (3.2.1)*-*(3.2.2) *converges in* $\mathbf{L}^1_{\text{loc}}$ *to a function in* $\mathbf{L}^\infty([0,T] \times \mathbb{R}; [0, \rho_{\max}])$ *such that* $\|\rho\|_{\mathbf{L}^1} = \|\rho_0\|_{\mathbf{L}^1}$ *.*

Proof. Lemma 3.3 ensures that the sequence of approximate solutions ρ_Δ is bounded in $\mathbf{L}^\infty$. Lemma 3.5 proves the $\mathbf{L}^1-$continuity in time of the sequence ρ_Δ, while Lemma 3.6 guarantees a bound on the spatial total variation in any interval $[a, b]$ not containing $x = 0$. Applying standard compactness results we have that for any interval $[a, b]$ not containing $x = 0$, there exists a subsequence, still denoted by ρ_Δ converging in $\mathbf{L}^1([0,T] \times [a, b]; [0, \rho_{\max}])$. Let us take a countable set of intervals $[a_i, b_i]$ such that $\cup_i [a_i, b_i] = \mathbb{R}^*$, using a standard diagonal process, we can extract a subsequence, still denoted by ρ_Δ, converging in $\mathbf{L}^1_{\text{loc}}([0,T] \times \mathbb{R}; [0, \rho_{\max}])$ and almost everywhere in $[0,T] \times \mathbb{R}$, to a function $\rho \in \mathbf{L}^\infty([0,T] \times \mathbb{R}; [0, \rho_{\max}])$. □

Lemma 3.9. *Let* $\rho(t, x)$ *be a weak solution constructed as the limit of approximations* ρ_Δ *generated by the scheme* (3.2.1) *and let* $c \in [0, \rho_{\max}]$*. Let* $\varphi \in \mathcal{D}(\mathbb{R}^* \times [0, T))$*. Then the following entropy inequality is satisfied:*

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}} (|\rho - c|\varphi_t \mathrm{d}x\mathrm{d}t + \int_0^T \int_{\mathbb{R}} \mathrm{sgn}(\rho - c)(f(t, x, \rho) - f(t, x, c))\varphi_x \, \mathrm{d}x\mathrm{d}t \\
&- \int_0^T \int_{\mathbb{R}} \mathrm{sgn}(\rho - c)\partial_x f(t, x, c)\varphi \, \mathrm{d}x\mathrm{d}t + \int_{-\infty}^{\infty} |\rho_0(x) - c|\varphi(0, x)\mathrm{d}x \geq 0. \quad (3.3.9)
\end{aligned}$$

Proof. Let φ be a test function of the type described in the statement of the lemma and set $\varphi_j^n = \varphi(t^n, x_j)$, let us denote $\Delta_- p_j = p_j - p_{j-1}$, we multiply the cell entropy inequality (3.3.8) by $\varphi_j^n \Delta x$, and then sum by parts to get

$$\begin{aligned}
S_1 + S_2 + S_3 \;=\;& \Delta x \Delta t \sum_{n \geq 0} \sum_{j \in \mathbb{Z}} \left|\rho_j^{n+1} - c\right| (\varphi_j^{n+1} - \varphi_j^n)/\Delta t + \Delta x \sum_j |\rho_j^0 - c|\varphi_j^0 \\
&+ \Delta x \Delta t \sum_{n \geq 0} \sum_{j \in \mathbb{Z}} \mathcal{F}^c_{j-1/2} \Delta_- \varphi_j^n / \Delta x \\
&- \Delta x \Delta t \sum_{n \geq 0} \sum_{j \in \mathbb{Z}} \mathrm{sgn}\left(\rho_j^{n+1} - c\right) c\, g(c)\, \Delta_- v_{j+1/2}\, \varphi_j^n / \Delta x \geq 0.
\end{aligned}$$

By Lebesgue's dominated convergence theorem as $\Delta := (\Delta x, \Delta t) \to 0$,

$$S_1 \to \int_0^T \int_{\mathbb{R}} |\rho - c|\varphi_t \mathrm{d}x\mathrm{d}t + \int_{-\infty}^{\infty} |\rho_0(x) - c|\varphi(0, x)\mathrm{d}x,$$

and

$$S_2 \to \int_0^T \int_{\mathbb{R}} \mathrm{sgn}\,(\rho - c)\,(f(t, x, \rho) - f(t, x, c))\varphi_x \, \mathrm{d}x\mathrm{d}t.$$

Now let us study the sum S_3 and we have

$$\begin{aligned} S_3 &= -\Delta x \Delta t \sum_{n\geq 0} \sum_{\substack{j\in\mathbb{Z}\\ j\leq -1}} \operatorname{sgn}\left(\rho_j^{n+1} - c\right) cg(c)\, \Delta_- v_{j+1/2}\, \varphi_j^n/\Delta x \\ &\quad -\Delta x \Delta t \sum_{n\geq 0} \sum_{\substack{j\in\mathbb{Z}\\ j\geq 1}} \operatorname{sgn}\left(\rho_j^{n+1} - c\right) cg(c)\, \Delta_- v_{j+1/2}\, \varphi_j^n/\Delta x \\ &\quad -\Delta x \Delta t \sum_{n\geq 0} \operatorname{sgn}\left(\rho_0^{n+1} - c\right) cg(c)\, \Delta_- v_{1/2}\, \varphi_0^n/\Delta x, \\ &= S_{31} + S_{32} + S_{33}. \end{aligned}$$

Observe that the support of the test function φ does not include the discontinuity flux point 0, for this reason we consider $\varphi_0 = 0$ according to our discretization, then the sum S_{33} is equal to zero because $\varphi_0 = 0$. Finally,

$$\begin{aligned} S_{31} + S_{32} \;\longrightarrow\; & -\int_0^T \int_{-\infty}^0 \operatorname{sgn}(\rho - c)\, \partial_x f(t,x,c)\varphi \,\mathrm{d}x\mathrm{d}t \\ & -\int_0^T \int_0^{\infty} \operatorname{sgn}(\rho - c)\, \partial_x f(t,x,c)\varphi \,\mathrm{d}x\mathrm{d}t. \end{aligned}$$

□

Lemma 3.10. *Let $\rho(t,x)$ be a weak solution constructed as the limit of approximations ρ_Δ generated by the scheme* (3.2.1) *and let $c \in [0, \rho_{\max}]$. Let $\varphi \in \mathbf{C}_c^1(\mathbb{R}\times[0,T))$. Then the following entropy inequality is satisfied:*

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}} |\rho - c|\, \varphi_t + \operatorname{sgn}(\rho - c)\,(f(t,x,\rho) - f(t,x,c))\, \partial_x\varphi \,\mathrm{d}x\,\mathrm{d}t \\ &+ \int_0^T \int_{\mathbb{R}_*} |\partial_x f(t,x,c)|\varphi \,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}} |\rho_0(x) - c|\, \varphi(0,x)\mathrm{d}x \\ &+ \int_0^T |(k_r - k_l)c\, g(c)\, \psi(\rho * \omega_\eta)|\, \varphi(t,0)\mathrm{d}t \geq 0. \end{aligned}$$

Proof. Let φ be a test function of the type described in the statement of the lemma and set $\varphi_j^n = \varphi(t^n, x_j)$. There exist $T > 0$ and $R > 0$ such that $\varphi(t,x) = 0$ for $t > T$ and $|x| > R$. Our starting point is the following cell entropy inequality which is a consequence of (3.3.8).

$$\left|\rho_j^{n+1} - c\right| \leq \left|\rho_j^n - c\right| - \lambda \Delta_- \mathcal{F}_{j+1/2}^c + \lambda \left|cg(c)\Delta_- v_{j+1/2}^n\right| \tag{3.3.10}$$

We multiply (3.3.10) by $\varphi_j^n \Delta x$, and then sum by parts to get

$$S_4 + S_5 + S_6 \;=\; \Delta x \Delta t \sum_{n\geq 0} \sum_{j\in\mathbb{Z}} \left|\rho_j^{n+1} - c\right| (\varphi_j^{n+1} - \varphi_j^n)/\Delta t + \Delta x \sum_j \left|\rho_j^0 - c\right| \varphi_j^0$$

$$
\begin{aligned}
&+\Delta x\Delta t\sum_{n\geq 0}\sum_{j\in\mathbb{Z}}\mathcal{F}^c_{j-1/2}(\Delta_-\varphi^n_j/\Delta x)\\
&+\Delta x\Delta t\sum_{n\geq 0}\sum_{j\in\mathbb{Z}}\left|cg(c)\Delta_- v^n_{j+1/2}\right|\varphi^n_j/\Delta x \geq 0.
\end{aligned}
$$

By Lebesgue's dominated convergence theorem as $\Delta := (\Delta x, \Delta t) \to 0$,

$$
S_4 \to \int_0^T\int_{\mathbb{R}}|\rho - c|\varphi_t \mathrm{d}x\mathrm{d}t + \int_{-\infty}^{\infty}|\rho_0(x) - c|\varphi(0,x)\mathrm{d}x.
$$

Following the same standard arguments as in Lemma (3.9),

$$
S_5 \to \int_0^T\int_{\mathbb{R}}\operatorname{sgn}(\rho - c)\left(f(t,x,\rho) - f(t,x,c)\right)\partial_x\varphi\,\mathrm{d}x\,\mathrm{d}t.
$$

Now we can rewrite the sum S_6

$$
\begin{aligned}
S_6 &= \Delta x\Delta t\sum_{n\geq 0}\sum_{j\in\mathbb{Z},\ j\leq -1}\left|cg(c)\Delta_- v^n_{j+1/2}\right|\varphi^n_j/\Delta x\\
&\quad +\Delta x\Delta t\sum_{n\geq 0}\sum_{j\in\mathbb{Z},\ j\geq 1}\left|cg(c)\Delta_- v^n_{j+1/2}\right|\varphi^n_j/\Delta x\\
&\quad +\Delta t\sum_{n\geq 0}\left|cg(c)\Delta_- v^n_{1/2}\right|\varphi^n_0\\
&= S_{61} + S_{62} + S_{63}.
\end{aligned}
$$

At this point, we can observe that as $\Delta := (\Delta x, \Delta t) \to 0$

$$
\begin{aligned}
S_{61} + S_{62} &\to \int_0^T\int_{\mathbb{R}\setminus\{0\}}|f(t,x,c)_x|\varphi\,\mathrm{d}x\,\mathrm{d}t,\\
S_{63} &\to \int_0^T|(k_r - k_l)c\,g(c)\,\psi(\rho*\omega_\eta)|\,\varphi(t,0)\mathrm{d}t.
\end{aligned}
$$

□

Theorem 3.11. *Let $\rho(t,x)$ be the limit of approximations ρ_Δ generated by the scheme* (3.2.1) *and let $c \in [0, \rho_{\max}]$. Then $\rho(t,x)$ is an entropy solution satisfying Definition 3.2.*

Proof. Let $0 \leq \varphi \in \mathbf{C}^1_c([0,T)\times\mathbb{R})$. We set $\varphi^n_j = \varphi(t^n, x_j)$. For $\varepsilon > 0$, define the set

$$
\sigma^\varepsilon_0 = \{(t,x) \in [0,T)\times\mathbb{R} | x \in (-\varepsilon,\varepsilon),\ t \in [0,T)\}.
$$

For each sufficiently small $\varepsilon > 0$ we can write the test function φ as a sum of two test functions, one having support away from 0 and the other with support in σ^ε_0. We take test functions ψ^ε, $\alpha^\varepsilon \in \mathbf{C}^1_c([0,T)\times\mathbb{R})$ such that

$$
\varphi(t,x) = \psi^\varepsilon(t,x) + \alpha^\varepsilon(t,x), \quad 0 \leq \psi^\varepsilon(t,x) \leq \varphi(t,x), \quad 0 \leq \alpha^\varepsilon(t,x) \leq \varphi(t,x),
$$

where ψ^ε has support located around the jump in 0

$$\mathrm{supp}(\psi^\varepsilon) \subseteq \sigma_0^\varepsilon, \qquad \psi^\varepsilon(t,0) = \varphi(t,0),$$

and α^ε vanishes around the jump, i.e.

$$\mathrm{supp}(\alpha^\varepsilon) \subseteq [0,T) \times \mathbb{R}^*.$$

We can take this decomposition in such way that

$$\alpha^\varepsilon \to \varphi \quad \text{in } \mathbf{L}^1([0,T)\times\mathbb{R}), \quad \psi^\varepsilon \to 0 \quad \text{in } \mathbf{L}^1([0,T)\times\mathbb{R}) \tag{3.3.11}$$

as $\varepsilon \to 0$. By applying Lemma 3.9 with the test function α^ε and Lemma 3.10 with ψ^ε, and summing the two entropy inequalities, using $\varphi = \psi^\varepsilon + \alpha^\varepsilon$ along with $\psi^\varepsilon(0,t) = \varphi(0,t)$ to get

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}} (|\rho - c|\varphi_t \mathrm{d}x\mathrm{d}t + \int_0^T \int_{\mathbb{R}} \mathrm{sgn}\,(\rho - c)\,(f(t,x,\rho) - f(t,x,c))\varphi_x \mathrm{d}x\mathrm{d}t \\
&- \int_0^T \int_{\mathbb{R}} \mathrm{sgn}\,(\rho - c)\, f(t,x,c)_x \alpha^\varepsilon \mathrm{d}x\mathrm{d}t + \int_0^T \int_{\mathbb{R}^*} |f(t,x,c))_x|\psi^\varepsilon \mathrm{d}x\mathrm{d}t \\
&+ \int_0^T |(k_r - k_l)c\, g(c)\, \psi(\rho * \omega_\eta)|\, \varphi(t,0)\mathrm{d}t + \int_{-\infty}^{\infty} |\rho_0(x) - c|\phi(0,x)\mathrm{d}x \geq 0.
\end{aligned}$$

Thanks to (3.3.11), we can complete the proof by sending $\varepsilon \to 0$. □

3.3.4 $\mathbf{L}^1$-Stability and uniqueness.

Theorem 3.12. *Assume the hypothesis (3.1.4). If ρ and $\tilde{\rho}$ are two entropy solutions of (3.1.1) in the sense of Definition (3.2), the following inequality holds*

$$\|\rho(t,\cdot) - \tilde{\rho}(t,\cdot)\|_{\mathbf{L}^1(\mathbb{R})} \leq \mathrm{e}^{K(T)t} \|\rho(0,\cdot) - \tilde{\rho}(0,\cdot)\|_{\mathbf{L}^1(\mathbb{R})}, \tag{3.3.12}$$

for almost every $0 < t < T$ and some suitable constant $K(T) > 0$.

Proof. Following [58, Theorem 2.1], for any two entropy solutions u and v we can derive the L^1 contraction property through the doubling of variables technique:

$$\begin{aligned}
&\iint_{\mathbb{R}^+\times\mathbb{R}} (|\rho - \tilde{\rho}|\phi_t + \mathrm{sgn}\,(\rho - \tilde{\rho})\,(f(t,x,\rho) - f(t,x,\tilde{\rho}))\phi_x)\,\mathrm{d}x\mathrm{d}t \\
&\leq K \iint_{\mathbb{R}^+\times\mathbb{R}} |\rho - \tilde{\rho}|\phi \mathrm{d}x\mathrm{d}t,
\end{aligned} \tag{3.3.13}$$

where $K = K(T)$, for any $0 \leq \phi \in \mathcal{D}(\mathbb{R}^+ \times \mathbb{R}^*)$. We remove the assumption in (3.3.13) that ϕ vanishes near 0, by introducing the following Lipschitz function for $h > 0$

$$\mu_h(x) = \begin{cases} \frac{1}{h}(x + 2h), & x \in [-2h, -h], \\ 1, & x \in [-h, h], \\ \frac{1}{h}(2h - x), & x \in [h, 2h], \\ 0, & |x| \geq 2h. \end{cases}$$

Now we can define $\Psi_h(x) = 1 - \mu_h(x)$, noticing that $\Psi_h \to 1$ in L^1 as $h \to 0$. Moreover, Ψ_h vanishes in a neighborhood of 0. For any $0 \le \Phi \in \mathbf{C}_c^\infty(\mathbb{R}^+ \times \mathbb{R})$, we can check that $\phi = \Phi\Psi_h$ is an admissible test function for (3.3.13). Using ϕ in (3.3.13) and integrating by parts we get

$$\begin{aligned}
&\iint_{\mathbb{R}^+\times\mathbb{R}} \left(|\rho-\tilde{\rho}|\Phi_t\Psi_h + \operatorname{sgn}(\rho-\tilde{\rho})\left(f(t,x,\rho)-f(t,x,\tilde{\rho})\right)\Phi_x\Psi_h\right)\mathrm{d}x\mathrm{d}t \\
&\underbrace{-\iint_{\mathbb{R}^+\times\mathbb{R}} \operatorname{sgn}(\rho-\tilde{\rho})(f(t,x,\rho)-f(t,x,\tilde{\rho}))\Phi(t,x)\Psi_h'(x)\mathrm{d}x\mathrm{d}t}_{J(h)} \\
&\le K\iint_{\mathbb{R}^+\times\mathbb{R}} |\rho-\tilde{\rho}|\Phi\Psi_h\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Sending $h \to 0$ we end up with

$$\begin{aligned}
&\iint_{\mathbb{R}^+\times\mathbb{R}} \left(|\rho-\tilde{\rho}|\Phi_t + \operatorname{sgn}(\rho-\tilde{\rho})\left(f(t,x,\rho)-f(t,x,\tilde{\rho})\right)\Phi_x\right)\mathrm{d}x\mathrm{d}t \\
&\qquad\qquad \le K\iint_{\mathbb{R}^+\times\mathbb{R}} |\rho-\tilde{\rho}|\Phi\mathrm{d}x\mathrm{d}t + \lim_{h\to 0} J(h).
\end{aligned}$$

We can write

$$\begin{aligned}
\lim_{h\to 0} J(h) &= \lim_{h\to 0}\frac{1}{h}\int_0^T\int_h^{2h} \operatorname{sgn}(\rho-\tilde{\rho})\left(f(t,x,\rho)-f(t,x,\tilde{\rho})\right)\mathrm{d}x\mathrm{d}t \\
&\quad - \lim_{h\to 0}\frac{1}{h}\int_0^T\int_{-2h}^{-h} \operatorname{sgn}(\rho-\tilde{\rho})\left(f(t,x,\rho)-f(t,x,\tilde{\rho})\right)\mathrm{d}x\mathrm{d}t \\
&= \int_0^T \left[\operatorname{sgn}(\rho-\tilde{\rho})\left(f(t,x,\rho)-f(t,x,\tilde{\rho})\right)\right]_{x=0^-}^{x=0^+}\Phi(t,0)\mathrm{d}t,
\end{aligned}$$

where we indicate the limits from the right and left at $x = 0$. The aim is to prove that the limit $\lim_{h\to 0} J(h) \le 0$. This is equivalent to proving

$$S := \left[\operatorname{sgn}(\rho-\tilde{\rho})\left(f(t,x,\rho)-f(t,x,\tilde{\rho})\right)\right]_{x=0^-}^{x=0^+} \le 0.$$

A simple application of the Rankine-Hugoniot condition yields $S \le 0$, see the proof of [58, Theorem 2.1], where we notice that in this setting there is no flux crossing. Therefore we conclude that $S \le 0$. In this way we know that (3.3.13) holds for any $0 \le \phi \in \mathbf{C}_c^\infty(\mathbb{R}^+ \times \mathbb{R})$. For $r > 1$, let $\gamma_r : \mathbb{R} \to \mathbb{R}$ be a C^∞ function which takes values in $[0, 1]$ and satisfies

$$\gamma_r(x) = \begin{cases} 1, & |x| \le r, \\ 0, & |x| \ge r+1. \end{cases}$$

Fix s_0 and s such that $0 < s_0 < s < T$. For any $\tau > 0$ and $k > 0$ with $0 < s_0+\tau < s+k < T$, let $\beta_{\tau,k} : [0, T] \to \mathbb{R}$ be a Lipschitz function that is linear on $[s_0, s_0+\tau[\cup[s, s+k]$ and satisfies

$$\beta_{\tau,k}(t) = \begin{cases} 0, & t \in [0, s_0] \cup [s+k, T], \\ 1, & t \in [s_0 + \tau, s]. \end{cases}$$

We can take the admissible test function via a standard regularization argument $\phi = \gamma_r(x)\beta_{\tau,k}(t)$. Using this test function in (3.3.13) we obtain

$$\begin{aligned} &\frac{1}{k}\int_s^{s+k}\int_{\mathbb{R}} |\rho(t,x) - \tilde\rho(t,x)|\gamma_r(x)\mathrm{d}x\mathrm{d}t - \frac{1}{\tau}\int_{s_0}^{s_0+k}\int_{\mathbb{R}} |\rho(t,x) - \tilde\rho(t,x)|\gamma_r(x)\mathrm{d}x\mathrm{d}t \\ &\leq K\int_{s_0}^{s_0+k}\int_{\mathbb{R}} |\rho - \tilde\rho|\gamma_r(x)\mathrm{d}x\mathrm{d}t \\ &\quad + \left\|\gamma_r'\right\|_\infty \int_{s_0}^{s+k}\int_{r\leq|x|\leq r+1} \mathrm{sgn}\,(\rho - \tilde\rho)\,(f(t,x,\rho) - f(t,x,\tilde\rho))\mathrm{d}x\mathrm{d}t. \end{aligned}$$

Sending $s_0 \to 0$, we get

$$\begin{aligned} \frac{1}{k}\int_s^{s+k}\int_{-r}^{r} |\rho(t,x) - \tilde\rho(t,x)|\gamma_r(x)\mathrm{d}x\,\mathrm{d}t \quad &\leq \quad \int_{-r}^{r} |\rho_0(x) - \tilde\rho_0(x)|\mathrm{d}x \\ &\quad + \frac{1}{\tau}\int_0^\tau\int_{-r}^{r} |\tilde\rho(t,x) - \tilde\rho_0(x)|\mathrm{d}x\mathrm{d}t \\ &\quad + \frac{1}{\tau}\int_0^\tau\int_{-r}^{r} |\rho(t,x) - \rho_0(x)|\mathrm{d}x\,\mathrm{d}t \\ &\quad + K\int_0^{t+\tau}\int_{\mathbb{R}} |\rho - \tilde\rho|\gamma_r(x)\mathrm{d}x\,\mathrm{d}t + o\left(\frac{1}{r}\right). \end{aligned}$$

Observe that the second and the third terms on the right-hand side of the inequality tends to zero as $\tau \to 0$ following the same argument in [58, Lemma B.1], because our initial condition is satisfied in the "weak" sense of the definition of our entropy condition. Sending $\tau \to 0$ and $r \to \infty$, we have

$$\begin{aligned} \frac{1}{k}\int_s^{s+k}\int_{\mathbb{R}} |\rho(t,x) - \tilde\rho(t,x)|\mathrm{d}x\,\mathrm{d}t \quad &\leq \quad \int_{\mathbb{R}} |\rho_0(x) - \tilde\rho_0(x)|\mathrm{d}x \\ &\quad + K\int_0^{s+k}\int_{\mathbb{R}} |\rho(t,x) - \tilde\rho(t,x)|\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

Sending $k \to 0$ and an application of Gronwall's inequality give us the statement. □

Lemma 3.13 (A Kružkov-type integral inequality). *For any two entropy solutions $\rho = \rho(t,x)$ and $\tilde\rho = \tilde\rho(t,x)$ the integral inequality* (3.3.13) *holds for any* $0 \leq \phi \in \mathbf{C}_c^\infty(\mathbb{R}^+ \times \mathbb{R} \setminus \{0\})$.

Proof. Let $0 \leq \phi \in \mathbf{C}_c^\infty\left((\mathbb{R}^+ \times \mathbb{R} \setminus \{0\})^2\right)$, $\phi = \phi(t,x,s,y)$, $\rho = \rho(t,x)$ and $\tilde\rho = \tilde\rho(s,y)$. From the definition of entropy solution for $\rho = \rho(t,x)$ with $\kappa = \tilde\rho(s,y)$ we get

$$-\iint_{\mathbb{R}^+\times\mathbb{R}} \left(|\rho - \tilde\rho|\phi_t + \mathrm{sgn}\,(\rho - \tilde\rho)\,(f(t,x,\rho) - f(t,x,\tilde\rho))\,\phi_x\right)\,\mathrm{d}t\,\mathrm{d}x$$

$$+\iint_{\mathbb{R}^+\times\mathbb{R}\setminus\{0\}} \operatorname{sgn}(\rho-\tilde\rho)\, f(t,x,\tilde\rho)_x \phi \,\mathrm{d}t\,\mathrm{d}x \le 0.$$

Integrating over $(s,y)\in\mathbb{R}^+\times\mathbb{R}$, we find

$$\begin{aligned}
&-\iiiint_{(\mathbb{R}^+\times\mathbb{R})^2} \left(|\rho-\tilde\rho|\phi_t + \operatorname{sgn}(\rho-\tilde\rho)\left(f(t,x,\rho)-f(t,x,\tilde\rho)\right)\phi_x\right)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}y\\
&+\iiiint_{(\mathbb{R}^+\times\mathbb{R}\setminus\{0\})^2} \operatorname{sgn}(\rho-\tilde\rho)\, f(t,x,\tilde\rho)_x\phi\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}y \le 0.
\end{aligned}\tag{3.3.14}$$

Similarly, for the entropy solution $\tilde\rho=\tilde\rho(s,y)$ with $\alpha(y)=\rho(t,x)$

$$\begin{aligned}
&-\iiiint_{(\mathbb{R}^+\times\mathbb{R})^2} \left(|\tilde\rho-\rho|\phi_s + \operatorname{sgn}(\tilde\rho-\rho)\left(f(s,y,\tilde\rho)-f(s,y,\rho)\right)\phi_x\right)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}y\\
&+\iiiint_{(\mathbb{R}^+\times\mathbb{R}-\{0\})^2} \operatorname{sgn}(\rho-\tilde\rho)\, f(t,x,\tilde\rho)_x\phi\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}y \le 0.
\end{aligned}\tag{3.3.15}$$

Note that we can write, for each $(t,x)\in\mathbb{R}^+\times\mathbb{R}\setminus\{0\}$,

$$\begin{aligned}
&\operatorname{sgn}(\rho-\tilde\rho)\,(f(t,x,\rho)-f(t,x,\tilde\rho))\phi_x - \operatorname{sgn}(\rho-\tilde\rho)\, f(t,x,\tilde\rho)_x\phi\\
&= \operatorname{sgn}(\rho-\tilde\rho)\,(f(t,x,\rho)-f(s,y,\tilde\rho))\phi_x - \operatorname{sgn}(\rho-\tilde\rho)\left[(f(t,x,\tilde\rho)-f(s,y,\tilde\rho))\phi\right]_x,
\end{aligned}$$

so that

$$\begin{aligned}
&-\iiiint_{(\mathbb{R}^+\times\mathbb{R})^2} \operatorname{sgn}(\rho-\tilde\rho)\,(f(t,x,\rho)-f(t,x,\tilde\rho))\phi_x\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}y\\
&+\iiiint_{(\mathbb{R}^+\times\mathbb{R}\setminus\{0\})^2} \operatorname{sgn}(\rho-\tilde\rho)\, f(t,x,\tilde\rho)_x\phi\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}y\\
=&-\iiiint_{(\mathbb{R}^+\times\mathbb{R})^2} \operatorname{sgn}(\rho-\tilde\rho)\,(f(t,x,\rho)-f(s,y,\tilde\rho))\phi_x\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}y\\
&+\iiiint_{(\mathbb{R}^+\times\mathbb{R}\setminus\{0\})^2} \operatorname{sgn}(\rho-\tilde\rho)\left[(f(t,x,\tilde\rho)-f(s,y,\tilde\rho))\phi\right]_x\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}y.
\end{aligned}$$

Similarly, writing, for each $(y,s)\in\mathbb{R}^+\times\mathbb{R}\setminus\{0\}$

$$\begin{aligned}
&\operatorname{sgn}(\tilde\rho-\rho)\,(f(s,y,\tilde\rho)-f(s,y,\rho))\phi_y - \operatorname{sgn}(\tilde\rho-\rho)\, f(s,y,\rho)_y\phi\\
&= \operatorname{sgn}(\rho-\tilde\rho)\,(f(s,y,\tilde\rho)-f(s,y,\rho))\phi_y - \operatorname{sgn}(\rho-\tilde\rho)\left[(f(t,x,\rho)-f(s,y,\rho))\phi\right]_x,
\end{aligned}$$

so that

$$\begin{aligned}
&-\iiiint_{(\mathbb{R}^+\times\mathbb{R})^2} \operatorname{sgn}(\rho-\tilde\rho)\,(f(s,y,\tilde\rho)-f(s,y,\rho))\phi_y\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}y\\
&+\iiiint_{(\mathbb{R}^+\times\mathbb{R}\setminus\{0\})^2} \operatorname{sgn}(\rho-\tilde\rho)\, f(s,y,\rho)_y\phi\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}y\\
=&-\iiiint_{(\mathbb{R}^+\times\mathbb{R})^2} \operatorname{sgn}(\rho-\tilde\rho)\,(f(t,x,\tilde\rho)-f(s,y,\rho))\phi_x\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}y
\end{aligned}$$

$$+\iiiint_{(\mathbb{R}^+\times\mathbb{R}\setminus\{0\})^2} \operatorname{sgn}(\rho-\tilde\rho)\left[(f(t,x,\rho)-f(s,y,\rho))\phi\right]_y \,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}y.$$

Let us introduce the notations

$$\begin{aligned}\partial_{t+s} &= \partial_t+\partial_s, \quad \partial_{x+y}=\partial_x+\partial_y,\\ \partial^2_{x+y} &= (\partial_x+\partial_y)^2=\partial_x^2+2\partial_x\partial_y+\partial_y^2.\end{aligned}$$

Adding (3.3.14) and (3.3.15) we obtain

$$\begin{aligned}&-\iiiint_{(\mathbb{R}^+\times\mathbb{R})^2}\Big(|\rho-\tilde\rho|\partial_{t+s}\phi\\ &+\operatorname{sgn}(\rho-\tilde\rho)\left(f(t,x,\rho)-f(s,y,\tilde\rho)\right)\partial_{x+y}\phi\Big)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}y\\ &+\iiiint_{\mathbb{R}^+\times\mathbb{R}\setminus\{0\}}\operatorname{sgn}(\rho-\tilde\rho)\left(\partial_x\left[(f(t,x,\tilde\rho)-f(s,y,\tilde\rho))\phi\right]\right.\\ &\quad\left.+\partial_y\left[(f(t,x,\rho)-f(s,y,\rho))\phi\right]\right)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}y\le 0. \qquad (3.3.16)\end{aligned}$$

We introduce a non-negative function $\delta\in\mathbf{C}_c^\infty(\mathbb{R})$, satisfying $\delta(\sigma)=\delta(-\sigma)$, $\delta(\sigma)=0$ for $|\sigma|\ge 1$, and $\int_{\mathbb{R}}\delta(\sigma)d\sigma=1$. For $u>0$ and $z\in\mathbb{R}$, let $\delta_p(z)=\frac{1}{p}\delta(\frac{z}{p})$. We take our test function $\phi=\phi(t,x,s,y)$ to be of the form

$$\Phi(t,x,s,y)=\phi\left(\frac{t+s}{2},\frac{x+y}{2}\right)\delta_p\left(\frac{x-y}{2}\right)\delta_p\left(\frac{t-s}{2}\right),$$

where $0\le\phi\in\mathbf{C}_c^\infty\left(\mathbb{R}^+\times\mathbb{R}\setminus 0\right)$ satisfies

$$\phi(t,x)=0,\quad \forall (t,x)\in[-h,h]\times[0,T],$$

for small $h>0$. By making sure that $p<h$, one can check that Φ belongs to $\mathbf{C}_c^\infty\left((\mathbb{R}^+\times\mathbb{R}\setminus\{0\})^2\right)$. We have

$$\begin{aligned}\partial_{t+s}\Phi(t,x,s,y)&=\partial_{t+s}\phi\left(\frac{t+s}{2},\frac{x+y}{2}\right)\delta_p\left(\frac{x-y}{2}\right)\delta_p\left(\frac{t-s}{2}\right),\\ \partial_{x+y}\Phi(t,x,s,y)&=\partial_{x+y}\phi\left(\frac{t+s}{2},\frac{x+y}{2}\right)\delta_p\left(\frac{x-y}{2}\right)\delta_p\left(\frac{t-s}{2}\right),\end{aligned}$$

and using Φ as test function in (3.3.16)

$$\begin{aligned}&-\iiiint_{(\mathbb{R}^+\times\mathbb{R})^2}\left(I_1(t,x,s,y)+I_2(t,x,s,y)\right)\delta_p\left(\frac{x-y}{2}\right)\delta_p\left(\frac{t-s}{2}\right)\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}y\\ &\le\iiiint_{(\mathbb{R}^+\times\mathbb{R}\setminus\{0\})^2}\left(I_3(t,x,s,y)+I_4(t,x,s,y)+I_5(t,x,s,y)\right)\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}y,\end{aligned}$$

where

$$I_1 = |\rho(t,x)-\tilde\rho(s,y)|\partial_{t+s}\phi\left(\frac{t+s}{2},\frac{x+y}{2}\right),$$

$$
\begin{aligned}
I_2 &= \operatorname{sgn}\left(\rho(t,x)-\tilde\rho(s,y)\right)\left(f(t,x,\rho)-f(s,y,\tilde\rho)\right)\partial_{x+y}\phi\left(\frac{t+s}{2},\frac{x+y}{2}\right),\\
I_3 &= -\operatorname{sgn}\left(\rho(t,x)-\tilde\rho(s,y)\right)\left(\partial_x f(t,x,\tilde\rho)-\partial_y f(s,y,\rho)\right)\phi\left(\frac{t+s}{2},\frac{x+y}{2},\right)\\
&\qquad\times\delta_p\left(\frac{x-y}{2}\right)\delta_p\left(\frac{t-s}{2}\right),\\
I_4 &= -\operatorname{sgn}\left(\rho(t,x)-\tilde\rho(s,y)\right)\delta_p\left(\frac{x-y}{2}\right)\delta_p\left(\frac{t-s}{2}\right)\\
&\qquad\times\left[\partial_x\phi\left(\frac{t+s}{2},\frac{x+y}{2},\right)\left(f(t,x,\tilde\rho)-f(s,y,\tilde\rho)\right)\right.\\
&\qquad\left.\times\partial_y\phi\left(\frac{t+s}{2},\frac{x+y}{2}\right)\left(f(t,x,\rho)-f(s,y,\rho)\right)\right],\\
I_5 &= \left(F(x,\rho(t,x),\tilde\rho(s,y))-F(y,\rho(t,x),\tilde\rho(s,y))\right)\phi\left(\frac{t+s}{2},\frac{x+y}{2}\right)\\
&\qquad\times\partial_x\delta_p\left(\frac{x-y}{2}\right)\delta_p\left(\frac{t-s}{2}\right),
\end{aligned}
$$

where $F(x,\rho,c) := \operatorname{sgn}\left((\rho-c)\right)\left(f(t,x,\rho)-f(t,x,c)\right)$. We now use the change of variables

$$
\tilde x=\frac{x+y}{2},\quad \tilde t=\frac{t+s}{2},\quad z=\frac{x-y}{2},\quad \tau=\frac{t-s}{2},
$$

which maps $(\mathbb{R}^+\times\mathbb{R})^2$ in $\Omega\subset\mathbb{R}^4$ and $(\mathbb{R}^+\times\mathbb{R}\setminus\{0\})^2$ in $\Omega_0\subset\mathbb{R}^4$, where

$$
\Omega=\{(\tilde x,\tilde t,z,\tau)\in\mathbb{R}^4:\ 0<\tilde t\pm\tau<T\},\qquad \Omega_0=\{(\tilde x,\tilde t,z,\tau)\in\Omega:\tilde x\pm z\neq 0\},
$$

respectively. With this changes of variables we can rewrite

$$
\partial_{t+s}\phi\left(\tfrac{t+s}{2},\tfrac{x+y}{2}\right)=\partial_{\tilde t}\phi(\tilde t,\tilde x),\qquad \partial_{x+y}\phi\left(\tfrac{t+s}{2},\tfrac{x+y}{2}\right)=\partial_{\tilde x}\phi(\tilde t,\tilde x).
$$

Now we can write

$$
\begin{aligned}
&-\iiiint_\Omega\left(I_1(\tilde t,\tilde x,\tau,z)+I_2(\tilde t,\tilde x,\tau,z)\right)\delta_p(z)\,\delta_p(\tau)\,\mathrm{d}\tilde t\,\mathrm{d}\tilde x\,\mathrm{d}\tau\,\mathrm{d}z\\
&\le\iiiint_{\Omega_0}\left(I_3(\tilde t,\tilde x,\tau,z)+I_4(\tilde t,\tilde x,\tau,z)+I_5(\tilde t,\tilde x,\tau,z)\right)\mathrm{d}\tilde t\,\mathrm{d}\tilde x\,\mathrm{d}\tau\,\mathrm{d}z,
\end{aligned}
$$

where

$$
\begin{aligned}
I_1(\tilde t,\tilde x,\tau,z) &= \left|\rho(\tilde t+\tau,\tilde x+z)-\tilde\rho(\tilde t-\tau,\tilde x-z)\right|\partial_{\tilde t}\phi\left(\tilde t,\tilde x\right),\\
I_2(\tilde t,\tilde x,\tau,z) &= \operatorname{sgn}\left(\rho(\tilde t+\tau,\tilde x+z)-\tilde\rho(\tilde t-\tau,\tilde x-z)\right)\\
&\quad\times\left(f(\tilde t+\tau,\tilde x+z,\rho)-f(\tilde t-\tau,\tilde x-z,\tilde\rho)\right)\partial_{\tilde x}\phi\left(\tilde t,\tilde x\right),\\
I_3(\tilde t,\tilde x,\tau,z) &= -\operatorname{sgn}\left(\rho(\tilde t+\tau,\tilde x+z)-\tilde\rho(\tilde t-\tau,\tilde x-z)\right)\\
&\quad\times\left(\partial_{\tilde x+z}f(\tilde t+\tau,\tilde x+z,\tilde\rho)-\partial_{\tilde x-z}f(\tilde t-\tau,\tilde x-z,\rho)\right)\phi\left(\tilde t,\tilde x\right)\delta_p(z)\,\delta_p(\tau),
\end{aligned}
$$

$$\begin{aligned} I_4(\tilde t,\tilde x,\tau,z) &= -\mathrm{sgn}\left(\rho(\tilde t+\tau,\tilde x+z)-\tilde\rho(\tilde t-\tau,\tilde x-z)\right) \\ &\quad \times\partial_{\tilde x}\phi\left(\tilde t,\tilde x\right)\delta_p\left(z\right)\delta_p\left(\tau\right)\left[\left(f(\tilde t+\tau,\tilde x+z,\tilde\rho)-f(\tilde t-\tau,\tilde x-z,\tilde\rho)\right)\right. \\ &\qquad \left.+\left(f(\tilde t+\tau,\tilde x+z,\rho)-f(\tilde t-\tau,\tilde x-z,\rho)\right)\right], \\ I_5(\tilde t,\tilde x,\tau,z) &= \big(F(\tilde x+z,\rho(\tilde t+\tau,\tilde x+z),\tilde\rho(\tilde t-\tau,\tilde x-z)) \\ &\quad -F(\tilde x-z,\rho(\tilde t+\tau,\tilde x+z),\tilde\rho(\tilde t-\tau,\tilde x-z))\big)\,\phi\left(\tilde t,\tilde x\right)\partial_z\delta_p\left(z\right)\delta_p\left(\tau\right). \end{aligned}$$

Employing Lebesgue's differentiation theorem, to obtain the following limits

$$\begin{aligned} &\lim_{p\to 0}\iiiint_{\Omega} I_1(\tilde t,\tilde x,\tau,z)\delta_p(z)\delta_p(\tau)\,\mathrm{d}\tilde t\,\mathrm{d}\tilde x\,\mathrm{d}\tau\,\mathrm{d}z \\ &\qquad = \iint_{\mathbb{R}^+\times\mathbb{R}} |\rho(t,x)-\tilde\rho(t,x)|\partial_t\phi(t,x)\mathrm{d}t\mathrm{d}x, \\ &\lim_{p\to 0}\iiiint_{\Omega} I_2(\tilde t,\tilde x,\tau,z)\delta_p(z)\delta_p(\tau)\,\mathrm{d}\tilde t\,\mathrm{d}\tilde x\,\mathrm{d}\tau\,\mathrm{d}z \\ &\qquad = \iint_{\mathbb{R}^+\times\mathbb{R}} \mathrm{sgn}\left(\rho(t,x)-\tilde\rho(t,x)\right)(f(t,x,\rho)-f(t,x,\tilde\rho))\,\partial_x\phi(t,x)\mathrm{d}t\mathrm{d}x. \end{aligned}$$

Let us consider the term I_3. Note that $I_3(\tilde t,\tilde x,\tau,z)=0$, if $\tilde x\in[-h,h]$, since then $\phi(\tilde t,\tilde x)=0$ for any $\tilde t$, or if $|z|\geq p$. On the other hand, if $\tilde x\notin[-h,h]$, then $\tilde x\pm z<0$ or $\tilde x\pm z>0$, at least when $|z|<p$ and $p<h$. Defining $U(t,x)=1-\omega_\eta*\rho$ and $V(t,x)=1-\omega_\eta*\tilde\rho$, and sending $p\to 0$:

$$\begin{aligned} &\lim_{p\to 0}\iiiint_{\Omega_0} I_3(\tilde t,\tilde x,\tau,z)\,\mathrm{d}\tilde t\,\mathrm{d}\tilde x\,\mathrm{d}\tau\,\mathrm{d}z \\ &= \iint_{\mathbb{R}^+\times\mathbb{R}\backslash\{0\}} \mathrm{sgn}\left(\rho(t,x)-\tilde\rho(t,x)\right)\mathfrak{v}(x)\left(\tilde\rho g(\tilde\rho)\partial_x V-\rho g(\rho)\partial_x U\right)\phi\left(t,x\right)\,\mathrm{d}t\,\mathrm{d}x \\ &\leq k_r\|\partial_x V\|\|g'\|\iint_{\mathbb{R}^+\times\mathbb{R}\backslash\{0\}}|\rho-\tilde\rho|\phi(t,x)\,\mathrm{d}t\,\mathrm{d}x \\ &\quad +k_r\iint_{\mathbb{R}^+\times\mathbb{R}\backslash\{0\}}|\rho g(\rho)||\partial_x V-\partial_x U|\,\mathrm{d}t\,\mathrm{d}x \\ &\leq K_1\iint_{\mathbb{R}^+\times\mathbb{R}\backslash\{0\}}|\rho-\tilde\rho|\phi(t,x)\,\mathrm{d}t\,\mathrm{d}x, \end{aligned}$$

where

$$\mathfrak{v}(x)=\begin{cases} k_l, & \text{if } x<0,\\ k_r, & \text{if } x>0.\end{cases}$$

In fact,

$$\begin{aligned} |\partial_x V-\partial_x U| &\leq \left\|\omega_\eta'\right\|\,\|u(t,\cdot)-v(t,\cdot)\|_{L^1} \\ &\quad +\omega_\eta(0)\left(|u-v|(t,x+\eta)+|u-v|(t,x)\right). \end{aligned}$$

The term I_4 converges to zero as $p \to 0$. Finally, the term I_5

$$\lim_{p\to 0} \iiiint_{\Omega_0} I_5(\tilde{t},\tilde{x},\tau,z)\,\mathrm{d}\tilde{t}\,\mathrm{d}\tilde{x}\,\mathrm{d}\tau\,\mathrm{d}z \le K_2 \iint_{\mathbb{R}^+\times\mathbb{R}\setminus\{0\}} |\rho-\tilde{\rho}|\phi(t,x)\,\mathrm{d}t\,\mathrm{d}x. \tag{3.3.17}$$

□

3.4 Numerical examples

In this section, we propose some numerical tests in order to illustrate the dynamics of the non-local model (3.1.1) with flux function discontinuous at $x = 0$ and compare it with the local case. We solve the equation (3.1.1) in an interval containing $x = 0$ using the numerical scheme described in subsection (3.2) for different values of Δx. For each integration, we set Δt such that it satisfies the CFL condition (3.3.1), and for all tests we choose $\omega(x) = \frac{2(x-\eta)}{\eta^2}$ for $0 \le x \le \eta$ and absorbing boundary conditions. The reference solution is computed with $\Delta x = 1/1280$.

3.4.1 Example 5.

We consider the initial condition

$$\rho_0(x) = \begin{cases} 0.9 & x \in [-0.5, 1.5] \\ 0.1 & \text{otherwise,} \end{cases}$$

$\psi(\rho) = 1-\rho$, which satisfies the hypothesis (3.1.4) and $\eta = 0.4$. In **Case I** we take $k_l = 3$ and $k_r = 1$, i.e. $v_l(\rho) > v_r(\rho)$. In Fig 3.4.1(Left) we display the approximated solution for $\Delta x = 1/320$ at different final times $T = 0.5, 1.0, 1.5, 2.0$. We can observe the formation of a stationary shock wave at $x = 0$ and a queue travelling backward. We observe that the solution satisfies the maximum principle according with Lemma 3.3. In **Case II** we take $k_l = 1$ and $k_r = 3$, i.e. $v_l(\rho) < v_r(\rho)$. In Fig 3.4.1(Right) we display the numerical solution for $\Delta x = 1/320$ at different final times $T = 1.0, 2.0$. We can observe the formation of a rarefaction wave at the right of $x = 0$ and the density diminishes at the left of $x = 0$. The $\mathbf{L}^1$-error for different Δx at $T = 2$ are computed in Table 3.4.1.

3.4.2 Example 6: Limit $\eta \to 0^+$.

In this example, we investigate the numerical convergence of the approximate solution computed with the numerical scheme (3.2.1)-(3.2.2) to the solution of the local conservation law with discontinuous flux under hypothesis (3.1.4), as the support of the kernel function ω_η tends to 0^+. In particular, we show numerical solutions at final time $T = 2$, with $\Delta x = 1/1600$ and $\eta = \{0.1, 0.02, 0.005\}$. To evaluate the convergence, we compute the $\mathbf{L}^1$ distance between the approximate solution of the non-local problem with

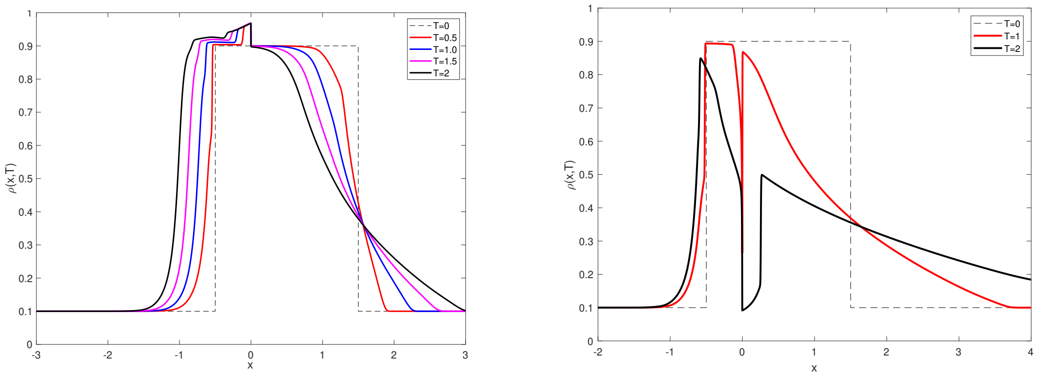

Figure 3.4.1: Example 5: Dynamics of model (3.1.1) (Left) Case $v_l(\rho) > v_r(\rho)$, (Right) Case $v_l(\rho) < v_r(\rho)$

	Cases I		Cases II	
Δx	$\mathbf{L}^1$-error	E.O.A.	$\mathbf{L}^1$-error	E.O.A.
$\frac{1}{40}$	$5.7e^{-2}$	–	$9.8e^{-2}$	–
$\frac{1}{80}$	$2.8e^{-2}$	1.0	$5.0e^{-2}$	1.0
$\frac{1}{160}$	$1.4e^{-2}$	1.0	$2.3e^{-2}$	1.1
$\frac{1}{320}$	$6.5e^{-3}$	1.1	$1.1e^{-2}$	1.1

Table 3.4.1: Example 5. $\mathbf{L}^1$-error and Experimental Order of Accuracy at time $T = 2$.

a given η and the results of the classical Godunov scheme for the corresponding local problem. In Table 3.4.2, we can observe than the $\mathbf{L}^1$ distance goes to zero when $\eta \to 0^+$. The results are illustrated in Fig 3.4.2.

	$\mathbf{L}^1$ distance		
η	0.1	0.02	0.005
Case I	7.4e-2	2.2e-2	6.3e-3
Case II	8.4e-2	2.8e-2	7.8e-3

Table 3.4.2: Example 6. $\mathbf{L}^1$ distance between the approximate solutions to the non-local problem and the local problem for different values of η at $T = 2$ with $\Delta x = 1/1600$.

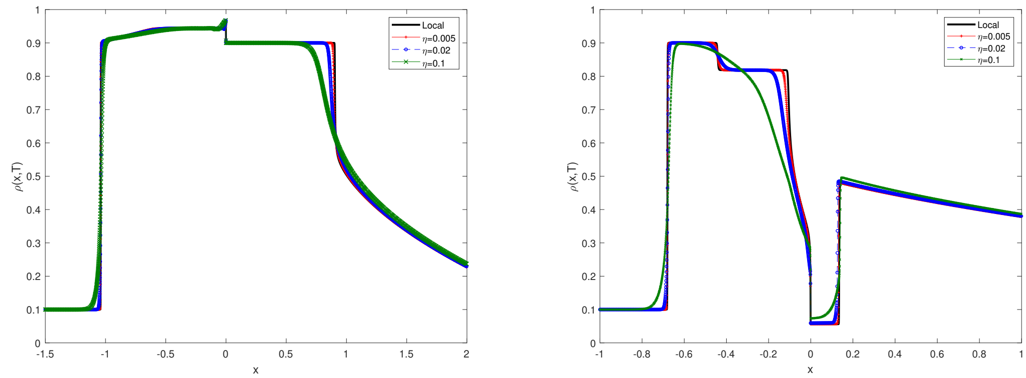

Figure 3.4.2: Example 6. Limit $\eta \to 0^+$, numerical approximations at final time $T = 0.7$ with $\Delta x = 1/3200$. (Left) Case I, (Right) Case II.

3.5 Conclusions of Chapter 3

In this chapter, we have studied a non-local conservation law whose flux function is of the form $H(-x)\rho g(\rho)v_l(\omega_\eta * \rho) + H(x)\rho g(\rho)v_r(\omega_\eta * \rho)$, with a single spatial discontinuity at $x = 0$ and the velocity functions satisfy the hypothesis (3.1.4). We have approximated the problem through a numerical scheme which takes advantage of the form in which the flow is written, and we have provided $\mathbf{L}^\infty$ and $\mathbf{BV}$ estimates for the approximate solutions. Thanks to these estimates, we have proved the well-posedness, i.e., existence and uniqueness of a weak entropy solutions. Numerical simulations illustrate the dynamics of the studied model and corroborate the convergence of the numerical scheme. The limit model as the kernel support tends to zero is numerically investigated.

Chapter 4

Nonlocal reaction traffic flow model with on-off ramps

4.1 Introduction

4.1.1 Scope

Models of conservation laws with nonlocal flux have been used to describe traffic flow dynamics in which drivers adapt their velocity with respect to what happens in front of them [14, 42, 51, 44, 79]. In this type of models, the flux depends on a downstream convolution term between the density or the velocity of vehicles and a kernel function with support on the negative axis. On the other hand, ramps are an important element of traffic systems and develops some complex traffic phenomena, see [53, 57, 67, 80, 81, 82, 84], therefore is important to study the behaviour of traffic on roads with ramps. In this chapter, we propose a new nonlocal traffic model which includes the effects of the inflow and output flow over the on- and off- ramps respectively. We start by considering a modified local reaction traffic model proposed in [67],

$$\rho_t + (\rho v(\rho))_x = S_{\text{on}} - S_{\text{off}}, \tag{4.1.1}$$

where the non-negative functions S_{on} and S_{off} are the source and sink term, respectively, defined by

$$S_{\text{on}}(t,x,\rho) = \mathbf{1}_{\text{on}}(x) q_{\text{on}}(t) \left(1 - \frac{\rho}{\rho_{\max}}\right), \tag{4.1.2}$$

$$S_{\text{off}}(t,x,\rho) = \mathbf{1}_{\text{off}}(x) q_{\text{off}}(t) \frac{\rho}{\rho_{\max}}, \tag{4.1.3}$$

where $q_{\text{on}} \in \mathbb{R}^+$, and $q_{\text{off}} \in \mathbb{R}^+$ the rate (number of vehicles per unit time per unit length) of the on- and off-ramp respectively, as in [81, 82]

$$q_{\text{on}}(t) = \frac{q_{\text{on}}^{\text{ramp}}(t)}{L_{\text{on}}}, \quad q_{\text{off}}(t) = \frac{q_{\text{off}}^{\text{ramp}}(t)}{L_{\text{off}}},$$

with $q_{\text{on}}^{\text{ramp}}(t)$ the expected inflow flow of the on-ramp and $q_{\text{off}}^{\text{ramp}}(t)$ the expected output flow of the off-ramp, L_{on} and L_{off} are the lengths of the on- and off-ramps respectively, whose spatial position are described by the indicator functions $\mathbf{1}_{\text{on}}(x)$ and $\mathbf{1}_{\text{off}}(x)$, defined as

$$\mathbf{1}_{\text{on}}(x) = \begin{cases} 1 & x \in \Omega_{\text{on}} := [\underline{x}_{\text{on}}, \overline{x}_{\text{on}}], \\ 0 & \text{otherwise}, \end{cases} \qquad \mathbf{1}_{\text{off}}(x) = \begin{cases} 1 & x \in \Omega_{\text{off}} := [\underline{x}_{\text{off}}, \overline{x}_{\text{off}}], \\ 0 & \text{otherwise}. \end{cases}$$

For simplicity we consider $L_{\text{on}} = L_{\text{off}} = L$ in the whole paper.

In order to obtain a nonlocal version of the model (4.1.1), we first rewrite the flux function $f(\rho) = \rho v(\rho)$ in its non-local version, where drivers react adapting their velocity with respect to what happens in front of them, see [6, 14, 42, 51],

$$f(\rho) = \rho v(\rho * \omega_\eta), \quad \text{with} \quad (\rho * \omega_\eta)(t,x) = \int_x^{x+\eta} \rho(t,y)\omega_\eta(y-x)\mathrm{d}y.$$

On the on-ramp the idea is that at position x the flow merging in the traffic way is inversely proportional to the average density around position $x + \delta$, see Fig. 4.1.1 , i.e, we write

$$S_{\text{on}}(t,x,\rho,\rho * \omega_{\eta,\delta}) = \mathbf{1}_{\text{on}}(x) q_{\text{on}}(t) \left(1 - \frac{\rho * \omega_{\eta,\delta}}{\rho_{\max}}\right), \tag{4.1.4}$$

where

$$(\rho * \omega_{\eta,\delta})(t,x) = \int_{x-\eta+\delta}^{x+\eta+\delta} \rho(t,y)\omega_{\eta,\delta}(y-x)\mathrm{d}y,$$

with $\eta \in [0,1]$ and $\delta \in [-\eta, \eta]$. Similarly to [42], here the parameter η represents the radius of the support of the kernel function $\omega_{\eta,\delta}$, while δ is the point at which the maximum is attained. This choice of the kernel models the fact that drivers on the on-ramp can see what happens on the backward and forward on the main road. However, in the numerical test section we will see that the choice of the non-local term (4.1.4) does not guarantee that the proposed model satisfies a Maximum Principle, see Example 3. In order to overcome this difficulty, we consider a first variant of (4.1.4) taking

$$S_{\text{on}}(t,x,\rho,\rho * \omega_{\eta,\delta}) = \mathbf{1}_{\text{on}}(x) q_{\text{on}}(t) \left(1 - \frac{\rho}{\rho_{\max}}\right)\left(1 - \frac{\rho * \omega_{\eta,\delta}}{\rho_{\max}}\right). \tag{4.1.5}$$

Note that this term contains a product which differentiates it from the original model, this choice is also assumed in the multilane model studied in [43]. An alternative to avoid the double product in the previous equation (4.1.5) is the following

$$S_{\text{on}}(t,x,\rho,\rho * \omega_{\eta,\delta}) = \mathbf{1}_{\text{on}}(x) q_{\text{on}}(t) \left(1 - \max\left\{\frac{\rho}{\rho_{\max}}; \frac{\rho * \omega_{\eta,\delta}}{\rho_{\max}}\right\}\right). \tag{4.1.6}$$

In both models with (4.1.5) and (4.1.6), if the main road is crowded only few vehicles can enter to the main road.

The purpose of this work is the study of the well-posedness of a nonlocal reaction traffic flow model with source term given by (4.1.5) and (4.1.6).

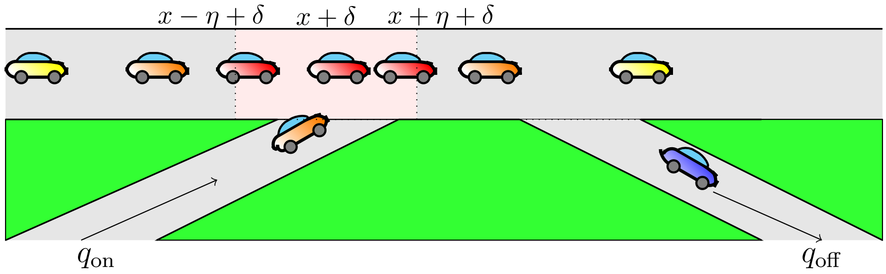

Figure 4.1.1: Illustration of the model setting. Vehicles on the on ramp are located at point $x+\delta$, and by means of the kernel function $\omega_{\eta,\delta}$ they can see to right and left on the main road.

4.1.2 Related work

In [11, 14, 21, 42, 26, 51, 44] the authors studied a nonlocal conservation law to model vehicular traffic flow in the case $S_{\text{on}} = S_{\text{off}} = 0$, i.e., without on- and off-ramps. The need to design more realistic models has led to the development of multi-lane vehicular traffic models among which we can highlight the following. In [56] is introduced a new local model for multilane dense vehicular traffic by means of a system of a weakly coupled scalar conservation laws. In [49], the authors consider the model proposed in [56] but with a more general source terms and they allow for the presence of space discontinuities both in the speed law and in the number of lanes; in these two local models the source term accounts for the lane change rate and the key assumption is that the drivers prefer to drive faster, and that the tendency of a vehicle to change the lanes is proportional to the difference in velocity between neighboring lanes. In [43] the authors introduce a multilane model with local and non-local flux combined with a source term that also incorporates a nonlocality; here, the non-local source term describes the lane changing rate depending on a (nonlinear) evaluation of the velocity. In particular, the lane changing rate is proportional to the difference in the velocity between two adjacent lanes, but the velocities are evaluated in a neighbourhood of the current position, moreover, this rate is proportional also to the density in the receiving lane, meaning that if that lane is crowded only a few vehicles can actually change lane.

Regarding to vehicular traffic flow models taking into account the presence of ramps we can mention [67], where the authors study the (local) first order nonlinear conservation law (4.1.1). In [82] a (local) second order model is proposed to study the effects of on- and off-ramps on a main road traffic during two rush periods. Likewise, other works about the study of effects of ramps in vehicular traffic flow models are referenced in [82]. In particular, in [40] the authors consider a Lighthill-Whitham-Richards (LWR) traffic flow model on a junction composed by one mainline, an on-ramp and an off-ramp, which are connected by a node. Moreover, in [54] a non-local gas-kinetic traffic model including ramps is proposed, the model allows to simulate synchronized congested traffic and reproduces realistic phenomena of vehicular traffic by variations of the on-ramp flow. In [68] a new modeling methodology for merging and diverging flows is studied, the methodology includes coupling effects between main and ramps flows and a new formulation for the modeling of traffic friction is also introduced.

4.1.3 Outline of the chapter

This work is organized as follows: In Section 4.2 we present the proposed mathematical model with all the considered assumptions on it. Afterwards, we introduce an upwind-type scheme with two different source terms and derive important properties such as maximum principle, $\mathbf{L^1}-$ bound and $\mathbf{BV}$ estimates. Furthermore, we derive the $\mathbf{L^1}-$Lipschitz continuous dependence of solutions to (4.2.1) on the initial data and the terms q_{on} and q_{off} in Section 4.3. In Section 4.4, we present numerical examples illustrating the behavior of the solutions of our model.

4.2 Mathematical model

The main goal of this work is to study the well-posedness of the non-local reaction traffic model

$$\rho_t + (\rho v(\rho * \omega_\eta))_x = S_{\text{on}}(\cdot,\cdot,\rho,\rho * \omega_{\eta,\delta}) - S_{\text{off}}(\cdot,\cdot,\rho), \quad x \in \mathbb{R}, \tag{4.2.1}$$

where $S_{\text{on}}(\cdot,\cdot,\rho,\rho * \omega_{\eta,\delta})$ defined in (4.1.5) or (4.1.6), S_{off} defined by (4.1.3) and initial condition

$$\rho(x,0) = \rho_0(x) \in \left(\mathbf{L^1} \cap \mathbf{BV}\right)(\mathbb{R},[0,\rho_{\max}]). \tag{4.2.2}$$

From now on we call Model 0 the equations (4.2.1)-(4.1.4)-(4.2.2), Model 1 the equations (4.2.1)-(4.1.5)-(4.2.2), and Model 2 (4.2.1)-(4.1.6)-(4.2.2). Let us assume the following assumptions.

Assumptions 4.2.1. *We assume*

(i) $q_{\text{on}}^{\text{ramp}} \in \mathbf{L}^\infty(\mathbb{R}^+;\mathbb{R}^+)$, $q_{\text{off}}^{\text{ramp}} \in \mathbf{L}^\infty(\mathbb{R}^+;\mathbb{R}^+)$.

(ii) $v \in \mathbf{C}^2([0,\rho_{\max}];\mathbb{R}^+)$, $v'(\rho) \leq 0$, $\rho \in [0,\rho_{\max}]$.

(iii) $\omega_\eta \in \mathbf{C}^1([0,\eta];\mathbb{R}^+)$ with $\omega_\eta'(x) \leq 0$, $\int_0^\eta \omega_\eta(x)\mathrm{d}x = 1$, $\forall \eta > 0$.

(iv) $\omega_{\eta,\delta} \in (\mathbf{C}^1 \cap \mathbf{L^1})([\delta-\eta,\delta+\eta];\mathbb{R}^+)$ with $\omega'(x)_{\eta,\delta} \geq 0$ *for* $x \in [\delta-\eta,0]$, $\omega'(x)_{\eta,\delta} \leq 0$ *for* $x \in [0,\delta+\eta]$, *and* $\int_{\delta-\eta}^{\delta+\eta} \omega_{\eta,\delta}(x)\mathrm{d}x = 1$, $\forall \eta > 0$.

We recall the definition of weak entropy solution for (4.2.1).

Definition 4.1. *Let* $\rho_0 \in (\mathbf{L^1} \cap \mathbf{BV})(\mathbb{R}; [0, \rho_{\max}])$*. We say that* $\rho \in \mathbf{C}([0,T]; \mathbf{L^1}(\mathbb{R}; [0, \rho_{\max}]))$*, with* $\rho(t, \cdot) \in \mathbf{BV}(\mathbb{R}; [0, \rho_{\max}])$ *for* $t \in [0,T]$*, is a weak solution to* (4.2.1) *with initial datum* ρ_0 *if for any* $\varphi \in \mathbf{C}^1_c([0,T[\times\mathbb{R};\mathbb{R})$

$$\int_0^T \int_{\mathbb{R}} (\rho\varphi_t + \rho V \varphi_x)\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\Omega_{\text{on}}} S_{\text{on}}\varphi\,\mathrm{d}x\,\mathrm{d}t \\ - \int_0^T \int_{\Omega_{\text{off}}} S_{\text{off}}\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}} \rho_0(x)\varphi(0,x)\,\mathrm{d}x = 0,$$

where $V(t,x) = v((\rho * \omega)(t,x))$ *and* S_{on} *is as in* (4.1.5) *or* (4.1.6)*.*

Definition 4.2. *Let* $\rho_0 \in (\mathbf{L^1} \cap \mathbf{BV})(\mathbb{R}; [0, \rho_{\max}])$*. We say that* $\rho \in \mathbf{C}([0,T]; \mathbf{L^1}(\mathbb{R}; [0, \rho_{\max}]))$*, with* $\rho(t, \cdot) \in \mathbf{BV}(\mathbb{R}; [0, \rho_{\max}])$ *for* $t \in [0,T]$*, is a entropy weak solution to* (4.2.1) *with initial datum* ρ_0 *if for any* $\varphi \in \mathbf{C}^1_c([0,T[\times\mathbb{R};\mathbb{R})$ *and for all* $k \in \mathbb{R}$

$$\int_0^T \int_{\mathbb{R}} (|\rho - k|\varphi_t + |\rho - k|\, V\varphi_x - \text{sgn}(\rho - k)kV_x\varphi)\,\mathrm{d}x\,\mathrm{d}t \\ + \int_0^T \int_{\Omega_{\text{on}}} \text{sgn}(\rho - k)S_{\text{on}}\varphi \mathrm{d}x\,\mathrm{d}t - \int_0^T \int_{\Omega_{\text{off}}} \text{sgn}(\rho - k)S_{\text{off}}\varphi \mathrm{d}x\,\mathrm{d}t \\ + \int_{\mathbb{R}} |\rho_0 - k|\varphi(0,x)\mathrm{d}x \geq 0.$$

Our main result is given by the following theorem, which states the well-posedness of problem (4.2.1) to (4.2.2) with source term given by (4.1.5) or (4.1.6). In order to simplify the computations we consider $\rho_{\max} = 1$ from now on.

Theorem 4.3. *Let* $\rho_0 \in \left(\mathbf{L^1} \cap \mathbf{BV}\right)(\mathbb{R}; [0,1])$*. Let Assumptions 4.2.1 hold. Then, for all* $T > 0$*, the problem* (4.2.1) *has a unique solution* $\rho \in \mathbf{C}^0\left([0,T]; \mathbf{L^1}(\mathbb{R}; [0,1])\right)$ *in the sense of Definition 4.2. Moreover, the following estimates hold: for any* $t \in [0,T]$

$$\begin{aligned} &\|\rho(t)\|_{\mathbf{L^1}(\mathbb{R})} \leq \mathcal{R}_1(t), \\ &0 \leq \rho(t,x) \leq 1, \\ &TV(\rho(t)) \leq e^{t\mathcal{H}}\left(TV(\rho_0) + t\mathcal{Q}_T\right), \end{aligned}$$

where

$$\begin{aligned} \mathcal{R}_1 = \|\rho_0\|_{\mathbf{L^1}(\mathbb{R})} + \|q_{\text{on}}^{\text{ramp}}(\cdot)\|_{\mathbf{L^1}([0,t])} - \min_{x\in\Omega_{\text{on}}} \|q_{\text{on}}^{\text{ramp}}(\cdot)\rho(\cdot,x)\|_{\mathbf{L^1}([0,t])} \\ - \min_{x\in\Omega_{\text{off}}} \left\|q_{\text{off}}^{\text{ramp}}(\cdot)\rho(\cdot,x)\right\|_{\mathbf{L^1}([0,t])}, \end{aligned} \tag{4.2.3}$$

$$\mathcal{Q}_T = 2\left(\|q_{\text{on}}\|_{\mathbf{L}^\infty([0,T])} + \|q_{\text{off}}\|_{\mathbf{L}^\infty([0,T])}\right), \tag{4.2.4}$$

$$\mathcal{H} = 2\,\|q_{\text{on}}\|_{\mathbf{L}^\infty([0,T])} + \|q_{\text{off}}\|_{\mathbf{L}^\infty([0,T])} + \omega_\eta(0)\mathcal{L}, \tag{4.2.5}$$

$$\mathcal{L} = \left(\|v\|_{\mathbf{L}^\infty([0,1])} + \|v'\|_{\mathbf{L}^\infty([0,1])}\right). \tag{4.2.6}$$

4.3 Existence of an entropy solution

4.3.1 Numerical discretization

We take a space step Δx such that $\eta = N\Delta x$, for some $N \in \mathbb{N}$, and a time step Δt subject to a CFL condition which will be specified later. For any $j \in \mathbb{Z}$, let $x_{j-1/2} = j\Delta x$ be a cells interfaces, $x_j = \left(j + \frac{1}{2}\right)\Delta x$ the cells centers. We consider ramps with length L and take $L = \ell \Delta x$, for some $\ell \in \mathbb{Z}^+$ such that $\underline{x}_{\rm on} = x_{\underline{k}_{\rm on}+1/2}$, $\overline{x}_{\rm on} = x_{\underline{k}_{\rm on}+1/2+\ell}$, $\underline{x}_{\rm off} = x_{\underline{k}_{\rm off}+1/2}$ and $\overline{x}_{\rm off} = x_{\underline{k}_{\rm off}+1/2+\ell}$, for some $\underline{k}_{\rm on}, \underline{k}_{\rm off} \in \mathbb{Z}$. With this notation, we define the subdomains $\Omega_{\rm on} = [\underline{x}_{\rm on}, \overline{x}_{\rm on}]$, $\Omega_{\rm off} = [\underline{x}_{\rm off}, \overline{x}_{\rm off}]$, and we put $\Omega^k_{\rm on} = [\underline{k}_{\rm on} + 1, \underline{k}_{\rm on} + \ell]$ and $\Omega^k_{\rm off} = [\underline{k}_{\rm off} + 1, \underline{k}_{\rm off} + \ell]$.
We fix $T > 0$, and set $N_T \in \mathbb{N}$ such that $N_T \Delta t \le T < (N_T + 1)\,\Delta t$ and define the time mesh as $t^n = n\Delta t$ for $n = 0, \dots, N_T$. Set $\lambda = \Delta t/\Delta x$. The initial data is approximated for $j \in \mathbb{Z}$, as follows:

$$\rho^0_j = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} \rho_0(x)\mathrm{d}x.$$

We define a piecewise constant approximate solution $\rho_\Delta(t,x)$ to (4.2.1) as

$$\rho_\Delta(t,x) = \rho^n_j, \quad \text{for } \begin{cases} t \in \left[t^n, t^{n+1}\right[\\ x \in]x_{j-1/2}, x_{j+1/2}], \end{cases} \quad \text{where } \begin{array}{l} n = 0,\dots,N_T - 1, \\ j \in \mathbb{Z}. \end{array} \tag{4.3.1}$$

The $S_{\rm on}$ terms (4.1.5) and (4.1.6) are discretized via

$$S_{\rm on}\left(t^{n+1/2}, x_j, \rho_j^{n+1/2}, R^{n+1/2}_{{\rm on},j}\right) = \mathbf{1}_{{\rm on},j} q^{n+1/2}_{\rm on}(1 - \rho^{n+1/2}_j)(1 - R^{n+1/2}_{{\rm on},j}), \tag{4.3.2}$$

$$S_{\rm on}\left(t^{n+1/2}, x_j, \rho_j^{n+1/2}, R^{n+1/2}_{{\rm on},j}\right) = \mathbf{1}_{{\rm on},j} q^{n+1/2}_{\rm on}\left(1 - \max\left\{\rho^{n+1/2}_j, R^{n+1/2}_{{\rm on},j}\right\}\right). \tag{4.3.3}$$

The $S_{\rm off}$ term is discretizated via

$$S_{\rm off}\left(t^{n+1/2}, x_j, \rho_j^{n+1/2}\right) = \mathbf{1}_{{\rm off},j} q^{n+1/2}_{\rm off}\rho^{n+1/2}_j, \tag{4.3.4}$$

where we denote

$$\mathbf{1}_{{\rm on},j} = \begin{cases} \dfrac{1}{\Delta x}\displaystyle\int_{x_{j-1/2}}^{x_{j+1/2}} \mathbf{1}_{\rm on}(x)\mathrm{d}x, & \underline{x}_{\rm on} \le x_j \le \overline{x}_{\rm on}, \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathbf{1}_{{\rm off},j} = \begin{cases} \dfrac{1}{\Delta x}\displaystyle\int_{x_{j-1/2}}^{x_{j+1/2}} \mathbf{1}_{\rm off}(x)\mathrm{d}x, & \underline{x}_{\rm off} \le x_j \le \overline{x}_{\rm off}, \\ 0 & \text{otherwise.} \end{cases}$$

$$q_{\mathrm{on}}^{n+1/2} = \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} q_{\mathrm{on}}(t)\mathrm{d}t, \qquad q_{\mathrm{off}}^{n+1/2} = \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} q_{\mathrm{off}}(t)\mathrm{d}t,$$

The approximate solution ρ_Δ is obtained via an upwind-type scheme together with operator splitting to account for the reaction term, see **Algorithm 4.3.1**

Algorithm 4.3.1 (Upwind scheme).

Input: approximate solution vector $\{\rho_j^n\}_{j\in\mathbb{Z}}$ for $t = t^n$

do $j \in \mathbb{Z}$

$$\rho_j^{n+1/2} \leftarrow \rho_j^n - \lambda\big(\rho_j^n v(R_{j+1/2}^n) - \rho_{j-1}^n v(R_{j-1/2}^n)\big) \tag{4.3.5}$$

enddo

do $j \in \mathbb{Z}$

$S_{\mathrm{on},j}^{n+1/2} \leftarrow S_{\mathrm{on}}\left(t^{n+1/2}, x_j, \rho_j^{n+1/2}, R_{\mathrm{on},j}^{n+1/2}\right)$, using (4.3.2) or (4.3.3),

$S_{\mathrm{off},j}^{n+1/2} \leftarrow S_{\mathrm{off}}\left(t^{n+1/2}, x_j, \rho_j^{n+1/2}\right)$, using (4.3.4),

$$\rho_j^{n+1} \leftarrow \rho_j^{n+1/2} + \Delta t S_{\mathrm{on},j}^{n+1/2} - \Delta t S_{\mathrm{off},j}^{n+1/2} \tag{4.3.6}$$

enddo

Output: approximate solution vector $\{\rho_j^{n+1}\}_{j\in\mathbb{Z}}$ for $t = t^{n+1} = t^n + \Delta t$.

The terms $R_{j+1/2}^n$, $R_{\mathrm{on},j}^{n+1/2}$ for $j \in \mathbb{Z}$ and $n = 0, \ldots, N_T - 1$ denote the discrete convolution operators in the velocity and source term and they are defined, respectively, by

$$R_{j+1/2}^n = \sum_{p=0}^{\lfloor \eta/\Delta x\rfloor - 1} \gamma_p \rho_{j+p+1}^n,$$

$$R_{\mathrm{on},j}^{n+1/2} = \sum_{h=\lfloor \frac{\delta-\eta}{\Delta x}\rfloor}^{\lfloor \frac{\delta+\eta}{\Delta x}\rfloor - 1} \hat{\gamma}_h \rho_{j+h}^{n+1/2}.$$

Here we denote $\gamma_p = \int_{x_{p-1/2}}^{x_{p+1/2}} \omega_\eta(y-x)\mathrm{d}y$, for $p \in [0, \lfloor \eta/\Delta x\rfloor - 1]$
and $\hat{\gamma}_h = \int_{x_{h-1/2}}^{x_{h+1/2}} \omega_{\eta,\delta}(y-x)\mathrm{d}y$, for $h \in [\lfloor(\delta-\eta)/\Delta x\rfloor, \lfloor(\delta+\eta)/\Delta x\rfloor - 1]$.

Remark 4.4. *If* $0 \le \rho_j^{n+1/2} \le 1$ *for all* $j \in \mathbb{Z}$*, then for all* $n \in \{0, \ldots, N_T - 1\}$, $\left\|R_{\mathrm{on}}^{n+1/2}\right\|_{\mathbf{L}^\infty(\Omega_{\mathrm{on}}^k)} \le 1$. *Indeed, we have that*

$$\left|R_{\mathrm{on},j}^{n+1/2}\right| \le \sum_{h=\lfloor \frac{\delta-\eta}{\Delta x}\rfloor}^{\lfloor \frac{\delta+\eta}{\Delta x}\rfloor - 1} \hat{\gamma}_h \left|\rho_{j+h+1}^{n+1/2}\right| \le \sum_{h=\lfloor \frac{\delta-\eta}{\Delta x}\rfloor}^{\lfloor \frac{\delta+\eta}{\Delta x}\rfloor - 1} \hat{\gamma}_h = 1.$$

Remark 4.5. *The discrete convolution operator* $R_{\mathrm{on},j}^{n+1/2}$ *satisfies*

$$\sum_{j\in\mathbb{Z}} \left| R_{\mathrm{on},j+1}^{n+1/2} - R_{\mathrm{on},j}^{n+1/2} \right| \leq \sum_{j\in\mathbb{Z}} \left| \rho_{j+1}^{n+1/2} - \rho_{j}^{n+1/2} \right|.$$

The proof of this property can be seen in [43, Lemma 3.2.]

4.3.2 Existence of solution Model 1

In order to prove the existence of a solution of the model (4.2.1)-(4.1.5), in the next lemmas we will show some properties of the approximate solutions constructed by the **Algorithm 4.3.1**.

Lemma 4.6 (Maximum principle)**.** *Let* $\rho_0 \in \mathbf{L}^\infty(\mathbb{R};[0,1])$. *Let Assumptions 4.2.1 and the following Courant-Friedrichs-Levy (CFL) condition hold*

$$\Delta t \leq \min\left\{ \frac{\Delta x}{\left(\gamma_0 \|v'\|_{\mathbf{L}^\infty([0,1])} + \|v\|_{\mathbf{L}^\infty([0,1])}\right)}, \frac{1}{\mathcal{Q}_T} \right\} \tag{4.3.7}$$

with $\mathcal{Q}_T$ *defined in* (4.2.4) *then for all* $t > 0$ *and* $x \in \mathbb{R}$ *the piece-wise constant approximate solution* ρ_Δ *constructed through* ***Algorithm 4.3.1*** *is such that*

$$0 \leq \rho_\Delta(t,x) \leq 1.$$

Proof. The proof is made by induction. Let us assume that $0 \leq \rho_j^n \leq 1$ for all $j \in \mathbb{Z}$. Consider the convective step (4.3.5) of **Algorithm 4.3.1**, by CFL condition (4.3.7) we have $0 \leq \rho_j^{n+1/2} \leq 1$ for $j \in \mathbb{Z}$ (see [44, Theorem 3.3]).

Now focus on the remaining step, involving the source term.

$$\begin{aligned}
\rho_j^{n+1} &= \rho_j^{n+1/2} + \Delta t \left(\mathbf{1}_{\mathrm{on},j} q_{\mathrm{on}}^{n+1/2} (1 - \rho_j^{n+1/2}) \left(1 - R_{\mathrm{on},j}^{n+1/2}\right) - \mathbf{1}_{\mathrm{off},j} q_{\mathrm{off}}^{n+1/2} \rho_j^{n+1/2} \right) \\
&\leq \rho_j^{n+1/2} + \Delta t \mathbf{1}_{\mathrm{on},j} q_{\mathrm{on}}^{n+1/2} \left(1 - \rho_j^{n+1/2}\right) - \Delta t \mathbf{1}_{\mathrm{off},j} q_{\mathrm{off}}^{n+1/2} \rho_j^{n+1/2} \\
&= \left(1 - \Delta t \left(\mathbf{1}_{\mathrm{on},j} q_{\mathrm{on}}^{n+1/2} + \mathbf{1}_{\mathrm{off},j} q_{\mathrm{off}}^{n+1/2} \right)\right) \rho_j^{n+1/2} + \Delta t \mathbf{1}_{\mathrm{on},j} q_{\mathrm{on}}^{n+1/2}.
\end{aligned}$$

Because of CFL condition (4.3.7), the last right-hand side is a convex combination of $\rho_j^{n+1/2}$ and one. Then $\rho_j^{n+1} \in \left[\rho_j^{n+1/2}, 1\right]$ and since $\rho_j^{n+1/2} \in [0,1]$, we therefore conclude that $0 \leq \rho_j^{n+1} \leq 1$, for $j \in \mathbb{Z}$.

□

Lemma 4.7 ($\mathbf{L^1}$ – *Bound*)**.** *Let* $\rho_0 \in \mathbf{L^1}(\mathbb{R},[0,1])$. *Let* (4.2.1) *and the CFL condition* (4.3.7) *hold. Then, the piece-wise constant approximate solution* ρ_Δ *constructed through* ***Algorithm 4.3.1*** *satisfies, for all* $T > 0$,

$$\|\rho_\Delta(T,\cdot)\|_{\mathbf{L^1}(\mathbb{R})} \leq \mathcal{C}_1(T),$$

with

$$\begin{aligned}\mathcal{C}_1(t) = \|\rho_0\|_{\mathbf{L^1}(\mathbb{R})} + \|q_{\text{on}}^{\text{ramp}}\|_{\mathbf{L^1}([0,t])} - \min_{x\in\Omega_{\text{on}}} \|q_{\text{on}}^{\text{ramp}}(\cdot)\rho_\Delta(\cdot,x)\|_{\mathbf{L^1}([0,t])} \\ - \min_{x\in\Omega_{\text{off}}} \left\|q_{\text{off}}^{\text{ramp}}(\cdot)\rho_\Delta(\cdot,x)\right\|_{\mathbf{L^1}([0,t])}.\end{aligned} \tag{4.3.8}$$

Proof. For the conservative form of the scheme (4.3.5), it is satisfied

$$\left\|\rho^{n+1/2}\right\|_{\mathbf{L^1}(\mathbb{R})} = \|\rho^n\|_{\mathbf{L^1}(\mathbb{R})}.$$

Now, we going to work $\mathbf{L^1}$ norm for relaxation step (4.3.6). By Remark 4.4 and CFL condition (4.3.7) we have

$$\left|\rho_j^{n+1}\right| \le \left|\rho_j^{n+1/2}\right| + \Delta t \mathbf{1}_{\text{on},j} q_{\text{on}}^{n+1/2}\left(1 - \left|\rho_j^{n+1/2}\right|\right) - \Delta t \mathbf{1}_{\text{off},j} q_{\text{off}}^{n+1/2}\left|\rho_j^{n+1/2}\right|, \tag{4.3.9}$$

multiplying this inequality by Δx and summing over all $j \in \mathbb{Z}$ we obtain

$$\begin{aligned}\left\|\rho^{n+1}\right\|_{\mathbf{L^1}(\mathbb{R})} &\le \left\|\rho^{n+1/2}\right\|_{\mathbf{L^1}(\mathbb{R})} + \Delta t q_{\text{on}}^{n+1/2}\Delta x\left(\sum_{j\in\Omega_{\text{on}}^k} \mathbf{1}_{\text{on},j} - \sum_{j\in\Omega_{\text{on}}^k} \mathbf{1}_{\text{on},j}\left|\rho_j^{n+1/2}\right|\right) \\ &\quad -\Delta t q_{\text{off}}^{n+1/2}\Delta x \sum_{j\in\Omega_{\text{off}}^k} \mathbf{1}_{\text{off},j}\left|\rho_j^{n+1/2}\right| \\ &\le \|\rho^n\|_{\mathbf{L^1}(\mathbb{R})} + \Delta t L q_{\text{on}}^{n+1/2}\left(1 - \min_{j\in\Omega_{\text{on}}^k} \rho_j^{n+1/2}\right) \\ &\quad -\Delta t L q_{\text{off}}^{n+1/2} \min_{j\in\Omega_{\text{off}}^k} \rho_j^{n+1/2} \\ &= \|\rho^n\|_{\mathbf{L^1}(\mathbb{R})} + \Delta t L q_{\text{on}}^{n+1/2} - \Delta t \min_{j\in\Omega_{\text{on}}^k} L q_{\text{on}}^{n+1/2}\rho_j^{n+1/2} \\ &\quad -\Delta t \min_{j\in\Omega_{\text{off}}^k} L q_{\text{off}}^{n+1/2}\rho_j^{n+1/2}.\end{aligned}$$

Thus, by a standard iterative procedure we can deduce

$$\begin{aligned}\|\rho^n\|_{\mathbf{L^1}(\mathbb{R})} &\le \|\rho_0\|_{\mathbf{L^1}(\mathbb{R})} + \|q_{\text{on}}^{\text{ramp}}\|_{\mathbf{L^1}([0,T])} - \min_{x\in\Omega_{\text{on}}} \|q_{\text{on}}^{\text{ramp}}(\cdot)\rho_\Delta(\cdot,x)\|_{\mathbf{L^1}([0,T])} \\ &\quad - \min_{x\in\Omega_{\text{off}}} \left\|q_{\text{off}}^{\text{ramp}}(\cdot)\rho_\Delta(\cdot,x)\right\|_{\mathbf{L^1}([0,T])}.\end{aligned}$$

□

4.3.3 BV estimates

We first prove the Lipschitz continuity of the source terms (4.3.2) in its second, third and fourth argument and (4.3.4) in its second and third argument.

Lemma 4.8. *The map S_{on} defined in* (4.3.2) *is Lipschitz continuous in second, third and fourth argument with Lipschitz constant $\|q_{\text{on}}\|_{\mathbf{L}^\infty([0,T])}$, and the map S_{off} defined in* (4.3.4) *is Lipschitz continuous in second and third argument with Lipschitz constant $\|q_{\text{off}}\|_{\mathbf{L}^\infty([0,T])}$.*

Proof. Let us start with term (4.3.2). We denote $\mathcal{S}_{\text{on}} = S_{\text{on}}(t,x,\rho,R_{\text{on}}) - S_{\text{on}}(t,\tilde{x},\tilde{\rho},\tilde{R}_{\text{on}})$, then

$$\begin{aligned}
|\mathcal{S}_{\text{on}}| &\leq |S_{\text{on}}(t,x,\rho,R_{\text{on}}) - S_{\text{on}}(t,x,\tilde{\rho},R_{\text{on}})| \\
&\quad + \left|S_{\text{on}}(t,x,\tilde{\rho},R_{\text{on}}) - S_{\text{on}}(t,x,\tilde{\rho},\tilde{R}_{\text{on}})\right| \\
&\quad + \left|S_{\text{on}}(t,x,\tilde{\rho},\tilde{R}_{\text{on}}) - S_{\text{on}}(t,\tilde{x},\tilde{\rho},\tilde{R}_{\text{on}})\right| \\
&= |\mathbf{1}_{\text{on}} q_{\text{on}} (1-R_{\text{on}})(\tilde{\rho}-\rho)| + \left|\mathbf{1}_{\text{on}} q_{\text{on}} (1-\tilde{\rho}) \left(\tilde{R}_{\text{on}} - R_{\text{on}}\right)\right| \\
&\quad + \left|\left(\mathbf{1}_{\text{on}} - \tilde{\mathbf{1}}_{\text{on}}\right) q_{\text{on}} (1-\tilde{\rho}) \left(1-\tilde{R}_{\text{on}}\right)\right| \\
&\leq \|q_{\text{on}}\|_{\mathbf{L}^\infty([0,T])} |\tilde{\rho}-\rho| + \|q_{\text{on}}\|_{\mathbf{L}^\infty([0,T])} \left|\tilde{R}_{\text{on}} - R_{\text{on}}\right| \\
&\quad + \|q_{\text{on}}\|_{\mathbf{L}^\infty([0,T])} \left|\mathbf{1}_{\text{on}} - \tilde{\mathbf{1}}_{\text{on}}\right| \\
&\leq \|q_{\text{on}}\|_{\mathbf{L}^\infty([0,T])} \left(|\tilde{\rho}-\rho| + \left|\tilde{R}_{\text{on}} - R_{\text{on}}\right| + \left|\mathbf{1}_{\text{on}} - \tilde{\mathbf{1}}_{\text{on}}\right|\right).
\end{aligned}$$

Now, we prove the Lipschitz continuity of S_{off} term (4.3.4). Denoting $\mathcal{S}_{\text{off}} = S_{\text{off}}(t,x,\rho) - S_{\text{off}}(t,\tilde{x},q_{\text{off}},\tilde{\rho})$, we get

$$\begin{aligned}
|\mathcal{S}_{\text{off}}| &\leq |S_{\text{off}}(t,x,\rho) - S_{\text{off}}(t,\tilde{x},\rho,)| + |S_{\text{off}}(t,\tilde{x},\rho) - S_{\text{off}}(t,\tilde{x},\tilde{\rho})| \\
&= \left|\mathbf{1}_{\text{off}} q_{\text{off}} \rho - \tilde{\mathbf{1}}_{\text{off}} q_{\text{off}} \rho\right| + \left|\tilde{\mathbf{1}}_{\text{off}} q_{\text{off}} \rho - \tilde{\mathbf{1}}_{\text{off}} q_{\text{off}} \tilde{\rho}\right| \\
&\leq \|q_{\text{off}}\|_{\mathbf{L}^\infty([0,T])} \left(\left|\mathbf{1}_{\text{off}} - \tilde{\mathbf{1}}_{\text{off}}\right| + |\rho-\tilde{\rho}|\right),
\end{aligned}$$

Thus, we have completed the proof. □

The Lipschitz continuity of the source term proved in Lemma 4.8 is one of the key ingredients in order to prove the following total variation bound on the numerical approximation.

Proposition 4.8.1 (**BV** estimate in space)**.** *Let $\rho_0 \in \left(\mathbf{L^1} \cap \mathbf{BV}\right)(\mathbb{R};[0,1])$. Let the Assumptions 4.2.1 and CFL condition* (4.3.7) *hold. Then, for $n = 0, \ldots, N_T - 1$ the following estimate holds*

$$\sum_{j\in\mathbb{Z}} \left|\rho_{j+1}^n - \rho_j^n\right| \leq e^{T\mathcal{H}} \left(TV(\rho_0) + T\mathcal{Q}_T\right),$$

with $\mathcal{Q}_T$ as in (4.2.4) *and $\mathcal{H}$ as in* (4.2.5).

Proof. Let us compute

$$
\begin{aligned}
\rho_{j+1}^{n+1} - \rho_j^{n+1} &= \rho_{j+1}^{n+1/2} - \rho_j^{n+1/2} + \Delta t \left[S_{\mathrm{on},j+1}^{n+1/2} - S_{\mathrm{on},j}^{n+1/2} \right] \\
&\quad - \Delta t \left[S_{\mathrm{off},j+1}^{n+1/2} - S_{\mathrm{off},j}^{n+1/2} \right].
\end{aligned}
$$

By the Lipschitz continuity of the source term proved in Lemma 4.8 and the property of the discrete convolution operator given in Remark 4.5, we get

$$
\begin{aligned}
\sum_{j\in\mathbb{Z}} \left| \rho_{j+1}^{n+1} - \rho_j^{n+1} \right| &\leq \left(1 + \Delta t \left\| q_{\mathrm{on}} \right\|_{\mathbf{L}^\infty([0,T])}\right) \sum_{j\in\mathbb{Z}} \left| \rho_{j+1}^{n+1/2} - \rho_j^{n+1/2} \right| \\
&\quad + \Delta t \left\| q_{\mathrm{on}} \right\|_{\mathbf{L}^\infty([0,T])} \sum_{j\in\Omega_{\mathrm{on}}^k} \left| \mathbf{1}_{\mathrm{on},j+1} - \mathbf{1}_{\mathrm{on},j} \right| \\
&\quad + \Delta t \left\| q_{\mathrm{on}} \right\|_{\mathbf{L}^\infty([0,T])} \sum_{j\in\mathbb{Z}} \left| R_{\mathrm{on},j+1}^{n+1/2} - R_{\mathrm{on},j}^{n+1/2} \right| \\
&\quad + \Delta t \left\| q_{\mathrm{off}} \right\|_{\mathbf{L}^\infty([0,T])} \sum_{j\in\mathbb{Z}} \left| \rho_{j+1}^{n+1/2} - \rho_j^{n+1/2} \right| \\
&\quad + \Delta t \left\| q_{\mathrm{off}} \right\|_{\mathbf{L}^\infty([0,T])} \sum_{j\in\Omega_{\mathrm{off}}} \left| \mathbf{1}_{\mathrm{off},j+1} - \mathbf{1}_{\mathrm{off},j} \right| \\
&\leq \left(1 + \Delta t \left(2 \left\| q_{\mathrm{on}} \right\|_{\mathbf{L}^\infty([0,T])} + \left\| q_{\mathrm{off}} \right\|_{\mathbf{L}^\infty([0,T])}\right)\right) \sum_{j\in\mathbb{Z}} \left| \rho_{j+1}^{n+1/2} - \rho_j^{n+1/2} \right| \\
&\quad + \Delta t \left\| q_{\mathrm{on}} \right\|_{\mathbf{L}^\infty([0,T])} \sum_{j\in\Omega_{\mathrm{on}}^k} \left| \mathbf{1}_{\mathrm{on},j+1} - \mathbf{1}_{\mathrm{on},j} \right| \\
&\quad + \Delta t \left\| q_{\mathrm{off}} \right\|_{\mathbf{L}^\infty([0,T])} \sum_{j\in\Omega_{\mathrm{off}}} \left| \mathbf{1}_{\mathrm{off},j+1} - \mathbf{1}_{\mathrm{off},j} \right| \\
&\leq \left(1 + \Delta t \left(2 \left\| q_{\mathrm{on}} \right\|_{\mathbf{L}^\infty([0,T])} + \left\| q_{\mathrm{off}} \right\|_{\mathbf{L}^\infty([0,T])}\right)\right) \sum_{j\in\mathbb{Z}} \left| \rho_{j+1}^{n+1/2} - \rho_j^{n+1/2} \right| \\
&\quad + \Delta t \mathcal{Q}_T. \qquad (4.3.10)
\end{aligned}
$$

Now, for convective part (4.3.5) we follow [44] and get

$$
\left| \rho_{j+1}^{n+1/2} - \rho_j^{n+1/2} \right| \leq (1 + \Delta t \omega_\eta(0) \mathcal{L}) \sum_{j\in\mathbb{Z}} \left| \rho_{j+1}^n - \rho_j^n \right|,
$$

with $\mathcal{L} = \left(\|v\|_{\mathbf{L}^\infty([0,1])} + \|v'\|_{\mathbf{L}^\infty([0,1])} \right)$. Substituting the inequality above in (4.3.10) we obtain

$$
\begin{aligned}
\sum_{j\in\mathbb{Z}} \left| \rho_{j+1}^{n+1} - \rho_j^{n+1} \right| \leq{}& \left(1 + \Delta t \left(2 \left\| q_{\mathrm{on}} \right\|_{\mathbf{L}^\infty([0,T])} + \left\| q_{\mathrm{off}} \right\|_{\mathbf{L}^\infty([0,T])}\right)\right) \\
&\times (1 + \Delta t \omega_\eta(0) \mathcal{L}) \sum_{j\in\mathbb{Z}} \left| \rho_{j+1}^n - \rho_j^n \right| + \Delta t \mathcal{Q}_T,
\end{aligned}
$$

which, applied recursively, yields

$$\sum_{j\in\mathbb{Z}} \left|\rho_{j+1}^n - \rho_j^n\right| \leq \mathrm{e}^{T\mathcal{H}} \left(TV(\rho_0) + T\mathcal{Q}_T\right), \tag{4.3.11}$$

with $\mathcal{H} = 2\left\|q_{\text{on}}\right\|_{\mathbf{L}^\infty([0,T])} + \left\|q_{\text{off}}\right\|_{\mathbf{L}^\infty([0,T])} + \omega_\eta(0)\mathcal{L}$.

□

Proposition 4.8.2 (**BV** estimate in space and time). *Let Assumptions 4.2.1 hold, $\rho_0 \in \left(\mathbf{L^1} \cap \mathbf{BV}\right)(\mathbb{R};[0,1])$. If the CFL condition (4.3.7) holds, then, for every $T>0$ the following discrete space and time total variation estimate is satisfied:*

$$TV(\rho_\Delta;[0,T]\times\mathbb{R}) \quad\leq\quad T\mathcal{C}_{xt}(T),$$

with

$$\mathcal{C}_{xt}(T) = e^{T\mathcal{H}}\left((1+2\mathcal{L})\left(TV(\rho_0)+T\mathcal{Q}_T\right)\right) + \frac{1}{2}\mathcal{Q}_T\mathcal{C}_1(T) + \|q_{\text{on}}^{\text{ramp}}\|_{\mathbf{L}^\infty([0,T])}\,. \tag{4.3.12}$$

Proof.

$$\begin{aligned} TV(\rho_\Delta;[0,T]\times\mathbb{R}) \quad = \quad & \sum_{n=0}^{N_T-1}\sum_{j\in\mathbb{Z}} \Delta t \left|\rho_{j+1}^n - \rho_j^n\right| + (T - N_T\Delta t)\sum_{j\in\mathbb{Z}}\left|\rho_{j+1}^{N_T} - \rho_j^{N_T}\right| \\ & + \sum_{n=0}^{N_T-1}\sum_{j\in\mathbb{Z}} \Delta x \left|\rho_j^{n+1} - \rho_j^n\right|. \end{aligned}$$

By **BV** estimate in space (4.3.11), we have

$$\begin{aligned} & \sum_{n=0}^{N_T-1}\sum_{j\in\mathbb{Z}} \Delta t \left|\rho_{j+1}^n - \rho_j^n\right| + (T - N_T\Delta t)\sum_{j\in\mathbb{Z}}\left|\rho_{j+1}^{N_T} - \rho_j^{N_T}\right| \\ \leq \quad & Te^{T\mathcal{H}}\left(TV(\rho_0) + T\mathcal{Q}_T\right). \end{aligned} \tag{4.3.13}$$

On the other hand, observe that

$$\left|\rho_j^{n+1} - \rho_j^n\right| \leq \left|\rho_j^{n+1} - \rho_j^{n+1/2}\right| + \left|\rho_j^{n+1/2} - \rho_j^n\right|. \tag{4.3.14}$$

We then estimate separately each term on the right hand side of the inequality (4.3.14). By the definition of the relaxation step (4.3.6), for the first term on right hand side of (4.3.14) we have

$$\begin{aligned} \left|\rho_j^{n+1} - \rho_j^{n+1/2}\right| \quad \leq \quad & \Delta t\left|S_{\text{on},j}^{n+1/2} - S_{\text{off},j}^{n+1/2}\right| \\ \leq \quad & \Delta t \mathbf{1}_{\text{on},j} q_{\text{on}}^{n+1/2}\left(1-\rho_j^{n+1/2}\right)\left(1 - R_{\text{on},j}^{n+1/2}\right) + \Delta t \mathbf{1}_{\text{off},j} q_{\text{off}}^{n+1/2}\rho_j^{n+1/2} \end{aligned}$$

$$\begin{aligned}
&\leq \Delta t q_{\text{on}}^{n+1/2}\left(\mathbf{1}_{\text{on},j}+\mathbf{1}_{\text{on},j}\left|\rho_j^{n+1/2}\right|\right)\\
&+\Delta t \mathbf{1}_{\text{off},j} q_{\text{off}}^{n+1/2}\left|\rho_j^{n+1/2}\right|, \qquad (4.3.15)
\end{aligned}$$

then multiplying by Δx and summing over all $j \in \mathbb{Z}$,

$$\begin{aligned}
\Delta x \sum_{j\in\mathbb{Z}}\left|\rho_j^{n+1}-\rho_j^{n+1/2}\right| &\leq \Delta t q_{\text{on}}^{n+1/2}\left(\Delta x \sum_{j\in\Omega_{\text{on}}^k}\mathbf{1}_{\text{on},j}+\Delta x \sum_{j\in\Omega_{\text{on}}^k}\mathbf{1}_{\text{on},j}\left|\rho_j^{n+1/2}\right|\right)\\
&+\Delta t q_{\text{off}}^{n+1/2}\Delta x \sum_{j\in\Omega_{\text{off}}^k}\mathbf{1}_{\text{off},j}\left|\rho_j^{n+1/2}\right|\\
&\leq \Delta t q_{\text{on}}^{n+1/2}\left(L+\left\|\rho^{n+1/2}\right\|_{\mathbf{L^1}(\mathbb{R})}\right)\\
&+\Delta t q_{\text{off}}^{n+1/2}\|\rho^{n+1/2}\|_{\mathbf{L^1}(\mathbb{R})}\\
&= \Delta t q_{\text{on}}^{n+1/2}\left(L+\|\rho^n\|_{\mathbf{L^1}(\mathbb{R})}\right)\\
&+\Delta t \left\|q_{\text{off}}\right\|_{\mathbf{L}^\infty([0,T])}\left\|\rho^n\right\|_{\mathbf{L^1}(\mathbb{R})}\\
&= \frac{1}{2}\Delta t \mathcal{Q}_T\|\rho^n\|_{\mathbf{L^1}(\mathbb{R})}+\Delta t\|q_{\text{on}}^{\text{ramp}}\|_{\mathbf{L}^\infty([0,T])}. \qquad (4.3.16)
\end{aligned}$$

Now we analyze the second term of the right hand side (4.3.14). Since the numerical flux defined in (4.3.5) is Lipschitz continuous in both arguments with Lipschitz constant $\mathcal{L}_2$, defined by (4.2.6), we obtain

$$\begin{aligned}
\left|\rho_j^{n+1/2}-\rho_j^n\right| &= \lambda\left|F_{j+1/2}(\rho_j^n,R_{j+1/2}^n)-F_{j-1/2}(\rho_{j-1}^n,R_{j-1/2}^n)\right|\\
&\leq \lambda\mathcal{L}\left(\left|\rho_j^n-\rho_{j-1}^n\right|+\left|R_{j+1/2}^n-R_{j-1/2}^n\right|\right),
\end{aligned}$$

multiplying by Δx, summing over all $j \in \mathbb{Z}$ and by Remark 4.5 we get

$$\Delta x \sum_{j\in\mathbb{Z}}\left|\rho_j^{n+1/2}-\rho_j^n\right| \leq 2\mathcal{L}\Delta t\sum_{j\in\mathbb{Z}}\left|\rho_{j+1}^n-\rho_j^n\right| \qquad (4.3.17)$$

Collecting together (4.3.16) and (4.3.17), and by using Lemma 4.7 and Proposition 4.8.1 we have,

$$\begin{aligned}
\Delta x \sum_{j\in\mathbb{Z}}\left|\rho_j^{n+1}-\rho_j^n\right| &\leq \frac{1}{2}\Delta t \mathcal{Q}_T\|\rho^n\|_{\mathbf{L^1}(\mathbb{R})}\\
&+\Delta t\left\|q_{\text{on}}^{\text{ramp}}\right\|_{\mathbf{L}^\infty([0,T])}+2\mathcal{L}\Delta t\sum_{j\in\mathbb{Z}}\left|\rho_{j+1}^n-\rho_j^n\right|\\
&\leq \frac{1}{2}\Delta t\mathcal{Q}_T\mathcal{C}_1(T)+\Delta t\left\|q_{\text{on}}^{\text{ramp}}\right\|_{\mathbf{L}^\infty([0,T])}\\
&+2\mathcal{L}\Delta t e^{T\mathcal{H}}\left(TV(\rho_0)+T\mathcal{Q}_T\right). \qquad (4.3.18)
\end{aligned}$$

Then, collecting together (4.3.13) and (4.3.18) we get

$$\begin{aligned} & \sum_{n=0}^{N_T-1} \sum_{j\in\mathbb{Z}} \Delta t \left|\rho_{j+1}^n - \rho_j^n\right| + (T - N_T\Delta t) \sum_{j\in\mathbb{Z}} \left|\rho_{j+1}^{N_T} - \rho_j^{N_T}\right| \\ & + \sum_{n=0}^{N_T-1} \sum_{j\in\mathbb{Z}} \Delta x \left|\rho_j^{n+1} - \rho_j^n\right| \\ \leq \; & T e^{T\mathcal{H}} \left((1+2\mathcal{L})\left(TV(\rho_0) + T\mathcal{Q}_T\right)\right) + \frac{1}{2} T \mathcal{Q}_T \mathcal{C}_1(T) + T \left\|q_{\text{on}}^{\text{ramp}}\right\|_{\mathbf{L}^\infty([0,T])}. \end{aligned}$$

□

4.3.4 Discrete Entropy Inequality

In order to define an entropy inequality we define, for $\kappa \in [0,1]$, and the numerical fluxes

$$G_{j+1/2}(u) = uv(R_{j+1/2}), \quad \mathcal{F}_{j+1/2}^\kappa(u) = G_{j+1/2}(u \vee \kappa) - G_{j+1/2}(u \wedge \kappa),$$

with $a \vee b = \max\{a,b\}$, and $a \wedge b = \min\{a,b\}$.

Lemma 4.9. *Let $\rho_0 \in \left(\mathbf{L^1} \cap \mathbf{BV}\right)(\mathbb{R};[0,1])$. Let the Assumptions 4.2.1 and CFL condition* (4.3.7) *hold. Then, the approximate solution ρ_Δ constructed by **Algorithm 4.3.1** satisfies the following discrete entropy inequality: for $j \in \mathbb{Z}$, for $n = 0, \ldots, N_T - 1$ and for any $\kappa \in [0,1]$,*

$$\begin{aligned} & \left|\rho_j^{n+1} - \kappa\right| - \left|\rho_j^n - \kappa\right| + \lambda\left(\mathcal{F}_{j+1/2}^k\left(\rho_j^n\right) - \mathcal{F}_{j+1/2}^k\left(\rho_{j-1}^n\right)\right) \\ & -\Delta t \operatorname{sgn}\left(\rho_j^{n+1} - \kappa\right)\left(S_{\text{on}}\left(t^{n+1/2}, x_j, \rho_j^{n+1/2}, R_{\text{on},j}^{n+1/2}\right) - S_{\text{off}}\left(t^{n+1/2}, x_j, \rho_j^{n+1/2}\right)\right) \\ & +\lambda \operatorname{sgn}\left(\rho_j^{n+1} - \kappa\right)\kappa\left(v\left(R_{j+1/2}^n\right) - v\left(R_{j-1/2}^n\right)\right) \leq 0. \end{aligned}$$

Proof. We set

$$\begin{aligned} \mathcal{G}_j(u,w) &= w - \lambda\left(G_{j+1/2}(w) - G_{j-1/2}(u)\right) \\ &= w - \lambda\left(wv(R_{j+1/2}) - uv(R_{j-1/2})\right). \end{aligned}$$

Clearly $\rho_j^{n+1/2} = \mathcal{G}_j(\rho_{j-1}^n, \rho_j^n)$.
The map $\mathcal{G}_j$ is a monotone non-decreasing function with respect to each variable under the CFL condition (4.3.7) since we have

$$\frac{\partial \mathcal{G}}{\partial w} = 1 - \lambda v(R_{j+1/2}) \geq 0, \qquad \frac{\partial \mathcal{G}}{\partial u} = \lambda v(R_{j-1/2}).$$

Moreover, we have the following identity

$$\mathcal{G}_j(\rho_{j-1}^n \vee \kappa, \rho_j^n \vee \kappa) - \mathcal{G}_j(\rho_{j-1}^n \wedge \kappa, \rho_j^n \wedge \kappa) = \left|\rho_j^n - \kappa\right| - \lambda\left(\mathcal{F}_{j+1/2}^k\left(\rho_j^n\right) - \mathcal{F}_{j-1/2}^k\left(\rho_{j-1}^n\right)\right).$$

Then, by monotonicity, the definition of scheme (4.3.5) and by using $|a+b| \geq |a| + \mathrm{sgn}(a)b$, we get

$$\begin{aligned}
& \mathcal{G}_j(\rho^n_{j-1} \vee \kappa, \rho^n_j \vee \kappa) - \mathcal{G}_j(\rho^n_{j-1} \wedge \kappa, \rho^n_j \wedge \kappa) \\
\geq\ & \mathcal{G}_j(\rho^n_{j-1}, \rho^n_j) \vee \mathcal{G}_j(\kappa,\kappa) - \mathcal{G}_j(\rho^n_{j-1}, \rho^n_j) \wedge \mathcal{G}_j(\kappa,\kappa) \\
=\ & \left|\mathcal{G}_j(\rho^n_{j-1}, \rho^n_j) - \mathcal{G}_j(\kappa,\kappa)\right| \\
=\ & \left|\rho_j^{n+1/2} - \mathcal{G}_j(\kappa,\kappa)\right| \\
=\ & \Big|\rho_j^{n+1} - \kappa + \lambda\kappa\left(v(R^n_{j+1/2}) - v(R^n_{j-1/2})\right) \\
& -\Delta t\left(S_{\mathrm{on}}\left(t^{n+1/2}, x_j, \rho_j^{n+1/2}, R^{n+1/2}_{\mathrm{on},j}\right) - S_{\mathrm{off}}\left(t^{n+1/2}, x_j, \rho_j^{n+1/2}\right)\right)\Big| \\
\geq\ & \left|\rho_j^{n+1} - \kappa\right| + \lambda\,\mathrm{sgn}\left(\rho_j^{n+1} - \kappa\right)\kappa\left(v(R^n_{j+1/2}) - v(R^n_{j-1/2})\right) \\
& -\Delta t\,\mathrm{sgn}\left(\rho_j^{n+1} - \kappa\right)\left(S_{\mathrm{on}}\left(t^{n+1/2}, x_j, \rho_j^{n+1/2}, R^{n+1/2}_{\mathrm{on},j}\right) - S_{\mathrm{off}}\left(t^{n+1/2}, x_j, \rho_j^{n+1/2}\right)\right).
\end{aligned}$$

□

The following Theorem states the $\mathbf{L^1}$-Lipschitz continuous dependence of solution to (4.2.1) on both the initial datum and the q_{on} and q_{off} functions.

Theorem 4.10 (Uniqueness)**.** *Let ρ and $\tilde\rho$ be two solutions to problem* (4.2.1) *in the sense of Definition 4.2, with initial data ρ_0, $\tilde\rho_0 \in \mathbf{L^1} \cap \mathbf{BV}(\mathbb{R};[0,1])$, with on-ramp rate q_{on}, $\tilde q_{\mathrm{on}}$ and off-ramp rate q_{off}, $\tilde q_{\mathrm{off}}$, respectively. Assume $v \in \mathbf{C}^2([0,1],\mathbb{R}^+)$. Then, for a.e. $t \in [0,T]$,*

$$\begin{aligned}
& \|\rho(t) - \tilde\rho(t)\|_{\mathbf{L^1}(\mathbb{R})} \\
& \leq \mathrm{e}^{\mathcal{C}T}\left(\|\rho_0 - \tilde\rho_0\|_{\mathbf{L^1}(\mathbb{R})} + L\left(\|q_{\mathrm{on}} - \tilde q_{\mathrm{on}}\|_{\mathbf{L^1}([0,t])} + \|q_{\mathrm{off}} - \tilde q_{\mathrm{off}}\|_{\mathbf{L^1}([0,T])}\right)\right).
\end{aligned}$$

Proof. The proof follows closely Theorem 5.6 of [43].

By using Kružkov's doubling of variables technique we get

$$\begin{aligned}
\|\rho(T,\cdot) - \tilde\rho(T,\cdot)\|_{\mathbf{L^1}(\mathbb{R})} \leq\ & \|\rho_0 - \tilde\rho_0\|_{\mathbf{L^1}(\mathbb{R})} + \int_0^T\int_{\Omega_{\mathrm{on}}}\left|\tilde{\mathcal{S}}_{\mathrm{on}}\right|\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_{\Omega_{\mathrm{off}}}\left|\tilde{\mathcal{S}}_{\mathrm{off}}\right|\mathrm{d}x\,\mathrm{d}t \\
& + \int_0^T\int_{\mathbb{R}}|\mathcal{V}|\,|\partial_x\rho(t,x)|\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_{\mathbb{R}}|\mathcal{V}_x|\,|\rho(t,x)|\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}$$

where

$$\begin{aligned}
\tilde{\mathcal{S}}_{\mathrm{on}} &= S_{\mathrm{on}}(t,x,q_{\mathrm{on}},\rho,R_{\mathrm{on}}) - S_{\mathrm{on}}\left(t,x,\tilde q_{\mathrm{on}},\tilde\rho,\tilde R_{\mathrm{on}}\right), \\
\tilde{\mathcal{S}}_{\mathrm{off}} &= S_{\mathrm{off}}(t,x,q_{\mathrm{on}},\rho) - S_{\mathrm{off}}(t,x,\tilde q_{\mathrm{on}},\tilde\rho), \\
\mathcal{V} &= v(R) - v(P),
\end{aligned}$$

$$\mathcal{V}_x = \partial_x v(R) - \partial_x v(P).$$

Let us now estimate all the terms appearing in the right hand side of the above inequality. We start bounding $\tilde{\mathcal{S}}_{\text{on}}$ and $\tilde{\mathcal{S}}_{\text{off}}$ terms:

$$\begin{aligned}\int_0^T \int_{\Omega_{\text{on}}} \left|\tilde{\mathcal{S}}_{\text{on}}\right| \mathrm{d}x\,\mathrm{d}t &= \int_0^T \int_{\Omega_{\text{on}}} \left|S_{\text{on}}\left(t,x,q_{\text{on}},\rho,R_{\text{on}}\right) - S_{\text{on}}\left(t,x,\tilde{q}_{\text{on}},\tilde{\rho},\tilde{R}_{\text{on}}\right)\right| \mathrm{d}x\,\mathrm{d}t \\ &\le \int_0^T \int_{\Omega_{\text{on}}} \left(\left|\tilde{\mathcal{S}}^1_{\text{on}}\right| + \left|\tilde{\mathcal{S}}^2_{\text{on}}\right| + \left|\tilde{\mathcal{S}}^3_{\text{on}}\right|\right) \mathrm{d}x\,\mathrm{d}tt,\end{aligned}$$

where

$$\begin{aligned}\tilde{\mathcal{S}}^1_{\text{on}} &= S_{\text{on}}\left(t,x,q_{\text{on}},\rho,R_{\text{on}}\right) - S_{\text{on}}\left(t,x,q_{\text{on}},\rho,\tilde{R}_{\text{on}}\right), \\ \tilde{\mathcal{S}}^2_{\text{on}} &= S_{\text{on}}\left(t,x,q_{\text{on}},\rho,\tilde{R}_{\text{on}}\right) - S_{\text{on}}\left(t,x,q_{\text{on}},\tilde{\rho},\tilde{R}_{\text{on}}\right), \\ \tilde{\mathcal{S}}^3_{\text{on}} &= S_{\text{on}}\left(t,x,q_{\text{on}},\tilde{\rho},\tilde{R}_{\text{on}}\right) - S_{\text{on}}\left(t,x,\tilde{q}_{\text{on}},\tilde{\rho},\tilde{R}_{\text{on}}\right).\end{aligned}$$

First we are going to bound $\tilde{\mathcal{S}}^1_{\text{on}}$ term ,

$$\begin{aligned}\left|\tilde{\mathcal{S}}^1_{\text{on}}\right| &= \left|\mathbf{1}_{\text{on}} q_{\text{on}} \left(1-\rho\right)\left(\left(1-R_{\text{on}}\right) - \left(1-\tilde{R}_{\text{on}}\right)\right)\right| \\ &\le \left\|q_{\text{on}}\right\|_{\mathbf{L}^\infty([0,T])} \left|\tilde{R}_{\text{on}} - R_{\text{on}}\right|,\end{aligned}$$

thus

$$\begin{aligned}\int_0^T \int_{\Omega_{\text{on}}} \left|\tilde{\mathcal{S}}^1_{\text{on}}\right| \mathrm{d}x\,\mathrm{d}t &\le \left\|q_{\text{on}}\right\|_{\mathbf{L}^\infty([0,T])} \int_0^T \int_{\Omega_{\text{on}}} \left|\tilde{R}_{\text{on}} - R_{\text{on}}\right| \mathrm{d}x\,\mathrm{d}t \\ &\le \left\|q_{\text{on}}\right\|_{\mathbf{L}^\infty([0,T])} \int_0^T \left\|\tilde{R}_{\text{on}} - R_{\text{on}}\right\|_{\mathbf{L}^1(\Omega_{\text{on}})} \mathrm{d}t.\end{aligned}$$

Observe that

$$\left\|R_{\text{on}} - \tilde{R}_{\text{on}}\right\|_{\mathbf{L}^1(\Omega_{\text{on}})} \le \left\|\rho(t,\cdot) - \tilde{\rho}(t,\cdot)\right\|_{\mathbf{L}^1(\Omega_{\text{on}})},$$

since $\int_{\mathbb{R}} \omega_\eta(x)\mathrm{d}x = 1$. Then,

$$\begin{aligned}\int_0^T \int_{\Omega_{\text{on}}} \left|\tilde{\mathcal{S}}^1_{\text{on}}\right| \mathrm{d}x\,\mathrm{d}t &\le \left\|q_{\text{on}}\right\|_{\mathbf{L}^\infty([0,T])} \int_0^T \left\|\rho(t,\cdot) - \tilde{\rho}(t,\cdot)\right\|_{\mathbf{L}^1(\Omega_{\text{on}})} \mathrm{d}t \\ &\le \left\|q_{\text{on}}\right\|_{\mathbf{L}^\infty([0,T])} \int_0^T \left\|\rho(t,\cdot) - \tilde{\rho}(t,\cdot)\right\|_{\mathbf{L}^1(\mathbb{R})} \mathrm{d}t.\end{aligned}$$

Now we are going to bound $\tilde{\mathcal{S}}^2_{\text{on}}$.

$$\left|\tilde{\mathcal{S}}^2_{\text{on}}\right| = \left|\mathbf{1}_{\text{on}} q_{\text{on}} \left(1-\tilde{R}_{\text{on}}\right)\left(1-\rho\right)\left(\tilde{\rho}-\rho\right)\right|$$

$$\leq \quad \|q_{\text{on}}\|_{\mathbf{L}^\infty([0,T])} \left|\rho - \tilde{\rho}\right|.$$

Integrating in time and space we have

$$\begin{aligned}\int_0^T \int_{\Omega_{\text{on}}} \left|\tilde{\mathcal{S}}^2_{\text{on}}\right| \mathrm{d}x\,\mathrm{d}t &\leq \|q_{\text{on}}\|_{\mathbf{L}^\infty([0,T])} \int_0^T \|\rho(t,\cdot) - \tilde{\rho}(t,\cdot)\|_{\mathbf{L}^1(\Omega_{\text{on}})} \mathrm{d}t \\ &\leq \|q_{\text{on}}\|_{\mathbf{L}^\infty([0,T])} \int_0^T \|\rho(t,\cdot) - \tilde{\rho}(t,\cdot)\|_{\mathbf{L}^1(\mathbb{R})} \mathrm{d}t.\end{aligned}$$

Bounding $\tilde{\mathcal{S}}^3_{\text{on}}$,

$$\begin{aligned}\left|\tilde{\mathcal{S}}^3_{\text{on}}\right| &= \left|\mathbf{1}_{\text{on}}\left(1-\tilde{\rho}\right)\left(1-\tilde{R}_{\text{on}}\right)\left(q_{\text{on}} - \tilde{q}_{\text{on}}\right)\right| \\ &\leq |q_{\text{on}} - \tilde{q}_{\text{on}}|,\end{aligned}$$

thus

$$\begin{aligned}\int_0^T \int_{\Omega_{\text{on}}} \left|\tilde{\mathcal{S}}^3_{\text{on}}\right| \mathrm{d}x\,\mathrm{d}t &\leq \int_0^T \int_{\Omega_{\text{on}}} |q_{\text{on}} - \tilde{q}_{\text{on}}| \,\mathrm{d}x\,\mathrm{d}t \\ &\leq L \left\|q_{\text{on}} - \tilde{q}_{\text{on}}\right\|_{\mathbf{L}^1([0,T])}.\end{aligned}$$

Therefore, we get the following estimate

$$\begin{aligned}&\int_0^T \int_{\Omega_{\text{on}}} \left|\tilde{\mathcal{S}}_{\text{on}}\right| \mathrm{d}x\,\mathrm{d}t \\ &\leq 2\|q_{\text{on}}\|_{\mathbf{L}^\infty([0,T])} \int_0^T \|\rho(t,\cdot) - \tilde{\rho}(t,\cdot)\|_{\mathbf{L}^1(\mathbb{R})} \mathrm{d}t + L \left\|q_{\text{on}} - \tilde{q}_{\text{on}}\right\|_{\mathbf{L}^1([0,T])}. \end{aligned} \tag{4.3.19}$$

Regarding $\tilde{\mathcal{S}}_{\text{off}}$ term, we proceed in a similar way like above and we get

$$\begin{aligned}\left|\tilde{\mathcal{S}}_{\text{off}}\right| &= \left|\mathbf{1}_{\text{off}} q_{\text{off}} \rho - \mathbf{1}_{\text{off}} \tilde{q}_{\text{off}} \tilde{\rho}\right| \\ &\leq \left|\tilde{\mathcal{S}}^1_{\text{off}}\right| + \left|\tilde{\mathcal{S}}^2_{\text{off}}\right|,\end{aligned}$$

where

$$\begin{aligned}\tilde{\mathcal{S}}^1_{\text{off}} &= S_{\text{off}}\left(t,x,q_{\text{off}},\rho\right) - S_{\text{off}}\left(t,x,q_{\text{off}},\tilde{\rho}\right), \\ \tilde{\mathcal{S}}^2_{\text{off}} &= S_{\text{off}}\left(t,x,q_{\text{off}},\tilde{\rho}\right) - S_{\text{off}}\left(t,x,\tilde{q}_{\text{off}},\tilde{\rho}\right).\end{aligned}$$

Then,

$$\begin{aligned}\int_0^T \int_{\Omega_{\text{off}}} \left|\tilde{\mathcal{S}}^1_{\text{off}}\right| \mathrm{d}x\,\mathrm{d}t &\leq \|q_{\text{off}}\|_{\mathbf{L}^\infty([0,T])} \int_0^T \|\rho(t,\cdot) - \tilde{\rho}(t,\cdot)\|_{\mathbf{L}^1(\Omega_{\text{off}})} \mathrm{d}t \\ &\leq \|q_{\text{off}}\|_{\mathbf{L}^\infty([0,T])} \int_0^T \|\rho(t,\cdot) - \tilde{\rho}(t,\cdot)\|_{\mathbf{L}^1(\mathbb{R})} \mathrm{d}t,\end{aligned}$$

and

$$\int_0^T \int_{\Omega_{\text{off}}} \left|\tilde{\mathcal{S}}_{\text{off}}^2\right| \mathrm{d}x \, \mathrm{d}t \le L \left\| q_{\text{off}} - \tilde{q}_{\text{off}} \right\|_{\mathbf{L^1}([0,T])} .$$

Thus, we get

$$\begin{aligned}
&\int_0^T \int_{\Omega_{\text{off}}} \left|\mathcal{S}_{\text{off}}\right| \mathrm{d}x \, \mathrm{d}t \\
&\le \left\| q_{\text{off}} \right\|_{\mathbf{L^\infty}([0,T])} \int_0^T \left\| \rho(t,\cdot) - \tilde{\rho}(t,\cdot) \right\|_{\mathbf{L^1}(\mathbb{R})} \mathrm{d}t + L \left\| q_{\text{off}} - \tilde{q}_{\text{off}} \right\|_{\mathbf{L^1}([0,T])} . \quad (4.3.20)
\end{aligned}$$

Next, focus on $\mathcal{V}$, by using the following estimate

$$|\mathcal{V}| \le \omega_\eta(0) \left\| v' \right\|_{\mathbf{L^\infty}([0,1])} \left\| \rho(t,\cdot) - \tilde{\rho}(t,\cdot) \right\|_{\mathbf{L^1}(\mathbb{R})} ,$$

we obtain

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}} |\mathcal{V}| \left| \partial_x \rho(t,x) \right| \mathrm{d}x \, \mathrm{d}t \\
&\le \omega_\eta(0) \left\| v' \right\|_{\mathbf{L^\infty}([0,1])} \sup_{t \in [0,T]} \left\| \rho(t,\cdot) \right\|_{\mathbf{TV}(\mathbb{R})} \int_0^T \left\| \rho(t,\cdot) - \tilde{\rho}(t,\cdot) \right\|_{\mathbf{L^1}(\mathbb{R})} \mathrm{d}t . \quad (4.3.21)
\end{aligned}$$

Next, we pass to $\mathcal{V}_x$. Following [43] we compute

$$\begin{aligned}
|\mathcal{V}_x| \quad \le \quad & \Big(2 \left(\omega_\eta(0)\right)^2 \left\| v'' \right\|_{\mathbf{L^\infty}([0,1])} + \left\| v' \right\|_{\mathbf{L^\infty}([0,1])} \left\| \omega_\eta' \right\|_{\mathbf{L^\infty}([0,\eta])} \Big) \left\| \rho(t,\cdot) - \tilde{\rho}(t,\cdot) \right\|_{\mathbf{L^1}(\mathbb{R})} \\
& + \omega_\eta(0) \| v' \|_{\mathbf{L^\infty}([0,1])} \left(\left| \rho - \tilde{\rho} \right| (t, x+\eta) + \left| \rho - \tilde{\rho} \right| (t,x) \right) ,
\end{aligned}$$

thus

$$\int_0^T \int_{\mathbb{R}} |\mathcal{V}_x| \left| \rho(t,x) \right| \mathrm{d}x \, \mathrm{d}t \quad \le \quad \mathcal{W} \int_0^T \left\| \rho(t,\cdot) - \tilde{\rho}(t,\cdot) \right\|_{\mathbf{L^1}(\mathbb{R})} \mathrm{d}t, \qquad (4.3.22)$$

where

$$\begin{aligned}
\mathcal{W} \quad = \quad & \Big(2 \left(\omega_\eta(0)\right)^2 \left\| v'' \right\|_{\mathbf{L^\infty}([0,1])} + \left\| v' \right\|_{\mathbf{L^\infty}([0,1])} \left\| \omega_\eta' \right\|_{\mathbf{L^\infty}([0,\eta])} \Big) \mathcal{C}_1(t) \\
& + 2 \omega_\eta(0) \left\| v' \right\|_{\mathbf{L^\infty}([0,1])} .
\end{aligned}$$

Collecting together (4.3.19), (4.3.20), (4.3.21) and (4.3.22) we get

$$\left\| \rho(T,\cdot) - \tilde{\rho}(T,\cdot) \right\|_{\mathbf{L^1}(\mathbb{R})} \quad \le \quad \left\| \rho_0 - \tilde{\rho}_0 \right\|_{\mathbf{L^1}(\mathbb{R})}$$

$$
\begin{aligned}
&+L\left(\|q_{\text{on}}-\tilde{q}_{\text{on}}\|_{\mathbf{L^1}([0,t])}+\|q_{\text{off}}-\tilde{q}_{\text{off}}\|_{\mathbf{L^1}([0,t])}\right)\\
&+\mathcal{C}\int_0^T \|\rho(t,\cdot)-\tilde{\rho}(t,\cdot)\|_{\mathbf{L^1}(\mathbb{R})}\,\mathrm{d}t,
\end{aligned}
\tag{4.3.23}
$$

where

$$
\mathcal{C}=\mathcal{H}+\omega_\eta(0)\,\|v'\|_{\mathbf{L}^\infty([0,1])}\sup_{t\in[0,T]}\|\rho(t,\cdot)\|_{\mathbf{TV}(\mathbb{R})}+\mathcal{W}. \tag{4.3.24}
$$

An application of Gronwall Lemma to (4.3.23) completes the proof. □

4.3.5 Proof of Theorem 4.3

The convergence of the approximate solutions constructed by **Algorithm 4.3.1** towards the unique weak entropy solution can be proven by applying Helly's compactness theorem. The latter can be applied due to Lemma 4.6 and Proposition 4.8.2 and states that there exists a sub-sequence of approximate solution ρ_Δ that converges in $\mathbf{L^1}$ to a function $\rho\in\mathbf{L}^\infty([0,T]\times\mathbb{R};[0,1])$. Following a Lax-Wendroff type argument, we can show that the limit function ρ is a weak entropy solution of (4.2.1) in the sense of Definition 4.2. Together with the uniqueness result in Theorem 4.10. this concludes the proof of Theorem 4.3.

4.3.6 Existence for Model 2

In this section we consider the problem (4.2.1) with S_{on} (4.1.6). In **Algorithm 4.3.1** we substitute the term S_{on} in the reaction step (4.3.6) by (4.3.3), thus now the term (4.3.6) is given by

$$
\begin{aligned}
\rho_j^{n+1} \;=\;& \rho_j^{n+1/2}+\\
&\Delta t\mathbf{1}_{\text{on},j}q_{\text{on}}^{n+1/2}\left(1-\max\left\{\rho_j^{n+1/2},R_{\text{on},j}^{n+1/2}\right\}\right)-\Delta t\mathbf{1}_{\text{off},j}q_{\text{off}}^{n+1/2}\rho_j^{n+1/2}.
\end{aligned}
\tag{4.3.25}
$$

Lemma 4.11 (Maximum Principle). *Let $\rho_0\in\mathbf{L}^\infty(\mathbb{R};[0,1])$. Let the Assumptions 4.2.1 and CFL condition (4.3.7) hold, then for all $t>0$ and $x\in\mathbb{R}$ the piece-wise constant approximate solution ρ_Δ constructed through **Algorithm 4.3.1** is such that*

$$
0\le\rho_\Delta(t,x)\le 1.
$$

Proof. The proof is made by induction. We assume that $0\le\rho_j^n\le 1$ for all $j\in\mathbb{Z}$. Consider the step (4.3.5) of **Algorithm 4.3.1**, by CFL condition (4.3.7) we have $0\le\rho_j^{n+1/2}\le 1$ for $j\in\mathbb{Z}$.

Now focus on the remaining step, involving the source term.

$$
\rho_j^{n+1} \;=\; \rho_j^{n+1/2}+\Delta t\mathbf{1}_{\text{on},j}q_{\text{on}}^{n+1/2}\left(1-\max\left\{\rho_j^{n+1/2},R_{\text{on},j}^{n+1/2}\right\}\right)-\Delta t\mathbf{1}_{\text{off},j}q_{\text{off}}^{n+1/2}\rho_j^{n+1/2}
$$

$$
\begin{aligned}
&= \rho_j^{n+1/2} + \Delta t \mathbf{1}_{\mathrm{on},j} q_{\mathrm{on}}^{n+1/2} \left(1 - \frac{\rho_j^{n+1/2} + R_{\mathrm{on},j}^{n+1/2} + \left|\rho_j^{n+1/2} - R_{\mathrm{on},j}^{n+1/2}\right|}{2}\right) \\
&\quad -\Delta t \mathbf{1}_{\mathrm{off},j} q_{\mathrm{off}}^{n+1/2} \rho_j^{n+1/2} \\
&= \rho_j^{n+1/2} + \Delta t \mathbf{1}_{\mathrm{on},j} q_{\mathrm{on}}^{n+1/2} - \frac{\Delta t}{2} \mathbf{1}_{\mathrm{on},j} q_{\mathrm{on}}^{n+1/2} \rho_j^{n+1/2} - \frac{\Delta t}{2} \mathbf{1}_{\mathrm{on},j} q_{\mathrm{on}}^{n+1/2} R_{\mathrm{on},j}^{n+1/2} \\
&\quad - \frac{\Delta t}{2} \mathbf{1}_{\mathrm{on},j} q_{\mathrm{on}}^{n+1/2} \left|\rho_j^{n+1/2} - R_{\mathrm{on},j}^{n+1/2}\right| - \Delta t \mathbf{1}_{\mathrm{off},j} q_{\mathrm{off}}^{n+1/2} \rho_j^{n+1/2} \\
&\leq \rho_j^{n+1/2} + \Delta t \mathbf{1}_{\mathrm{on},j} q_{\mathrm{on}}^{n+1/2} - \frac{\Delta t}{2} \mathbf{1}_{\mathrm{on},j} q_{\mathrm{on}}^{n+1/2} \rho_j^{n+1/2} - \cancel{\frac{\Delta t}{2} \mathbf{1}_{\mathrm{on},j} q_{\mathrm{on}}^{n+1/2} R_{\mathrm{on},j}^{n+1/2}} \\
&\quad + \cancel{\frac{\Delta t}{2} \mathbf{1}_{\mathrm{on},j} q_{\mathrm{on}}^{n+1/2} \left|R_{\mathrm{on},j}^{n+1/2}\right|} - \frac{\Delta t}{2} \mathbf{1}_{\mathrm{on},j} q_{\mathrm{on}}^{n+1/2} \left|\rho_j^{n+1/2}\right| - \Delta t \mathbf{1}_{\mathrm{off},j} q_{\mathrm{off}}^{n+1/2} \rho_j^{n+1/2} \\
&= \rho_j^{n+1/2} + \Delta t \mathbf{1}_{\mathrm{on},j} q_{\mathrm{on}}^{n+1/2} - \Delta t \mathbf{1}_{\mathrm{on},j} q_{\mathrm{on}}^{n+1/2} \rho_j^{n+1/2} - \Delta t \mathbf{1}_{\mathrm{off},j} q_{\mathrm{off}}^{n+1/2} \rho_j^{n+1/2} \\
&= \left(1 - \Delta t \left(\mathbf{1}_{\mathrm{on},j} q_{\mathrm{on}}^{n+1/2} + \mathbf{1}_{\mathrm{off},j} q_{\mathrm{off}}^{n+1/2}\right)\right) \rho_j^{n+1/2} + \Delta t \mathbf{1}_{\mathrm{on},j} q_{\mathrm{on}}^{n+1/2},
\end{aligned}
$$

now we can proceed as in Lemma 4.6. □

Lemma 4.12. *Let* $\rho_0 \in \mathbf{L^1}(\mathbb{R}, [0,1])$. *Let the Assumptions* (4.2.1) *and the CFL condition* (4.3.7) *hold. Then, the piece-wise constant approximate solution* ρ_Δ *constructed through* ***Algorithm 4.3.1*** *satisfies,*

$$\|\rho_\Delta(t)\|_{\mathbf{L^1}(\mathbb{R})} \leq \mathcal{C}_1(t),$$

where $\mathcal{C}_1$ *like in* (4.3.8).

Proof. By (4.3.26) and CFL condition (4.3.7) we have

$$\left|\rho_j^{n+1}\right| \leq \left|\rho_j^{n+1/2}\right| + \Delta t \mathbf{1}_{\mathrm{on},j} q_{\mathrm{on}}^{n+1/2} \left(1 - \left|\rho_j^{n+1/2}\right|\right) - \Delta t \mathbf{1}_{\mathrm{off},j} q_{\mathrm{off}}^{n+1/2} \left|\rho_j^{n+1/2}\right|,$$

this cases reduce to (4.3.9) and we can proceed as in Lemma 4.7. □

4.3.7 BV estimates

Lemma 4.13. *The map* S_{on} *given in* (4.3.25) *is Lipschitz continuous in second, third and fourth argument with Lipschitz constant* $\|q_{\mathrm{on}}\|_{\mathbf{L}^\infty([0,T])}$.

Proof.

$$\left|S_{\mathrm{on}}(t,x,\rho,R_{\mathrm{on}}) - S_{\mathrm{on}}(t,\tilde{x},\tilde{\rho},\tilde{R}_{\mathrm{on}})\right| \leq \mathcal{S}_1 + \mathcal{S}_2 + \mathcal{S}_3,$$

where

$$
\begin{aligned}
\mathcal{S}_1 &= \left|S_{\mathrm{on}}(t,x,\rho,R_{\mathrm{on}}) - S_{\mathrm{on}}(t,x,\tilde{\rho},R_{\mathrm{on}})\right| \\
\mathcal{S}_2 &= \left|S_{\mathrm{on}}(t,x,\tilde{\rho},R_{\mathrm{on}}) - S_{\mathrm{on}}(t,x,\tilde{\rho},\tilde{R}_{\mathrm{on}})\right|
\end{aligned}
$$

$$\mathcal{S}_3 = \left| S_{\rm on}(t,x,\tilde\rho,\tilde R_{\rm on}) - S_{\rm on}(t,\tilde x,\tilde\rho,\tilde R_{\rm on}) \right| .$$

by the definition of $S_{\rm on}$ term and by using the estimation $|\max(a_1,b)-\max(a_2,b)| \leq |a_1-a_2|$ we have

$$\begin{aligned}
\mathcal{S}_1 &\leq \|q_{\rm on}\|_{\mathbf{L}^\infty([0,T])} \left| 1-\max\{\rho,R_{\rm on}\} - (1-\max\{\tilde\rho,R_{\rm on}\}) \right| \\
&= \|q_{\rm on}\|_{\mathbf{L}^\infty([0,T])} \left| \max\{\tilde\rho,R_{\rm on}\} - \max\{\rho,R_{\rm on}\} \right| \\
&\leq \|q_{\rm on}\|_{\mathbf{L}^\infty([0,T])} \left|\tilde\rho-\rho\right| .
\end{aligned}$$

Pass now to $\mathcal{S}_2$:

$$\begin{aligned}
\mathcal{S}_2 &\leq \|q_{\rm on}\|_{\mathbf{L}^\infty([0,T])} \left| \max\left\{\tilde\rho,\tilde R_{\rm on}\right\} - \max\{\tilde\rho,R_{\rm on}\} \right| \\
&\leq \|q_{\rm on}\|_{\mathbf{L}^\infty([0,T])} \left| R_{\rm on}-\tilde R_{\rm on}\right| .
\end{aligned}$$

Next, we analyze the $\mathcal{S}_3$ term:

$$\begin{aligned}
\mathcal{S}_3 &= \left| \mathbf{1}_{\rm on} q_{\rm on} \left(1-\max\left\{\tilde\rho,\tilde R_{\rm on}\right\}\right) - \tilde{\mathbf{1}}_{\rm on} q_{\rm on} \left(1-\max\left\{\tilde\rho,\tilde R_{\rm on}\right\}\right) \right| \\
&\leq \|q_{\rm on}\|_{\mathbf{L}^\infty([0,T])} \left|\mathbf{1}_{\rm on}-\tilde{\mathbf{1}}_{\rm on}\right| \left|1-\max\left\{\tilde\rho,\tilde R_{\rm on}\right\}\right| \\
&\leq \|q_{\rm on}\|_{\mathbf{L}^\infty([0,T])} \left|\mathbf{1}_{\rm on}-\tilde{\mathbf{1}}_{\rm on}\right| .
\end{aligned}$$

□

Proposition 4.13.1 (**BV** estimate in space). *Let $\rho_0 \in \left(\mathbf{L^1}\cap\mathbf{BV}\right)(\mathbb{R};[0,1])$. Let the Assumptions 4.2.1 and CFL condition* (4.3.7) *hold. Then, for $n = 0,\ldots,N_T-1$ the following estimate holds*

$$\sum_{j\in\mathbb{Z}} \left|\rho^n_{j+1}-\rho^n_j\right| \leq e^{T\mathcal{H}}\left(TV(\rho^0)+T\mathcal{Q}_T\right),$$

with $\mathcal{H}$ like in (4.2.5).

Proof. Due to the results obtained in Lemma 4.13, the proof is analogous to that one of Proposition 4.8.1. □

Proposition 4.13.2 (**BV** estimate in space and time). *Let the Assumptions 4.2.1 hold, $\rho_0 \in \left(\mathbf{L^1}\cap\mathbf{BV}\right)(\mathbb{R};[0,1])$. If the CFL condition* (4.3.7) *holds, then, for every $T>0$ the following discrete space and time total variation estimate is satisfied:*

$$TV(\rho_\Delta;[0,T]\times\mathbb{R}) \leq T\mathcal{C}_{xt}(T),$$

with $\mathcal{C}_{xt}(T)$ defined in (4.3.12).

Proof. For this proof we need to compute the following estimate,

$$\begin{aligned}\left|\rho_j^{n+1}-\rho_j^{n+1/2}\right| &\leq \Delta t\left|S_{\text{on},j}^{n+1/2}-S_{\text{off},j}^{n+1/2}\right| \\ &= \Delta t\left|\mathbf{1}_{\text{on},j}q_{\text{on}}\left(1-\max\left\{\rho_j^{n+1/2},R_{\text{on},j}^{n+1/2}\right\}\right)-\mathbf{1}_{\text{off},j}q_{\text{off}}\rho_j^{n+1/2}\right|. \\ &\leq \Delta t\mathbf{1}_{\text{on},j}\left\|q_{\text{on}}\right\|_{\mathbf{L}^\infty([0,T])}\left|1-\max\left\{\rho_j^{n+1/2},R_{\text{on},j}^{n+1/2}\right\}\right) \\ &\quad +\Delta t\mathbf{1}_{\text{off},j}\left\|q_{\text{off}}\right\|_{\mathbf{L}^\infty([0,T])}\left|\rho_j^{n+1/2}\right|\end{aligned}$$

Here we need to consider two cases, which are described below:

Case 1: $\max\left\{\rho_j^{n+1/2},R_{\text{on},j}^{n+1/2}\right\}=\rho_j^{n+1/2}$. In this case we get the following estimate

$$\begin{aligned}\left|\rho_j^{n+1}-\rho_j^{n+1/2}\right| &\leq \Delta t\mathbf{1}_{\text{on},j}\left\|q_{\text{on}}\right\|_{\mathbf{L}^\infty([0,T])}\left|1-\rho_j^{n+1/2}\right| \\ &\quad +\Delta t\mathbf{1}_{\text{off},j}\left\|q_{\text{off}}\right\|_{\mathbf{L}^\infty([0,T])}\left|\rho_j^{n+1/2}\right| \\ &\leq \Delta t\mathbf{1}_{\text{on},j}\left\|q_{\text{on}}\right\|_{\mathbf{L}^\infty([0,T])}\left(1+\left|\rho_j^{n+1/2}\right|\right) \\ &\quad +\Delta t\mathbf{1}_{\text{off},j}\left\|q_{\text{off}}\right\|_{\mathbf{L}^\infty([0,T])}\left|\rho_j^{n+1/2}\right| \\ &\leq \Delta t\|q_{\text{on}}\|_{\mathbf{L}^\infty([0,T])}\left(\mathbf{1}_{\text{on},j}+\mathbf{1}_{\text{on},j}\left|\rho_j^{n+1/2}\right|\right) \\ &\quad +\Delta t\left\|q_{\text{off}}\right\|_{\mathbf{L}^\infty([0,T])}\mathbf{1}_{\text{off},j}\left|\rho_j^{n+1/2}\right|.\end{aligned}$$

Case 2: $\max\left\{\rho_j^{n+1/2},R_{\text{on},j}^{n+1/2}\right\}=R_{\text{on},j}^{n+1/2}$. Observe that since $R_{\text{on,j}}^{n+1/2}\leq 1$, this implies that $0\leq\left|1-R_{\text{on,j}}^{n+1/2}\right|\leq 1\leq 1+\left|\rho_j^{n+1/2}\right|$, from what we get the following estimate

$$\begin{aligned}\left|\rho_j^{n+1}-\rho_j^{n+1/2}\right| &\leq \Delta t\mathbf{1}_{\text{on},j}\left\|q_{\text{on}}\right\|_{\mathbf{L}^\infty([0,T])}\left|1-R_{\text{on},j}^{n+1/2}\right| \\ &\quad +\Delta t\mathbf{1}_{\text{off},j}\left\|q_{\text{off}}\right\|_{\mathbf{L}^\infty([0,T])}\left|\rho_j^{n+1/2}\right| \\ &\leq \Delta t\|q_{\text{on}}\|_{\mathbf{L}^\infty([0,T])}\left(\mathbf{1}_{\text{on},j}+\mathbf{1}_{\text{on},j}\left|\rho_j^{n+1/2}\right|\right) \\ &\quad +\Delta t\left\|q_{\text{off}}\right\|_{\mathbf{L}^\infty([0,T])}\mathbf{1}_{\text{off},j}\left|\rho_j^{n+1/2}\right|.\end{aligned}$$

Note that both cases reduces to (4.3.15) and therefore the rest of the proof is analogous to Proposition 4.8.2. □

4.4 Numerical experiments

In this section we present some numerical examples to describe the effects that the ramps have on a road. We solve Model 1 and Model 2 by means **Algorithm 4.3.1** with the term S_{on} computed as (4.3.2) and (4.3.3), respectively. In all numerical examples below, we consider one on-ramp and one off-ramp, both ramps with length $L = 0.1$, the on-ramp is located from $x = 1.0$ until $x = 1.1$, the off-ramp is located from $x = 3$ until $x = 3.1$ and we consider the following kernel functions

$$\begin{aligned}\omega_\eta(x) &:= 2\frac{\eta - x}{\eta^2}\chi_{[0,\eta]}(x),\\ \omega_{\eta,\delta}(x) &:= \frac{1}{\eta^6}\frac{16}{5\pi}\left(\eta^2 - (x-\delta)^2\right)^{5/2}\chi_{[-\eta+\delta,\eta+\delta]}(x),\end{aligned}$$

for convective and reactive terms respectively, with $\eta \in [0,1]$ and $\delta \in [-\eta, \eta]$.

4.4.1 Example 7. Dynamic of Model 1 vs. Model 2

In this example we show numerically the behavior of the density of vehicles in a main road with the presence of one on-ramp and one off-ramp. We solve (4.2.1) numerically in the interval $[-1, 9]$ in simulated times $T = 0.5,\ T = 2,\ T = 5,\ T = 7$. We consider $\Delta x = 1/1000$, $\eta = 0.05$, $\delta = -0.01$, a constant initial condition $\rho_0(x) = 0.3$, and the rate of the on- and off-ramp are given by $q_{\text{on}}(t) = 1.2,\ q_{\text{off}}(t) = 0.8$, respectively.
In Fig.4.4.1 we can see that when vehicles enter the ramp, the density of vehicles on the main road increases and a shock wave with negative speed is formed, after that, a rarefaction wave appears and when some vehicles leave the main road through off-ramp a shock wave with positive speed is formed. In particular we can observe a difference between the maximum density that is reached in each model, which may be due to the presence of the term $1 - \rho$ in the Model 1.

4.4.2 Example 8. Limit $\eta \to 0$ in Model 2

In this example we take a look at the limit case $\eta \to 0$ and investigate the convergence of the Model 2 to the solution of the local problem (4.1.1)-(4.1.3). In particular, we consider the initial condition $\rho_0(x) = 0.3$ for $x \in [0,1]$, $q_{\text{on}}(t) = 1.2,\ q_{\text{off}}(t) = 0.8$ at $T = 5$ with fixed $\Delta x = 1/1000$ and $\eta \in \{0.1, 0.05, 0.01, 0.004\}$, and $\delta = 0$. To evaluate the convergence, we compute the $\mathbf{L^1}$ distance between the approximate solution obtained for the proposed upwind-type scheme by means **Algorithm 4.3.1** with a given η and the result of a classical Godunov scheme for the corresponding local problem. In Table 4.4.1, we can observe that the $\mathbf{L^1}$ distance goes to zero when $\eta \to 0$. The results are illustrated in Fig.4.4.2.

η	0.1	0.05	0.01	0.004
$\mathbf{L^1}$ distance	2.8e-1	1.6e-1	3.6e-2	1.1e-2

Table 4.4.1: Example 8. $\mathbf{L^1}$ distance between the approximate solutions to the non-local problem and the local problem for different values of η at $T = 5$ with $\Delta x = 1/1000$.

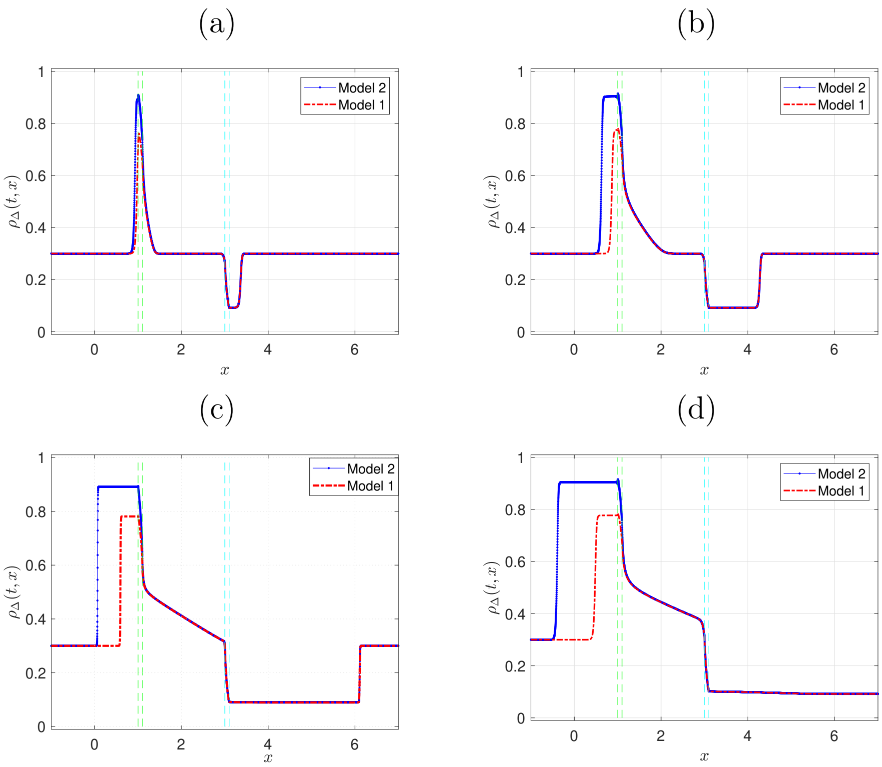

Figure 4.4.1: Example 7. Numerical approximations of the problem (4.2.1). Dynamic of Model 1 vs. Model 2 at (a)$T = 0.5$, (b)$T = 2$, (c)$T = 5$, (d)$T = 7$.

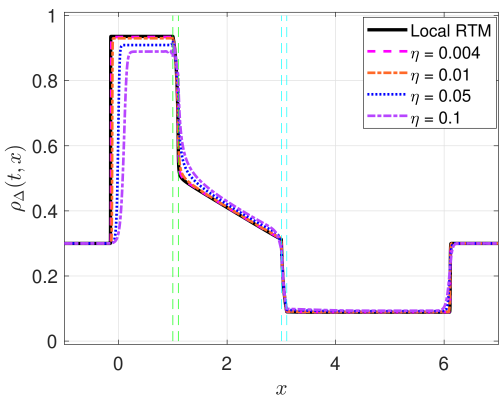

Figure 4.4.2: Example 8. Numerical approximations of the problem (4.2.1) at $T = 5$. Comparison of local and non-local versions of the model (4.2.1) with $\delta = 0$ and different values for η.

4.4.3 Example 9. Maximum principle

In this example we verify that the **Algorithm 4.3.1** with the terms S_{on} (4.3.2) and (4.3.3) satisfy the maximum principle, i.e., we verify numerically that Lemmas 4.6 and 4.11 respectively, are fulfilled. On the other hand, we also verify that the **Algorithm 4.3.1** with a discretization of the term S_{on} (4.1.4), which we called Model 0, does not satisfy a maximum principle. For this purpose we consider the initial condition given by

$$\rho_0(x) = \begin{cases} 0.1 & \text{if} \quad x \leq 1.1 \\ 1.0 & \text{if} \quad x > 1.1, \end{cases}$$

$q_{\text{on}}(t) = 1,\ q_{\text{off}}(t) = 0.2$ at $T = 0.3$, with $\Delta x = 1/100$, $\eta = 0.05$, and $\delta = -0.01$. We can see in Fig.4.4.3 (a) that the Model 0 does not satisfy a maximum principle unlike Model 1 and Model 2. The Fig 4.4.3 (b) is a zoom of (a) in which we can appreciate in a better form that Model 0 does not satisfy a maximum principle.

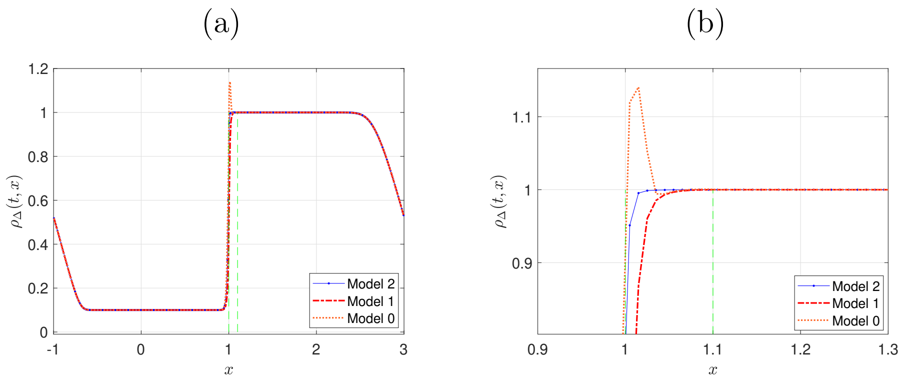

Figure 4.4.3: Example 9. Numerical approximation at time $T = 0.3$. (a) Model 1, Model 2 satisfying a maximum principle and Model 0 not satisfying a maximum principle. (b) Zoom of a part of (a).

4.4.4 Example 10. Free main road

In this example we consider a free main road, i.e, we consider a initial condition $\rho_0 = 0$, boundary conditions $\rho_0(t) = 0.4$ for all $t > 0$ and absorbing conditions at $x = 5$. We also consider the rate of the on-ramp $q_{\text{on}}(t) = \frac{1}{2}(\sin(\pi t) + 1)$ and the rate of the off-ramp $q_{\text{off}}(t) = 0.2$. We solve (4.2.1) numerically in the interval $[-1, 5]$ in different times, namely $T = 1,\ T = 2,\ T = 5,\ T = 7$ and consider $\Delta x = 1/1000$, $\eta = 0.1$, $\delta = -0.02$. In Fig.4.4.4 we can see the dynamic of the model 4.2.1 approximated by means of Model 1 and Model 2.

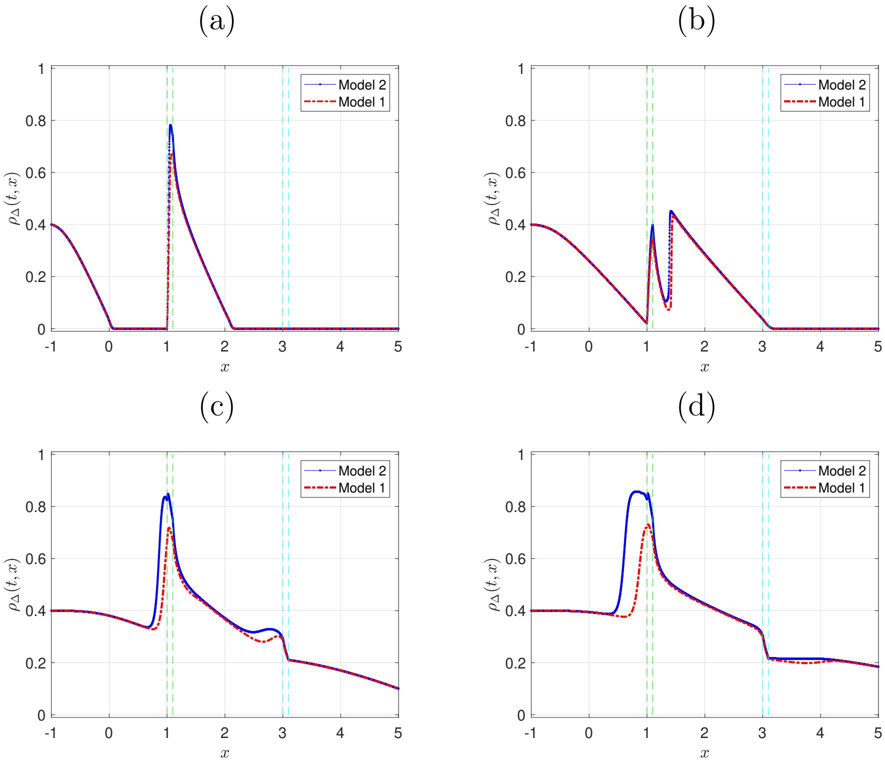

Figure 4.4.4: Example 10. Dynamic of the model (4.2.1). Behavior of the numerical solution computed with **Algorithm 4.3.1** by means of Model 1 and Model 2 at time (a)$T = 1$, (b)$T = 2$, (c)$T = 5$, (d)$T = 7$.

4.5 Conclusion of Chapter 4

In this chapter we introduced a nonlocal balance law to model vehicular traffic flow including on- and off-ramps. We presented three different models called Model 0, Model 1 and Model 2 and we proved existence and uniqueness of solutions for Model 1 and Model 2. We approximated the problem through a upwind-type numerical scheme, providing a Maximum principle, $\mathbf{L^1}$ and $\mathbf{BV}$ estimates for approximate solutions. Numerical simulations illustrate the dynamics of the studied models and show that Model 0 does not satisfy a maximum principle. A limit model as the kernel support tends to zero is numerically investigated.

Chapter 5

Stability estimates for nonlocal balance laws arising in traffic modelling

5.1 Introduction

5.1.1 Scope

In traffic flow modeling, nonlocal conservation laws are intended to describe the behaviour of drivers that adapt their velocity with respect to what happens in front of them, see [14, 21, 26, 43, 44]; this type of situation, the classical LWR (Lighthill-Whitham [66] and Richards [72]) is not able to model. In the same way, in order to extend the LWR model to more real situations, in [23] is introduced a nonlocal balance law which is intended to model vehicular traffic flow on a main road with on- and off-ramps and is given by

$$\rho_t + (\rho v(\rho * \omega_\eta))_x = S_{\text{on}}(\cdot, \cdot, \rho, \rho * \omega_{\eta,\delta}) - S_{\text{off}}(\cdot, \cdot, \rho), \quad x \in \mathbb{R} \tag{5.1.1}$$

where S_{on} and S_{off} describe traffic flow entering and exiting through an on- and off-ramp, respectively, and the convolution term in S_{on} is defined as follows,

$$(\rho * \omega_{\eta,\delta})(t, x) = \int_{x-\eta+\delta}^{x+\eta+\delta} \rho(t, y)\omega_{\eta,\delta}(y - x)\mathrm{d}y,$$

with $\eta \in [0, 1]$ and $\delta \in [-\eta, \eta]$, here the parameter η represents the radius of the support of the kernel function $\omega_{\eta,\delta}$, while δ is the point at which the maximum is attained. This choice of the kernel models the fact that drivers on the on-ramp can see what happens on the backward and forward on the main road.

It is well known that on-ramp merging has a great impact on traffic efficiency, if one concentrates on highway networks the reduction of the capacity is often due to on- and off-ramps, for this reason in this chapter we are interested in to study the dependence of solutions to (5.1.1) on the convolution kernel given in the source term S_{on}. The strategies that we employ are inspired by the results obtained in [23, 28]. Particularly, we adopt the results about existence and uniqueness to (5.1.1) presented in [23] and we propose to study the dependence of solution to (5.1.1) varying the kernel function on the source term.

5.1.2 Outline of this chapter.

This chapter is organized as follows: In Section 4.2 we recall the mathematical model in which we will focus our study as well as the definitions of weak and entropy weak solution. In section 5.3 we present the main result of this chapter, which deals with $\mathbf{L^1}-$ Lipschitz continuous dependence of solutions to (5.1.1) on to the initial datum, the on-ramp rate, the off-ramp rate and the kernel function in source term. Finally, in Section 5.4 we present a numerical example in order to illustrate the behaviour of solutions to model when the kernel function in source term vary.

5.2 Mathematical model

We will consider the equation (5.1.1) with terms S_{on} and S_{off} defined as

$$S_{\text{on}}(t,x,\rho,\rho*\omega_{\eta,\delta}) = \mathbf{1}_{\text{on}}(x)q_{\text{on}}(t)\left(1-\frac{\rho}{\rho_{\max}}\right)\left(1-\frac{\rho*\omega_{\eta,\delta}}{\rho_{\max}}\right), \quad (5.2.1)$$

$$S_{\text{off}}(t,x,\rho) = \mathbf{1}_{\text{off}}(x)q_{\text{off}}(t)\frac{\rho}{\rho_{\max}}, \quad (5.2.2)$$

with $\rho_{\max}=1$ for the sake of simplicity, and we also endow to nonlocal traffic reaction model (5.1.1) with a initial condition, as follows,

$$\rho(x,0)=\rho_0(x)\in\left(\mathbf{L^1}\cap\mathbf{BV}\right)(\mathbb{R};[0,\rho_{\max}]). \quad (5.2.3)$$

In order to get our main goal, let us assume the following assumptions on the parameters of model (5.1.1).

Assumptions 5.2.1. *We assume*

(i) $q_{\text{on}}^{\text{ramp}}\in\mathbf{L}^\infty(\mathbb{R}^+;\mathbb{R}^+), q_{\text{off}}^{\text{ramp}}\in\mathbf{L}^\infty(\mathbb{R}^+;\mathbb{R}^+)$.

(ii) $v\in\mathbf{C}^2([0,\rho_{\max}];\mathbb{R}^+),\ v'(\rho)\le 0,\ \rho\in[0,\rho_{\max}]$.

(iii) $\omega_\eta\in\mathbf{C}^1([0,\eta];\mathbb{R}^+)$ with $\omega_\eta'(x)\le 0,\ \int_0^\eta\omega_\eta(x)\mathrm{d}x=1,\ \forall\eta>0$.

(iv) $\omega_{\eta,\delta}\in(\mathbf{C}^1\cap\mathbf{L^1})([\delta-\eta,\delta+\eta];\mathbb{R}^+)$ with $\omega'(x)_{\eta,\delta}\ge 0$ *for* $x\in[\delta-\eta,0]$, $\omega'(x)_{\eta,\delta}\le 0$ *for* $x\in[0,\delta+\eta]$, *and* $\int_{\delta-\eta}^{\delta+\eta}\omega_{\eta,\delta}(x)\mathrm{d}x=1,\ \forall\eta>0$.

We will consider solutions in a weak sense, as follows,

Definition 5.1. *Let* $\rho_0 \in (\mathbf{L^1} \cap \mathbf{BV})(\mathbb{R}; [0, \rho_{\max}])$. *We say that* $\rho \in \mathbf{C}([0, T]; \mathbf{L^1}(\mathbb{R}; [0, \rho_{\max}]))$, *with* $\rho(t, \cdot) \in \mathbf{BV}(\mathbb{R}; [0, \rho_{\max}])$ *for* $t \in [0, T]$, *is a weak solution to* (5.1.1)-(4.2.2) *if for any* $\varphi \in \mathbf{C}_c^1([0, T[\times\mathbb{R}; \mathbb{R})$

$$\begin{aligned}\int_0^T \int_{\mathbb{R}} (\rho\varphi_t + \rho V \varphi_x) \, \mathrm{d}x\mathrm{d}t + \int_0^T \int_{\Omega_{\mathrm{on}}} S_{\mathrm{on}}\varphi \mathrm{d}x\mathrm{d}t \\ - \int_0^T \int_{\Omega_{\mathrm{off}}} S_{\mathrm{off}}\varphi \mathrm{d}x\mathrm{d}t + \int_{\mathbb{R}} \rho_0(x)\varphi(0, x)\mathrm{d}x = 0,\end{aligned}$$

where $V(t, x) = v((\rho * \omega)(t, x))$.

Also, we will consider entropy weak solution in Kružkov sense, as follows,

Definition 5.2. *Let* $\rho_0 \in (\mathbf{L^1} \cap \mathbf{BV})(\mathbb{R}; [0, \rho_{\max}])$. *We say that* $\rho \in \mathbf{C}([0, T]; \mathbf{L^1}(\mathbb{R}; [0, \rho_{\max}]))$, *with* $\rho(t, \cdot) \in \mathbf{BV}(\mathbb{R}; [0, \rho_{\max}])$ *for* $t \in [0, T]$, *is a entropy weak solution to* (5.1.1) *with initial datum* ρ_0 *if for any* $\varphi \in \mathbf{C}_c^1([0, T[\times\mathbb{R}; \mathbb{R})$ *and for all* $k \in \mathbb{R}$

$$\begin{aligned}\int_0^T \int_{\mathbb{R}} (|\rho - k|\varphi_t + |\rho - k| \, V\varphi_x - \mathrm{sgn}(\rho - k)kV_x\varphi) \, \mathrm{d}x\mathrm{d}t \\ + \int_0^T \int_{\Omega_{\mathrm{on}}} \mathrm{sgn}(\rho - k)S_{\mathrm{on}}\varphi \mathrm{d}x\mathrm{d}t - \int_0^T \int_{\Omega_{\mathrm{off}}} \mathrm{sgn}(\rho - k)S_{\mathrm{off}}\varphi \mathrm{d}x\mathrm{d}t \\ + \int_{\mathbb{R}} |\rho_0 - k|\varphi(0, x)\mathrm{d}x \geq 0,\end{aligned}$$

where Ω_{on} and Ω_{off} are the spatial position of on-ramp and off-ramp on the main road, respectively.

5.3 Main Result

Before giving the main result of this chapter, we first recall the main theorem in [23, Theorem 2.1]

Theorem 5.3. *Let* $\rho_0 \in \left(\mathbf{L^1} \cap \mathbf{BV}\right)(\mathbb{R}; [0, 1])$. *Let the Assumptions 5.2.1 hold. Then, for all* $T > 0$, *the problem* (5.1.1) *has a unique solution* $\rho \in \mathbf{C}^0\left([0, T]; \mathbf{L^1}(\mathbb{R}; [0, 1])\right)$ *in the sense of Definition 4.2. Moreover, the following estimates hold: for any* $t \in [0, T]$

$$\begin{aligned}&\|\rho(t)\|_{\mathbf{L^1}(\mathbb{R})} \leq \mathcal{R}_1(t), \\ &0 \leq \rho(t, x) \leq 1, \\ &TV(\rho(t)) \leq e^{t\mathcal{H}} \left(TV(\rho_0) + t\mathcal{Q}_T\right),\end{aligned}$$

where

$$\begin{aligned}\mathcal{R}_1 = \|\rho_0\|_{\mathbf{L^1}(\mathbb{R})} + \|q_{\mathrm{on}}^{\mathrm{ramp}}(\cdot)\|_{\mathbf{L^1}([0,t])} - \min_{x \in \Omega_{\mathrm{on}}} \|q_{\mathrm{on}}^{\mathrm{ramp}}(\cdot)\rho(\cdot, x)\|_{\mathbf{L^1}([0,t])} \\ - \min_{x \in \Omega_{\mathrm{off}}} \left\|q_{\mathrm{off}}^{\mathrm{ramp}}(\cdot)\rho(\cdot, x)\right\|_{\mathbf{L^1}([0,t])},\end{aligned} \tag{5.3.1}$$

$$\begin{aligned}\mathcal{Q}_T &= 2\left(\|q_{\text{on}}\|_{\mathbf{L}^\infty([0,T])} + \|q_{\text{off}}\|_{\mathbf{L}^\infty([0,T])}\right), &(5.3.2)\\ \mathcal{H} &= 2\|q_{\text{on}}\|_{\mathbf{L}^\infty([0,T])} + \|q_{\text{off}}\|_{\mathbf{L}^\infty([0,T])} + \omega_\eta(0)\mathcal{L}, &(5.3.3)\\ \mathcal{L} &= \left(\|v\|_{\mathbf{L}^\infty([0,1])} + \|v'\|_{\mathbf{L}^\infty([0,1])}\right). &(5.3.4)\end{aligned}$$

The following theorem is the main result of this chapter and it states the $\mathbf{L^1}$- Lipschitz continuous dependence of solutions to (5.1.1) on to the initial datum, the on-ramp rate, the off-ramp rate and the kernel function.

Theorem 5.4. *Let ρ and $\tilde{\rho}$ be two solutions to problem* (5.1.1) *in the sense of Definition 4.2, with initial data $\rho_0,\ \tilde{\rho}_0 \in \mathbf{L^1} \cap \mathbf{BV}(\mathbb{R};[0,1])$, with on-ramp rates q_{on}, $\tilde{q}_{\text{on}}$, off-ramp rates q_{off}, $\tilde{q}_{\text{off}}$ and kernel functions $\omega_{\eta,\delta}$, $\tilde{\omega}_{\eta,\delta}$, respectively. Assume $v \in \mathbf{C}^2([0,1],\mathbb{R}^+)$. Then, for a.e. $t \in [0,T]$,*

$$\begin{aligned}\|\rho(t) - \tilde{\rho}(t)\|_{\mathbf{L^1}(\mathbb{R})} \leq\ & e^{\mathcal{C}T}\Bigg(\|\rho_0 - \tilde{\rho}_0\|_{\mathbf{L^1}(\mathbb{R})}\\ &+L\left(\|q_{\text{on}} - \tilde{q}_{\text{on}}\|_{\mathbf{L^1}([0,t])} + \|q_{\text{off}} - \tilde{q}_{\text{off}}\|_{\mathbf{L^1}([0,T])}\right)\\ &+r(T)\|\omega_{\eta,\delta} - \tilde{\omega}_{\eta,\delta}\|_{\mathbf{L^1}(\mathbb{R})}\Bigg),\end{aligned}$$

where $r(T)$ depends on $\mathbf{L^1}-$norms of initial conditions, expected inflow flow of the on-ramp and expected output flow of the off-ramp and $\mathcal{C}$ is defined as in [23, (3.24)].

Proof. Since ρ and $\tilde{\rho}$ are entropy weak solutions to (5.1.1) then

$$\begin{cases}\rho_t + (\rho V(t,x))_x = S_{\text{on}}(t,x,q_{\text{on}},\rho,R_{\text{on}}) - S_{\text{off}}(t,x,q_{\text{off}},\rho)\\ \rho(0,x) = \rho_0(x),\end{cases}$$

$$\begin{cases}\tilde{\rho}_t + (\tilde{\rho}\tilde{V}(t,x))_x = S_{\text{on}}\left(t,x,\tilde{q}_{\text{on}},\tilde{\rho},\tilde{R}_{\text{on}}\right) - S_{\text{off}}(t,x,\tilde{q}_{\text{off}},\tilde{\rho})\\ \tilde{\rho}(0,x) = \tilde{\rho}_0(x),\end{cases}$$

in a distributional sense, where $V(t,x) = v((\rho*\omega_\eta)(t,x))$, $R_{\text{on}} = (\rho*\omega_{\eta,\delta})(t,x)$, $\tilde{V}(t,x) = v((\tilde{\rho}*\omega_\eta)(t,x))$, $\tilde{R}_{\text{on}} = (\tilde{\rho}*\tilde{\omega}_{\eta,\delta})(t,x)$. Following the argument in [23] and using Kružkov's doubling of variables technique we get

$$\begin{aligned}&\|\rho(T,\cdot) - \tilde{\rho}(T,\cdot)\|_{\mathbf{L^1}(\mathbb{R})}\\ &\leq \|\rho_0 - \tilde{\rho}_0\|_{\mathbf{L^1}(\mathbb{R})} + \int_0^T\int_{\Omega_{\text{on}}}\left|\tilde{\mathcal{S}}_{\text{on}}\right|\mathrm{d}x\mathrm{d}t + \int_0^T\int_{\Omega_{\text{off}}}\left|\tilde{\mathcal{S}}_{\text{off}}\right|\mathrm{d}x\mathrm{d}t\\ &\quad + \int_0^T\int_{\mathbb{R}}|\mathcal{V}|\,|\rho_x(t,x)|\,\mathrm{d}x\mathrm{d}t + \int_0^T\int_{\mathbb{R}}|\mathcal{V}_x|\,|\rho(t,x)|\,\mathrm{d}x\mathrm{d}t, \qquad (5.3.5)\end{aligned}$$

where

$$\tilde{\mathcal{S}}_{\text{on}} = S_{\text{on}}(t,x,q_{\text{on}},\rho,R_{\text{on}}) - S_{\text{on}}\left(t,x,\tilde{q}_{\text{on}},\tilde{\rho},\tilde{R}_{\text{on}}\right),$$

$$\begin{aligned}\tilde{\mathcal{S}}_{\text{off}} &= S_{\text{off}}\left(t,x,q_{\text{on}},\rho\right)-S_{\text{off}}\left(t,x,\tilde{q}_{\text{on}},\tilde{\rho}\right),\\ \mathcal{V} &= v(R)-v(P),\\ \mathcal{V}_x &= v_x(R)-v_x(P).\end{aligned}$$

Except for the second term, all terms appearing at the right-hand side of (5.3.5) are computed as in [23, Theorem 3.1],

$$\begin{aligned}&\int_0^T\int_{\Omega_{\text{off}}}\left|\mathcal{S}_{\text{off}}\right|\mathrm{d}x\mathrm{d}t\\ &\quad\le \left\|q_{\text{off}}\right\|_{\mathbf{L}^\infty([0,T])}\int_0^T\left\|\rho(t,\cdot)-\tilde{\rho}(t,\cdot)\right\|_{\mathbf{L^1}(\mathbb{R})}\mathrm{d}t+L\left\|q_{\text{off}}-\tilde{q}_{\text{off}}\right\|_{\mathbf{L^1}([0,T])},\end{aligned}$$

$$\begin{aligned}&\int_0^T\int_{\mathbb{R}}\left|\mathcal{V}\right|\left|\rho_x(t,x)\right|\mathrm{d}x\mathrm{d}t\\ &\quad\le \omega_\eta(0)\left\|v'\right\|_{\mathbf{L}^\infty([0,1])}\sup_{t\in[0,T]}TV(\rho(t,\cdot))\int_0^T\left\|\rho(t,\cdot)-\tilde{\rho}(t,\cdot)\right\|_{\mathbf{L^1}(\mathbb{R})}\mathrm{d}t,\end{aligned}$$

$$\int_0^T\int_{\mathbb{R}}\left|\mathcal{V}_x\right|\left|\rho(t,x)\right|\mathrm{d}x\mathrm{d}t \;\le\; \mathcal{W}\int_0^T\left\|\rho(t,\cdot)-\tilde{\rho}(t,\cdot)\right\|_{\mathbf{L^1}(\mathbb{R})}\mathrm{d}t.$$

Regarding the second term in (5.3.5), we have

$$\begin{aligned}\int_0^T\int_{\Omega_{\text{on}}}\left|\tilde{\mathcal{S}}_{\text{on}}\right|\mathrm{d}x\mathrm{d}t &= \int_0^T\int_{\Omega_{\text{on}}}\left|S_{\text{on}}\left(t,x,q_{\text{on}},\rho,R_{\text{on}}\right)-S_{\text{on}}\left(t,x,\tilde{q}_{\text{on}},\tilde{\rho},\tilde{R}_{\text{on}}\right)\right|\mathrm{d}x\mathrm{d}t\\ &\le \int_0^T\int_{\Omega_{\text{on}}}\left(\left|\tilde{\mathcal{S}}^1_{\text{on}}\right|+\left|\tilde{\mathcal{S}}^2_{\text{on}}\right|+\left|\tilde{\mathcal{S}}^3_{\text{on}}\right|\right)\mathrm{d}x\mathrm{d}t,\end{aligned}$$

where

$$\begin{aligned}\tilde{\mathcal{S}}^1_{\text{on}} &= S_{\text{on}}\left(t,x,q_{\text{on}},\rho,R_{\text{on}}\right)-S_{\text{on}}\left(t,x,q_{\text{on}},\rho,\tilde{R}_{\text{on}}\right),\\ \tilde{\mathcal{S}}^2_{\text{on}} &= S_{\text{on}}\left(t,x,q_{\text{on}},\rho,\tilde{R}_{\text{on}}\right)-S_{\text{on}}\left(t,x,q_{\text{on}},\tilde{\rho},\tilde{R}_{\text{on}}\right),\\ \tilde{\mathcal{S}}^3_{\text{on}} &= S_{\text{on}}\left(t,x,q_{\text{on}},\tilde{\rho},\tilde{R}_{\text{on}}\right)-S_{\text{on}}\left(t,x,\tilde{q}_{\text{on}},\tilde{\rho},\tilde{R}_{\text{on}}\right).\end{aligned}$$

We bound $\tilde{\mathcal{S}}^2_{\text{on}}$ and $\tilde{\mathcal{S}}^3_{\text{on}}$ writing

$$\int_0^T\int_{\Omega_{\text{on}}}\left|\tilde{\mathcal{S}}^2_{\text{on}}\right|\mathrm{d}x\mathrm{d}t \;\le\; \left\|q_{\text{on}}\right\|_{\mathbf{L}^\infty([0,T])}\int_0^T\left\|\rho(t,\cdot)-\tilde{\rho}(t,\cdot)\right\|_{\mathbf{L^1}(\mathbb{R})}\mathrm{d}t,$$

$$\int_0^T\int_{\Omega_{\text{on}}}\left|\tilde{\mathcal{S}}^3_{\text{on}}\right|\mathrm{d}x\mathrm{d}t \;\le\; \int_0^T\int_{\Omega_{\text{on}}}\left|q_{\text{on}}-\tilde{q}_{\text{on}}\right|\mathrm{d}x\mathrm{d}t$$

$$\leq \quad L \left\| q_{\text{on}} - \tilde{q}_{\text{on}} \right\|_{\mathbf{L^1}([0,T])} .$$

Next, we are going to bound $\tilde{\mathcal{S}}^1_{\text{on}}$ term,

$$\begin{aligned}\left|\tilde{\mathcal{S}}^1_{\text{on}}\right| &= \left|\mathbf{1}_{\text{on}} q_{\text{on}} (1-\rho)\left((1-R_{\text{on}}) - \left(1-\tilde{R}_{\text{on}}\right)\right)\right| \\ &\leq \left\|q_{\text{on}}\right\|_{\mathbf{L}^\infty([0,T])} \left|\tilde{R}_{\text{on}} - R_{\text{on}}\right|,\end{aligned}$$

thus

$$\int_0^T \int_{\Omega_{\text{on}}} \left|\tilde{\mathcal{S}}^1_{\text{on}}\right| \mathrm{d}x\mathrm{d}t \leq \left\|q_{\text{on}}\right\|_{\mathbf{L}^\infty([0,T])} \int_0^T \int_{\Omega_{\text{on}}} \left|\tilde{R}_{\text{on}} - R_{\text{on}}\right| \mathrm{d}x\mathrm{d}t,$$

and observe that

$$\begin{aligned}\int_{\Omega_{\text{on}}} \left|R_{\text{on}} - \tilde{R}_{\text{on}}\right| \mathrm{d}x &= \int_{\Omega_{\text{on}}} \left|\omega_{\eta,\delta} * \rho - \tilde{\omega}_{\eta,\delta} * \tilde{\rho}\right| \mathrm{d}x \\ &= \int_{\Omega_{\text{on}}} \left|\omega_{\eta,\delta} * \rho - \omega_{\eta,\delta} * \tilde{\rho} + \omega_{\eta,\delta} * \tilde{\rho} - \tilde{\omega}_{\eta,\delta} * \tilde{\rho}\right| \mathrm{d}x \\ &\leq \int_{\Omega_{\text{on}}} \left|\omega_{\eta,\delta} * \rho - \omega_{\eta,\delta} * \tilde{\rho}\right| + \left|\omega_{\eta,\delta} * \tilde{\rho} - \tilde{\omega}_{\eta,\delta} * \tilde{\rho}\right| \mathrm{d}x \\ &= \int_{\Omega_{\text{on}}} \left|\omega_{\eta,\delta} * (\rho - \tilde{\rho})\right| + \int_{\Omega_{\text{on}}} \left|(\omega_{\eta,\delta} - \tilde{\omega}_{\eta,\delta}) * \tilde{\rho}\right| \mathrm{d}x \\ &\leq \left\|\omega_{\eta,\delta}\right\|_{\mathbf{L^1}(\Omega_{\text{on}})} \left\|\rho - \tilde{\rho}\right\|_{\mathbf{L^1}(\Omega_{\text{on}})} + \left\|\omega_{\eta,\delta} - \tilde{\omega}_{\eta,\delta}\right\|_{\mathbf{L^1}(\Omega_{\text{on}})} \left\|\tilde{\rho}\right\|_{\mathbf{L^1}(\Omega_{\text{on}})} \\ &\leq \left\|\omega_{\eta,\delta}\right\|_{\mathbf{L^1}(\mathbb{R})} \left\|\rho - \tilde{\rho}\right\|_{\mathbf{L^1}(\mathbb{R})} + \left\|\omega_{\eta,\delta} - \tilde{\omega}_{\eta,\delta}\right\|_{\mathbf{L^1}(\mathbb{R})} \left\|\tilde{\rho}\right\|_{\mathbf{L^1}(\mathbb{R})} \\ &\leq \left\|\rho - \tilde{\rho}\right\|_{\mathbf{L^1}(\mathbb{R})} + \left\|\omega_{\eta,\delta} - \tilde{\omega}_{\eta,\delta}\right\|_{\mathbf{L^1}(\mathbb{R})} \mathcal{R}_1(t),\end{aligned}$$

since $\int_{\mathbb{R}} \omega_{\eta,\delta}(x)\mathrm{d}x = 1$. Here $\mathcal{R}_1$ is defined as in (4.2.3). Then,

$$\begin{aligned}\int_0^T \int_{\Omega_{\text{on}}} \left|\tilde{\mathcal{S}}^1_{\text{on}}\right| \mathrm{d}x\mathrm{d}t &\leq \left\|q_{\text{on}}\right\|_{\mathbf{L}^\infty([0,T])} \int_0^T \left\|\rho(t,\cdot) - \tilde{\rho}(t,\cdot)\right\|_{\mathbf{L^1}(\mathbb{R})} \mathrm{d}t \\ &\quad + \int_0^T \left\|\omega_{\eta,\delta} - \tilde{\omega}_{\eta,\delta}\right\|_{\mathbf{L^1}(\mathbb{R})} \mathcal{R}_1(t)\mathrm{d}t \\ &= \left\|q_{\text{on}}\right\|_{\mathbf{L}^\infty([0,T])} \int_0^T \left\|\rho(t,\cdot) - \tilde{\rho}(t,\cdot)\right\|_{\mathbf{L^1}(\mathbb{R})} \mathrm{d}t \\ &\quad + \left\|\omega_{\eta,\delta} - \tilde{\omega}_{\eta,\delta}\right\|_{\mathbf{L^1}(\mathbb{R})} \int_0^T \mathcal{R}_1(t)\mathrm{d}t.\end{aligned}$$

Therefore, the inequality (5.3.5) is equivalent to following inequality

$$\begin{aligned}&\left\|\rho(T,\cdot) - \tilde{\rho}(T,\cdot)\right\|_{\mathbf{L^1}(\mathbb{R})} \\ &\leq \left\|\rho_0 - \tilde{\rho}_0\right\|_{\mathbf{L^1}(\mathbb{R})} + L\left(\left\|q_{\text{on}} - \tilde{q}_{\text{on}}\right\|_{\mathbf{L^1}([0,t])} + \left\|q_{\text{off}} - \tilde{q}_{\text{off}}\right\|_{\mathbf{L^1}([0,t])}\right)\end{aligned}$$

$$+\mathcal{C}\int_0^T \|\rho(t,\cdot)-\tilde{\rho}(t,\cdot)\|_{\mathbf{L^1}(\mathbb{R})}\,\mathrm{d}t+\|\omega_{\eta,\delta}-\tilde{\omega}_{\eta,\delta}\|_{\mathbf{L^1}(\mathbb{R})}\int_0^T \mathcal{R}_1(t)\mathrm{d}t$$
$$=\|\rho_0-\tilde{\rho}_0\|_{\mathbf{L^1}(\mathbb{R})}+L\left(\|q_{\text{on}}-\tilde{q}_{\text{on}}\|_{\mathbf{L^1}([0,t])}+\|q_{\text{off}}-\tilde{q}_{\text{off}}\|_{\mathbf{L^1}([0,t])}\right)$$
$$+\mathcal{C}\int_0^T \|\rho(t,\cdot)-\tilde{\rho}(t,\cdot)\|_{\mathbf{L^1}(\mathbb{R})}\,\mathrm{d}t+r(T)\,\|\omega_{\eta,\delta}-\tilde{\omega}_{\eta,\delta}\|_{\mathbf{L^1}(\mathbb{R})}\,.$$

An application of Gronwall's Lemma to the above quantity completes the proof. □

5.4 Numerical examples

In this section we use [23, Algorithm 3.1] in order to compute approximate solutions to (5.1.1)-(4.2.2) with the source term (5.2.1).

5.4.1 Example 11

We simulate an optimization problem in traffic merging, which consist in to investigate what is the optimal value to δ keeping fixed η in the kernel function $\omega_{\eta,\delta}$, it means, according to the meaning of that kernel function, where a driver should look, located on the on-ramp, in order to avoid creating more congestion on the main road. For this end, we consider one on-ramp with length $L = 0.1$, located from $x = 1.0$ until $x = 1.1$, and we consider the following kernel functions

$$\begin{aligned}
\omega_\eta(x) &:= 2\frac{\eta-x}{\eta^2}\chi_{[0,\eta]}(x),\\
\omega_{\eta,\delta}(x) &:= \frac{1}{\eta^6}\frac{16}{5\pi}\left(\eta^2-(x-\delta)^2\right)^{5/2}\chi_{[-\eta+\delta,\eta+\delta]}(x),
\end{aligned}$$

for convective and reactive terms, respectively, with $\eta = 0.5$ and $\delta \in [-\eta,\eta]$, for $x \in [-1,4]$ at simulated time $T = 6$ and velocity function given by $v(\rho) = 1-\rho$. We also consider $\Delta x = 1/200$, a constant initial condition $\rho_0(x) = 0.3$, and the rate of the on-ramp is given by $q_{\text{on}}(t) = 1.2$. Following [28], as a metric of traffic congestion we consider the two following functionals

$$J(T) = \int_0^T \mathrm{d}|\rho_x(t,\cdot)|\mathrm{d}t,$$

$$\Psi(T;a,b) = \int_0^T\int_a^b \varphi(\rho(t,x))\mathrm{d}x\mathrm{d}t,$$

where

$$\varphi(r) = \begin{cases} 0, & r < 0.75,\\ 10r-7.5, & 0.75 \le r \le 0.85,\\ 1, & 0.85 < r \le 1.\end{cases}$$

The functional J defined measures the integral with respect to time of the spatial total variation of the traffic density while the functional Ψ measures the queue of the solution in the interval $[a, b] = [-1, 4]$.
Figure 5.4.1 shows the values of the functionals J and Ψ when we vary the value of δ and keeping fixed η. We can observe that the minimum value of J and Ψ is given when $\delta = 0.1$. In Figure 5.4.2 we are comparing the solutions of (5.1.1)-(4.2.2) for different values of δ, keeping fixed η, namely, we consider $\delta \in \{-0.5, 0.1, 0.5\}$ and $\eta = 0.5$. We can see that the solution for $\delta = 0.1$ (optimal δ to Ψ) produces a smaller queue and therefore less increase in density on the main road when vehicles enter through the on-ramp than the other values considered for δ.

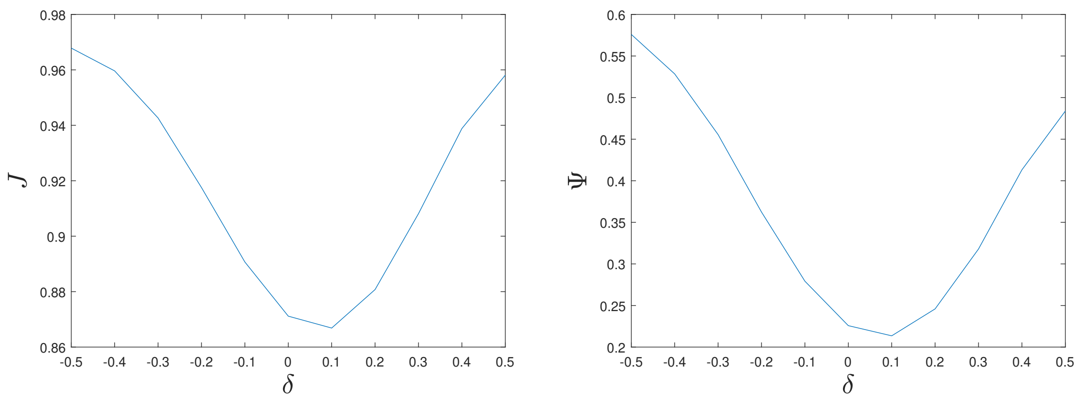

Figure 5.4.1: Left: Functional J with $\eta = 0.5$ and $\delta \in [-\eta, \eta]$. Right: Functional Ψ with $\eta = 0.5$ and $\delta \in [-\eta, \eta]$

5.5 Conclusions of Chapter 5

In this chapter we proved the stability of entropy weak solutions of a scalar nonlocal balance law with nonlocal source term arising in traffic modelling with a on-ramp introduced in [23]. We got an estimate of the dependence of the solution with respect to the kernel function in the source term, the on-ramp rate, the off-ramp rate and the initial datum. Stability was obtained from the entropy condition through doubling of variable technique. Finally, following [28], we shown a numerical simulation illustrating the dependencies above for two cost functionals derived from traffic flow applications.

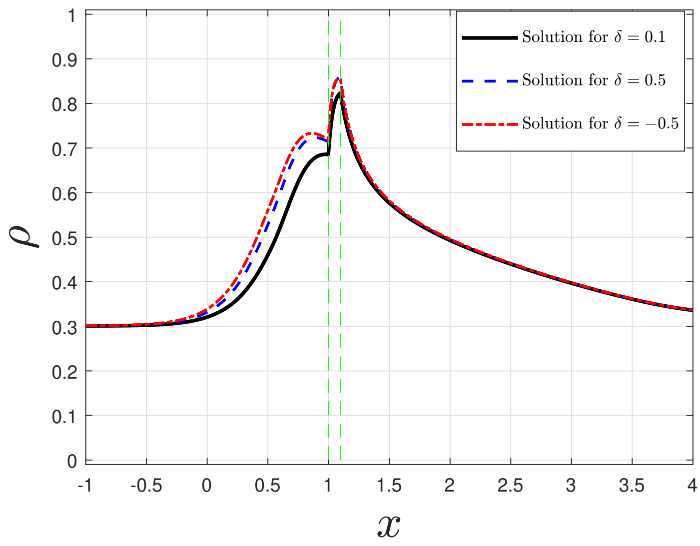

Figure 5.4.2: Solution of (5.1.1)-(4.2.2) for $\eta = 0.5$ and varying $\delta \in \{-0.5, 0.1, 0.5\}$

Chapter 6

Two-way nonlocal traffic model

6.1 Introduction

6.1.1 Motivation

Our main objective is to model vehicular traffic flow on a two-lane and two-way road where drivers have a preferred lane, the lane on their right, and the left one is used only for overtaking slower vehicles, see Figure 6.1.1. Lanes are labeled lane 1 and lane 2 and we denote by ρ_1 and ρ_2 the density of cars traveling from the left to right on the lane 1 and lane 2, respectively; by $\tilde{\rho}_1$ and $\tilde{\rho}_2$ the density of cars traveling from the right to left on the lane 2 and lane 1, respectively. In order to extend the classical LWR (Lighthill - Whitham [66] and Richards [72]) traffic model to a two-lane two-way road where overtaking of cars is allowed, we should consider that the velocity in each lane depends not only on the density of the preferred class, but also on the density of the other class that comes in the opposite direction making overtake, which leaves the following model

$$\begin{cases} \partial_t \rho_1 + \partial_x(\rho_1 v_1(\rho_1 + (\rho_{\max} - \rho_1)\chi_\varepsilon(\tilde{\rho}_2 * \omega_\eta))) &= 0, \\ \partial_t \rho_2 + \partial_x(\rho_2 v_2(\rho_2 + (\rho_{\max} - \rho_2)\chi_\varepsilon(\tilde{\rho}_1 * \omega_\eta))) &= 0, \\ \partial_t \tilde{\rho}_1 - \partial_x(\tilde{\rho}_1 v_1(\tilde{\rho}_1 + (\rho_{\max} - \tilde{\rho}_1)\chi_\varepsilon(\rho_2 * \hat{\omega}_\eta))) &= 0, \\ \partial_t \tilde{\rho}_2 - \partial_x(\tilde{\rho}_2 v_2(\tilde{\rho}_2 + (\rho_{\max} - \tilde{\rho}_2)\chi_\varepsilon(\rho_1 * \hat{\omega}_\eta))) &= 0, \end{cases} \tag{6.1.1}$$

where $\chi_\varepsilon(\cdot)$ is a regularization of the indicator function, this term models the fact that the vehicles must slow down in the presence of vehicles downstream in the same lane but traveling in opposite direction. Regularizing the indicator function in the flux functions of this model becomes necessary since if we consider the indicator function without regularization, the velocity functions of (6.1.1) become discontinuous, e.g. if $v(\rho_1, \tilde{\rho}_2) := v_1(\rho_1 + (\rho_{\max} - \rho_1)\chi(\tilde{\rho}_2 * \omega_\eta)))$, then we have $v(\rho_1, 0) = v_1(\rho_1) > 0$ and $v(\rho_1, \tilde{\rho}_2) = v_1(\rho_{\max}) = 0$ for $\tilde{\rho}_2 > 0$.

Now, in order to model overtaking and returning maneuvers we endow (6.1.1) with source terms which are defined following some rules explain now. We impose the following rules in order to vehicles can overtake, i.e., at the position x, a vehicle can overtake if and only if the following condition are fulfilled,

1. the particular velocity $v_i(\rho_i)$ is greater than the velocity of the average of cars in front of it;

2. there are no vehicles traveling in opposite direction on the other lane and neither on the same lane (overtaking from the other lane),

which is expressed mathematically as follows,

$$S_{\mathrm{O}}(\rho_1,\rho_2,\tilde{\rho}_1,\tilde{\rho}_2)=\begin{cases}K_1(\rho_{\max}-\rho_2)\rho_1, & \text{if } \ v_1(\omega_\eta^1*\rho_1)<v_1(\rho_1), \quad \omega_\delta^2*\tilde{\rho}_1=0 \\ & \text{and } \ \omega_\delta^2*\tilde{\rho}_2=0, \\ & \\ 0, & \text{otherwise,}\end{cases} \tag{6.1.2}$$

here we have used the notation in [43, 56] in order to describe the lane change, where $K_1>0$ is a constant. At position x, the nonlocal terms $\omega_\eta^1*\rho_1$ and $\omega_\delta^2*\tilde{\rho}_i$, $i=1,2$, describe the average of cars traveling in the same direction, in front of the drivers on the lane 1, and average of cars traveling in opposite direction, respectively; we also assume that $\delta>\eta$. In addition, we enforce the following rule for overtaking vehicles can return to the preferred lane,

(3) the particular velocity $v_i(\rho_i)$ is less than the velocity of the average of cars in front of them,

this condition is imposed in the sense that vehicles returning to the preferred lane do not instantly overtake again. This condition can be formulated as

$$S_{\mathrm{R}}(\rho_1,\rho_2,\tilde{\rho}_1,\tilde{\rho}_2)=\begin{cases}K_2(\rho_{\max}-\rho_1)\rho_2, & \text{if } \ v_1(\omega_\eta^1*\rho_1)>v_1(\rho_1), \quad \text{or} \quad \omega_\delta^2*\tilde{\rho}_1>0 \\ & \text{or } \ \omega_\delta^2*\tilde{\rho}_2>0, \\ & \\ 0, & \text{otherwise,}\end{cases} \tag{6.1.3}$$

where $K_2>0$ is a constant. Likewise, we define the terms $\tilde{S}_{\mathrm{O}}(\tilde{\rho}_1,\tilde{\rho}_2,\rho_1,\rho_2)$ and $\tilde{S}_{\mathrm{R}}(\tilde{\rho}_1,\tilde{\rho}_2,\rho_1,\rho_2)$ as follows

$$\tilde{S}_{\mathrm{O}}(\tilde{\rho}_1,\tilde{\rho}_2,\rho_1,\rho_2)=\begin{cases}K_1(\rho_{\max}-\tilde{\rho}_2)\tilde{\rho}_1, & \text{if } \ v_2(\hat{\omega}_\eta^1*\tilde{\rho}_1)<v_2(\tilde{\rho}_1), \quad \hat{\omega}_\delta^2*\rho_1=0 \\ & \text{and } \ \hat{\omega}_\delta^2*\rho_2=0, \\ & \\ 0, & \text{otherwise,}\end{cases} \tag{6.1.4}$$

$$\tilde{S}_{\mathrm{R}}(\tilde{\rho}_1,\tilde{\rho}_2,\rho_1,\rho_2) = \begin{cases} K_2(\rho_{\max}-\tilde{\rho}_1)\tilde{\rho}_2, & \text{if } \; v_2(\hat{\omega}^1_\eta * \tilde{\rho}_1) > v_2(\tilde{\rho}_1), \quad \text{or} \quad \hat{\omega}^2_\delta * \rho_1 > 0 \\ & \text{or} \quad \hat{\omega}^2_\delta * \rho_2 > 0, \\ 0, & \text{otherwise}, \end{cases} \tag{6.1.5}$$

where $\hat{\omega}^1_\eta(x) := \omega^1_\eta(-x)$, for all $x \in \{-\eta,\dots,0\}$ and $\hat{\omega}^2_\delta(x) := \omega^2_\delta(-x)$ for all $x \in \{-\delta,\dots,0\}$. At position x, the nonlocal terms $\hat{\omega}^1_\eta * \tilde{\rho}_1$ and $\hat{\omega}^2_\delta * \rho_i$, $i = 1, 2$, describe the average of cars, in the same direction, in front of the drivers on the lane 2, and average of cars traveling in opposite direction, respectively.

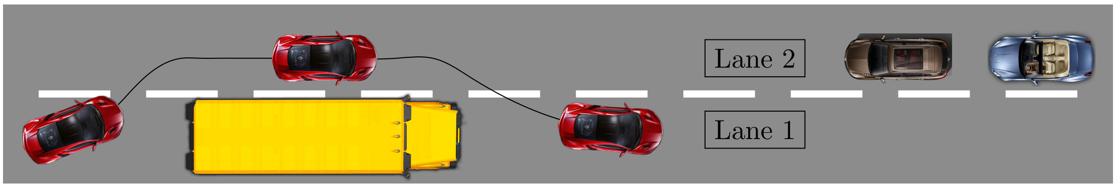

Figure 6.1.1: Illustration of the model setting. The red car overtakes a slower vehicle, the bus, using the lane 2 and then return to their preferred lane, the lane 1.

6.1.2 Related Work

Macroscopic models of vehicular traffic flow with nonlocal fluxes have been extensively studied recently [12, 14, 20, 23, 21, 26, 51]. In this kind of models, the velocity function depends on a weighted mean of the downstream traffic density and their importance lies in the fact that it allows describe traffic flow dynamics in which drivers adapt their velocity with respect to what happens to the cars in front of them. Most of work mentioned above consider a one-directional road with a class of vehicles, but since it is necessary to model more realistic situations, the study on nonlocal models has been extended to multi-class and multilane settings; for example, in [26] is studied a system of nonlocal conservation laws that model multi-class traffic flow for which the authors proved the existence of weak solutions for small times, this solutions are approximate by means of a Godunov type scheme. Holden and Risebro [56] proposed a (local) weakly coupled system of hyperbolic conservation laws with a source term in order to model the vehicular traffic flow on a road with multiple lanes where the velocity depends only on the density in the same lane. In that model it is assumed that the tendency of drivers to change to a neighboring lane is proportional to the difference in velocity between lanes; the authors proved some bounds for the solutions of the model and show the convergence to a weak solution. In [43] the authors proposed and studied a multilane traffic model based on nonlocal balance laws where the nonlocal source term was used to describe the lane change rate. In that paper also was proposed a Godunov type scheme in order to approximate the solutions of the model and was proved compactness estimates in order to show the well posedness of

model. More recently, in [27] it has been proposed a system of conservation laws with nonlocal fluxes, coupled in the velocity functions, in which is described two populations moving in opposite directions and the authors proved existence of weak solutions for sufficiently small times.
We remark that this model differs from the nonlocal multilane model presented in [43] in several aspects, namely, in [43] the authors consider a one-way multilane model only, thus the velocity on a lane depends only on density of vehicles in the same lane traveling in the same direction, while we consider a two-way model in which is allowed to overtake, that leads to a velocity function that depends not only on the preferred class but also of the classes traveling in opposite direction on the same lane. Likewise, in [43] the source terms take into account a nonlocal evaluation of the velocity influencing the lane changing rate; in our proposed model, instead, in the source term the criteria of overtaking and returning are defined in a nonlocal form, only.

6.1.3 Outline of the chapter

This chapter is organized as follows: In Section 6.2 we present the proposed mathematical model with all the considered assumptions on it. Afterwards, in Section 6.3 we introduce a HW-type numerical scheme and derive important properties such as positivity of approximate solutions, $\mathbf{L}^{\infty}-$ bound, $\mathbf{L^1}-$ bound and $\mathbf{BV}$ estimates in order to show the convergence of approximate solutions to a weak solution of the proposed model. In the Section 6.4, we present numerical examples illustrating the behavior of the solutions of our model.

6.2 Mathematical model

The main goal of this chapter is to study the well-posedness of the nonlocal system of equations

$$\begin{cases} \partial_t\rho_1 + \partial_x(\rho_1 v_1(\rho_1 + (\rho_{\max} - \rho_1)\chi_\varepsilon(\tilde{\rho}_2 * \omega_\eta))) = -S_{\mathrm{O}}(\rho_1,\rho_2,\tilde{\rho}_1,\tilde{\rho}_2) + S_{\mathrm{R}}(\rho_1,\rho_2,\tilde{\rho}_1,\tilde{\rho}_2) \\ \partial_t\rho_2 + \partial_x(\rho_2 v_2(\rho_2 + (\rho_{\max} - \rho_2)\chi_\varepsilon(\tilde{\rho}_1 * \omega_\eta))) = S_{\mathrm{O}}(\rho_1,\rho_2,\tilde{\rho}_1,\tilde{\rho}_2) - S_{\mathrm{R}}(\rho_1,\rho_2,\tilde{\rho}_1,\tilde{\rho}_2) \\ \partial_t\tilde{\rho}_1 - \partial_x(\tilde{\rho}_1 v_1(\tilde{\rho}_1 + (\rho_{\max} - \tilde{\rho}_1)\chi_\varepsilon(\rho_2 * \hat{\omega}_\eta))) = -\tilde{S}_{\mathrm{O}}(\tilde{\rho}_1,\tilde{\rho}_2,\rho_1,\rho_2) + \tilde{S}_{\mathrm{R}}(\tilde{\rho}_1,\tilde{\rho}_2,\rho_1,\rho_2) \\ \partial_t\tilde{\rho}_2 - \partial_x(\tilde{\rho}_2 v_2(\tilde{\rho}_2 + (\rho_{\max} - \tilde{\rho}_2)\chi_\varepsilon(\rho_1 * \hat{\omega}_\eta))) = \tilde{S}_{\mathrm{O}}(\tilde{\rho}_1,\tilde{\rho}_2,\rho_1,\rho_2) - \tilde{S}_{\mathrm{R}}(\tilde{\rho}_1,\tilde{\rho}_2,\rho_1,\rho_2), \end{cases} \tag{6.2.1}$$

where $\boldsymbol{\rho}_i = \left(\rho_i, \rho_{[i]}\right)$ and $\tilde{\boldsymbol{\rho}}_i = \left(\tilde{\rho}_i, \tilde{\rho}_{[i]}\right)$, for $i = 1, 2$, take values in the set

$$\Omega = \{\boldsymbol{\rho_i}, \tilde{\boldsymbol{\rho}}_i \in \mathbb{R}^2 : (0,0) \le \boldsymbol{\rho_i}, \tilde{\boldsymbol{\rho}}_i \le (\rho_{\max}, \rho_{\max}), \text{ for } i = 1, 2\},$$

and the notation $[\cdot]$, means $[1] = 2$ and $[2] = 1$. Next, the convolution terms in fluxes are defined as follows for $i = 1, 2$,

$$\tilde{\rho}_i * \omega_\eta \quad := \quad \int_x^{x+\eta} \omega_\eta(y - x)\tilde{\rho}_i(t, y)\mathrm{d}y,$$

$$\rho_i * \hat{\omega}_\eta \quad := \quad \int_{x-\eta}^{x} \hat{\omega}_\eta(y-x)\rho_i(t,y)\mathrm{d}y,$$

where $\hat{\omega}_\eta(x) = \omega_\eta(-x)$. The initial conditions satisfies

$$\begin{array}{ll} \rho_1(x,0) = \rho_1^0(x) \in (\mathbf{L^1} \cap \mathbf{BV})(\mathbb{R};[0,\rho_{\max}]), & \rho_2(x,0) = 0, \\ \tilde{\rho}_1(x,0) = \tilde{\rho}_1^0(x) \in (\mathbf{L^1} \cap \mathbf{BV})(\mathbb{R};[0,\rho_{\max}]), & \tilde{\rho}_2(x,0) = 0, \end{array} \tag{6.2.2}$$

where $\rho_2(x,0) = \tilde{\rho}_2(x,0) = 0$ means that there is no overtake initially. In addition, we consider the following assumptions.

Assumptions 6.2.1. *The nonlocal problem* (6.2.1) *is studied under the following assumptions:*

(i) $v_1, v_2 \in \mathbf{C}^1([0,1];\mathbb{R}^+)$, with $v_1'(\rho) \leq 0$, $v_2'(\rho) \leq 0$, $\rho \in [0,1]$.

(ii) $\omega_\eta^1 \in \mathbf{C}_c^1([0,\eta];\mathbb{R}^+)$ with $(\omega^1)_\eta'(x) \leq 0$, $\displaystyle\int_0^\eta \omega_\eta^1(x)\mathrm{d}x = 1$, $\forall \eta > 0$.

(iii) $\omega_\delta^2 \in \mathbf{C}_c^1([0,\delta];\mathbb{R}^+)$ with $(\omega^2)_\delta'(x) \leq 0$, $\displaystyle\int_0^\delta \omega_\delta^2(x)\mathrm{d}x = 1$, $\forall \delta > 0$.

(iv) $\mathrm{supp}(\omega_\eta^1) \subset \mathrm{supp}(\omega_\delta^2)$, *i.e,* $\delta > \eta$.

(v) $\hat{\omega}_\eta^1 \in \mathbf{C}_c^1([0,\eta];\mathbb{R}^+)$ with $(\hat{\omega}^1)_\eta'(x) \geq 0$, $\displaystyle\int_{-\eta}^0 \hat{\omega}_\eta^1(x)\mathrm{d}x = 1$, $\forall \eta > 0$.

(vi) $\hat{\omega}_\delta^2 \in \mathbf{C}_c^1([-\delta,0];\mathbb{R}^+)$ with $(\hat{\omega}^2)_\delta'(x) \geq 0$, $\displaystyle\int_{-\delta}^0 \hat{\omega}_\delta^2(x)\mathrm{d}x = 1$, $\forall \delta > 0$.

Solutions for (6.2.1)-(6.2.2) are intended in the following weak sense,

Definition 6.1 (Weak solution). *Let* $\rho_i^0, \tilde{\rho}_i^0 \in \left(\mathbf{L^1} \cap \mathbf{BV}\right)(\mathbb{R};[0,1])$, *for* $i = 1,2$. *We say that* $\rho_i, \tilde{\rho}_i \in \mathbf{C}^0\left([0,T];\mathbf{L^1}(\mathbb{R};[0,1])\right)$, *with* $\rho_i(t,\cdot) \in \mathbf{BV}(\mathbb{R};[0,1])$ *for* $t \in [0,T]$ *and* $i = 1,2$, *is a weak solution to* (6.2.1) *with initial data* $\rho_i^0, \tilde{\rho}_i^0$, $i = 1,2$, *if for any* $\varphi \in \mathbf{C}_c^1([0,T[\times\mathbb{R};\mathbb{R})$

$$\int_0^T \int_{\mathbb{R}} \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} \varphi_t \,\mathrm{d}x\mathrm{d}t + \int_0^T \int_{\mathbb{R}} \begin{pmatrix} \rho_1 v_1(\rho_1 + (\rho_{\max} - \rho_1)\chi_\varepsilon(\tilde{\rho}_2 * \omega_\eta)) \\ \rho_2 v_2(\rho_2 + (\rho_{\max} - \rho_2)\chi_\varepsilon(\tilde{\rho}_1 * \omega_\eta)) \end{pmatrix} \varphi_x \,\mathrm{d}x\mathrm{d}t$$
$$+ \int_0^T \int_{\mathbb{R}} \begin{pmatrix} -(S_{\mathrm{O}} - S_{\mathrm{R}}) \\ S_{\mathrm{O}} - S_{\mathrm{R}} \end{pmatrix} \varphi \,\mathrm{d}x\mathrm{d}t + \int_{\mathbb{R}} \varphi(0,x) \begin{pmatrix} \rho_1(0,x) \\ \rho_2(0,x) \end{pmatrix} \mathrm{d}x = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

and

$$\int_0^T \int_{\mathbb{R}} \begin{pmatrix} \tilde{\rho}_1 \\ \tilde{\rho}_2 \end{pmatrix} \varphi_t \,\mathrm{d}x\mathrm{d}t - \int_0^T \int_{\mathbb{R}} \begin{pmatrix} \tilde{\rho}_1 v_2(\tilde{\rho}_1 + (\rho_{\max} - \tilde{\rho}_1)\chi_\varepsilon(\rho_2 * \hat{\omega}_\eta)) \\ \tilde{\rho}_2 v_2(\tilde{\rho}_2 + (\rho_{\max} - \tilde{\rho}_2)\chi_\varepsilon(\rho_1 * \hat{\omega}_\eta)) \end{pmatrix} \varphi_x \,\mathrm{d}x\mathrm{d}t$$

$$+\int_0^T\int_{\mathbb{R}}\left(\begin{array}{c}-\left(\tilde{S}_{\mathrm{O}}-\tilde{S}_{\mathrm{R}}\right)\\ \tilde{S}_{\mathrm{O}}-\tilde{S}_{\mathrm{R}}\end{array}\right)\varphi\,\mathrm{d}x\mathrm{d}t+\int_{\mathbb{R}}\varphi(0,x)\left(\begin{array}{c}\tilde{\rho}_1(0,x)\\ \tilde{\rho}_2(0,x)\end{array}\right)\mathrm{d}x=\left(\begin{array}{c}0\\ 0\end{array}\right).$$

Our main result is given by the following theorem, which states existence of solutions to problem (6.2.1) - (6.2.2).

Theorem 6.2. *Let $\rho_i^0, \tilde{\rho}_i^0 \in (\mathbf{L}^\infty \cap \mathbf{BV})(\mathbb{R};\mathbb{R}^+)$ for $i = 1, 2$ and the Assumptions 6.2.1 hold. Then, for all $T > 0$, the problem* (6.2.1) *admits a weak solution on $[0, T] \times \mathbb{R}$ in the sense of the Definition 6.1.*

In order to prove the Theorem 6.2 we first propose a numerical scheme in the sense of finite volume method along with a operator splitting whereby we derive some important properties of model as well as compactness estimates that will allow us to use the Helly's Compactness Theorem.

6.3 Numerical scheme

6.3.1 Discretization of the model

We take a uniform space step Δx and a time step Δt subject to a Courant-Friedrichs-Levy (CFL) condition which will be specified later. For any $j \in \mathbb{Z}$, let $x_{j+1/2} = (j + 1/2)\Delta x$ be a cells interfaces and $x_j = j\Delta x$ the cells centers. We fix $T > 0$, and set $N_T \in \mathbb{N}$ such that $N_T\Delta t \leq T < (N_T + 1)\Delta t$ and define the time mesh as $t^n = n\Delta t$, for $n = 0, \ldots, N_T$. The initial data are approximated, for $j \in \mathbb{Z}$ and $i = 1, 2$ as follows

$$\rho_{i,j}^0 = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} \rho_i^0(x)\mathrm{d}x, \quad \text{and} \quad \tilde{\rho}_{i,j}^0 = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} \tilde{\rho}_i^0(x)\mathrm{d}x.$$

We denote

$$\omega_\eta^k := \int_{k\Delta x}^{(k+1)\Delta x} \omega_\eta(y)\mathrm{d}y \text{ for } k = 0, \ldots, N-1,$$

$$\hat{\omega}_\eta^k := \int_{(-k-1)\Delta x}^{-k\Delta x} \hat{\omega}_\eta(y)\mathrm{d}y \text{ for } k = 0, \ldots, N-1$$

and set the convolution term, for $i = 1, 2$

$$\tilde{R}_{[i]}(x_{j+1/2}, t^n) = (\omega_\eta * \tilde{\rho}_{[i],j})(x_{j+1/2}, t^n) \approx \sum_{k=0}^{N-1} \omega_\eta^k \tilde{\rho}_{[i],j+k+1}^n,$$

$$R_i(x_{j+1/2}, t^n) = (\hat{\omega}_\eta * \rho_{i,j})(x_{j+1/2}, t^n) \approx \sum_{k=0}^{N-1} \hat{\omega}_\eta^k \rho_{i,j-k}^n.$$

Likewise, we define a piecewise constant approximate solution
$\boldsymbol{\rho}^{\boldsymbol{\Delta}}(t,x) = (\rho_1^\Delta(t,x), \rho_2^\Delta(t,x))$ and $\tilde{\boldsymbol{\rho}}^{\boldsymbol{\Delta}}(t,x) = (\tilde{\rho}_1^\Delta(t,x), \tilde{\rho}_2^\Delta(t,x))$ as

$$\rho_i^\Delta(t,x) = \rho_{i,j}^n \text{ and } \tilde{\rho}_i^\Delta(t,x) = \tilde{\rho}_{i,j}^n, \quad (t,x) \in [t^n, t^{n+1}[\times]x_{j-1/2}, x_{j+1/2}], \quad i = 1, 2. \tag{6.3.1}$$

The convective terms in (6.2.1) are obtained via a finite volume Hilliges-Weidlich-type (HW) scheme [15, 55] defined by

$$F_{i,j+1/2} := \rho^n_{i,j} v_1(\rho^n_{i,j+1} + (\rho_{\max} - \rho^n_{i,j+1})\chi_\varepsilon(\tilde{R}^n_{[i],j+1/2})), \tag{6.3.2}$$
$$G_{[i],j+1/2} := \tilde{\rho}^n_{[i],j+1} v_2(\tilde{\rho}^n_{[i],j} + (\rho_{\max} - \tilde{\rho}^n_{[i],j})\chi_\varepsilon(R^n_{i,j+1/2})), \tag{6.3.3}$$

where $\tilde{R}_{[i],j+1/2} := \sum_{k=0}^{N-1} \omega^k_\eta \tilde{\rho}^n_{[i],j+k+1}$ and $R_{i,j+1/2} := \sum_{k=0}^{N-1} \hat{\omega}^k_\eta \rho^n_{i,j-k}$. Then, we put $\boldsymbol{F}^n_{j+1/2} = [F^n_{1,j+1/2}, F^n_{2,j+1/2}]$ and $\boldsymbol{G}^n_{j+1/2} = [G^n_{1,j+1/2}, G^n_{2,j+1/2}]$.

In order to compute the source terms in (6.2.1), we first introduce the following notations for the convolutions terms for kernel functions $\omega^1_\eta \in \mathbf{C}^1_c([0,\eta])$, $\hat{\omega}^1_\eta \in \mathbf{C}^1_c([-\eta,0])$ satisfying the Assumptions 6.2.1 for some $N_1 \in \mathbb{N}$ such that $\eta = \Delta x N_1$ and any piecewise constant function u^Δ

$$R_1(u^\Delta)^n_j := (\omega^1_\eta * u^\Delta)(x_j, t^n) = \int_{x_j}^{x_j+\eta} \omega^1_\eta(y - x_j) u^\Delta(t,y)\mathrm{d}y \approx \sum_{k=0}^{N_1} \gamma_k u^n_{j+k},$$

with coefficients

$$\gamma_0 = \int_0^{\Delta x/2} \omega^1_\eta(y)\mathrm{d}y, \tag{6.3.4}$$
$$\gamma_k = \int_{(k-1/2)\Delta x}^{(k+1/2)\Delta x} \omega^1_\eta(y)\mathrm{d}y, \ \textit{for } k = 1, \dots, N_1 - 1, \tag{6.3.5}$$
$$\gamma_{N_1} = \int_{\eta-\Delta x/2}^{\eta} \omega^1_\eta(y)\mathrm{d}y, \tag{6.3.6}$$

and we also define

$$\tilde{R}_1(u^\Delta)^n_j := (\hat{\omega}^1_\eta * u^\Delta)(x_j, t^n) = \int_{x_j-\eta}^{x_j} \hat{\omega}^1_\eta(x_j - y) u^\Delta(t,y)\mathrm{d}y \approx \sum_{k=0}^{N_1} \tilde{\gamma}_k u^n_{j+k},$$

with coefficients

$$\tilde{\gamma}_0 = \int_{-\Delta x/2}^{0} \hat{\omega}^1_\eta(y)\mathrm{d}y, \tag{6.3.7}$$
$$\tilde{\gamma}_i = \int_{-(k+1/2)\Delta x}^{-(k-1/2)\Delta x} \hat{\omega}^1_\eta(y)\mathrm{d}y, \ \textit{for } k = 1, \dots, N_1 - 1; \tag{6.3.8}$$
$$\tilde{\gamma}_{N_1} = \int_{-\eta}^{-(\eta-\Delta x/2)} \hat{\omega}^1_\eta(y)\mathrm{d}y. \tag{6.3.9}$$

Similarly, for $\omega^2_\delta \in \mathbf{C}^1_c([0,\delta])$ and $\hat{\omega}^2_\delta \in \mathbf{C}^1_c([-\delta,0])$ satisfying the Assumptions 6.2.1 for some $N_2 \in \mathbb{N}$ such that $\delta = \Delta x N_2$

$$R_2(u^\Delta)^n_j := (\omega^2_\delta * u^\Delta)(x_j, t^n) = \int_{x_j}^{x_j+\delta} \omega^2_\delta(y - x_j) u^\Delta(t,y)\mathrm{d}y \approx \sum_{k=0}^{N_2} \zeta_k u^n_{j+k},$$

with coefficients ζ_k as in (6.3.4), (6.3.5), (6.3.6) and likewise we define

$$\tilde{R}_2(u^{\Delta})^n_j := (\hat{\omega}^2_\delta * u^{\Delta})(x_j, t^n) = \int_{x_j-\delta}^{x_j} \hat{\omega}^2_\delta(x_j - y) u^{\Delta}(t, y) \mathrm{d}y \approx \sum_{k=0}^{N_2} \tilde{\zeta}_k u^n_{j+k}, \tag{6.3.10}$$

with coefficients $\tilde{\zeta}_k$ as in (6.3.7), (6.3.8) and (6.3.9). Finally for $j \in \mathbb{Z}$ and $n \in \mathbb{N}$ we can compute the source terms (6.1.2) and (6.1.3) as

$$S_{\mathrm{O}}(\boldsymbol{\rho}^{\Delta}, \tilde{\boldsymbol{\rho}}^{\Delta})_j = \begin{cases} K_1(\rho_{\max} - \rho^n_{2,j})\rho^n_{1,j}, & \text{if, } v_1(R_1(\rho_1^{\Delta})^n_j) < v_1(\rho^n_{1,j}), \quad R_2(\tilde{\rho}_1^{\Delta})^n_j = 0 \\ & \text{and } \quad R_2(\tilde{\rho}_2^{\Delta})^n_j = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{6.3.11}$$

$$S_{\mathrm{R}}(\boldsymbol{\rho}^{\Delta}, \tilde{\boldsymbol{\rho}}^{\Delta})_j = \begin{cases} K_2(\rho_{\max} - \rho^n_{1,j})\rho^n_{2,j}, & \text{if, } v_1(R_1(\rho_1^{\Delta})^n_j) > v_1(\rho^n_{1,j}), \text{ or } R_2(\tilde{\rho}_1^{\Delta})^n_j > 0 \\ & \text{or } R_2(\tilde{\rho}_2^{\Delta})^n_j > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{6.3.12}$$

In the same way, we can compute the source terms (6.1.4) and (6.1.5) as follows

$$\tilde{S}_{\mathrm{O}}(\tilde{\boldsymbol{\rho}}^{\Delta}, \boldsymbol{\rho}^{\Delta})_j = \begin{cases} K_1(\rho_{\max} - \tilde{\rho}^n_{2,j})\tilde{\rho}^n_{1,j}, & \text{if, } v_2(\tilde{R}_1(\tilde{\rho}_1^{\Delta})^n_j) < v_2(\tilde{\rho}^n_{1,j}), \quad \tilde{R}_2(\rho_1^{\Delta})^n_j = 0 \\ & \text{and } \quad \tilde{R}_2(\rho_2^{\Delta})^n_j = 0, \\ 0 & \text{otherwise,} \end{cases} \tag{6.3.13}$$

$$\tilde{S}_{\mathrm{R}}(\tilde{\boldsymbol{\rho}}^{\Delta}, \boldsymbol{\rho}^{\Delta})_j = \begin{cases} K_2(\rho_{\max} - \tilde{\rho}^n_{1,j})\tilde{\rho}^n_{2,j}, & \text{if, } v_2(\tilde{R}_1(\tilde{\rho}_1^{\Delta})^n_j) > v_1(\tilde{\rho}^n_{1,j}), \text{ or } \tilde{R}_2(\rho_1^{\Delta})^n_j > 0 \\ & \text{or } \tilde{R}_2(\rho_2^{\Delta})^n_j > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{6.3.14}$$

The values $\boldsymbol{\rho}^n_j = (\rho^n_{1,j}, \rho^n_{2,j})$ and $\tilde{\boldsymbol{\rho}}^n_j = (\tilde{\rho}^n_{1,j}, \tilde{\rho}^n_{2,j})$ are update by using Algorithm 6.3.1 composed of HW type scheme together with operator splitting, to account for the source terms.

Algorithm 6.3.1.

Input: approximate solution vectors $\boldsymbol{\rho}^n_j = (\rho^n_{1,j}, \rho^n_{2,j})$ and $\tilde{\boldsymbol{\rho}}^n_j = (\tilde{\rho}^n_{1,j}, \tilde{\rho}^n_{2,j})$ for $j \in \mathbb{Z}$ and $t = t^n$

do $j \in \mathbb{Z}$,

$$\boldsymbol{\rho}^{n+1/2}_j \leftarrow \boldsymbol{\rho}^n_j - \lambda\big(\boldsymbol{F}^n_{j+1/2} - \boldsymbol{F}^n_{j-1/2}\big), \tag{6.3.15}$$

$$\tilde{\boldsymbol{\rho}}^{n+1/2}_j \leftarrow \tilde{\boldsymbol{\rho}}^n_j + \lambda\big(\boldsymbol{G}^n_{j+1/2} - \boldsymbol{G}^n_{j-1/2}\big), \tag{6.3.16}$$

enddo

do $j \in \mathbb{Z}$,

$$S_j^{n+1/2} \leftarrow S_{\mathrm{O}}\left(\boldsymbol{\rho}^{n+1/2}, \tilde{\boldsymbol{\rho}}^{n+1/2}\right)_j - S_{\mathrm{R}}\left(\boldsymbol{\rho}^{n+1/2}, \tilde{\boldsymbol{\rho}}^{n+1/2}\right)_j, \tag{6.3.17}$$

$$\tilde{S}_j^{n+1/2} \leftarrow \tilde{S}_{\mathrm{O}}\left(\tilde{\boldsymbol{\rho}}^{n+1/2}, \boldsymbol{\rho}^{n+1/2}\right)_j - \tilde{S}_{\mathrm{R}}\left(\tilde{\boldsymbol{\rho}}^{n+1/2}, \boldsymbol{\rho}^{n+1/2}\right)_j, \tag{6.3.18}$$

$$\boldsymbol{\rho}_j^{n+1} \leftarrow \boldsymbol{\rho}_j^{n+1/2} + \Delta t[-S_j^{n+1/2}, S_j^{n+1/2}], \tag{6.3.19}$$

$$\tilde{\boldsymbol{\rho}}_j^{n+1} \leftarrow \tilde{\boldsymbol{\rho}}_j^{n+1/2} + \Delta t[-\tilde{S}_j^{n+1/2}, \tilde{S}_j^{n+1/2}]. \tag{6.3.20}$$

enddo

Output: approximate solution vectors $\boldsymbol{\rho}_j^{n+1} = (\rho_{1,j}^{n+1}, \rho_{2,j}^{n+1})$ and $\tilde{\boldsymbol{\rho}}_j^{n+1} = (\tilde{\rho}_{1,j}^{n+1}, \tilde{\rho}_{2,j}^{n+1})$ for $j \in \mathbb{Z}$ and $t = t^{n+1} = t^n + \Delta t$.

Notation: From now on, the conditions for overtaking or returning will be denoted as follows

- $S_{\mathrm{O}} > 0$ if $v_1(R_1(\rho_1^{\Delta})_j^n) < v_1(\rho_{1,j}^n)$, $\quad R_2(\tilde{\rho}_1^{\Delta})_j^n = 0$ and $\quad R_2(\tilde{\rho}_2^{\Delta})_j^n = 0$ and $S_{\mathrm{R}} > 0$ otherwise.
- $\tilde{S}_{\mathrm{O}} > 0$ if $v_2(\tilde{R}_1(\tilde{\rho}_1^{\Delta})_j^n) < v_2(\tilde{\rho}_{1,j}^n)$, $\quad \tilde{R}_2(\rho_1^{\Delta})_j^n = 0$ and $\quad \tilde{R}_2(\rho_2^{\Delta})_j^n = 0$, and $\tilde{S}_{\mathrm{R}} > 0$ otherwise.

Next, in the following lemma we present some properties of the discrete source terms, which will be useful later.

Lemma 6.3. *Let* $u_{i,j}^n, \tilde{u}_{i,j}^n \in [0,1]$, *for all* $j \in \mathbb{Z}$ *and* $i = 1,2$. *Consider the terms* $S_{\mathrm{O}}(\boldsymbol{u}, \tilde{\boldsymbol{u}})_j^n$, $S_{\mathrm{R}}(\boldsymbol{u}, \tilde{\boldsymbol{u}})_j^n$, $\tilde{S}_{\mathrm{O}}(\tilde{\boldsymbol{u}}, \boldsymbol{u})_j^n$ *and* $\tilde{S}_{\mathrm{R}}(\tilde{\boldsymbol{u}}, \boldsymbol{u})_j^n$, *then,*

1. *If* $S_{\mathrm{O}}(\boldsymbol{u}, \tilde{\boldsymbol{u}})_j > 0$, *then* $\tilde{S}_{\mathrm{O}}(\tilde{\boldsymbol{u}}, \boldsymbol{u})_j = 0$ *and* $\tilde{S}_{\mathrm{R}}(\tilde{\boldsymbol{u}}, \boldsymbol{u})_j = 0$. *Furthermore,* $\tilde{\boldsymbol{u}}_j^{n+1} = 0$, *for* $\tilde{\boldsymbol{u}}_j^{n+1}$ *given by* (6.3.19) *in Algorithm* (4.3.1), *for all* $j \in \mathbb{Z}$ *and* $i = 1,2$.
2. *If* $\tilde{S}_{\mathrm{O},j}(\tilde{\boldsymbol{u}}, \boldsymbol{u})_j > 0$, *then* $S_{\mathrm{O}}(\boldsymbol{u}, \tilde{\boldsymbol{u}})_j^n = 0$ *and* $S_{\mathrm{R}}(\boldsymbol{u}, \tilde{\boldsymbol{u}})_j^n = 0$. *Furthermore,* $\boldsymbol{u}_j^{n+1} = 0$, *for* $\boldsymbol{u}_j^{n+1}$ *given by* (6.3.20) *in Algorithm* (4.3.1), *for all* $j \in \mathbb{Z}$ *and* $i = 1,2$.

Proof. 1. If $v_1(R_1(u_1^{\Delta})_j^n) < v_1(u_{1,j}^n)$, $\quad R_2(\tilde{u}_1^{\Delta})_j^n = 0$ and $\quad R_2(\tilde{u}_2^{\Delta})_j^n = 0$, then $\tilde{u}_{1,k}^n = 0$ and $\tilde{u}_{2,k}^n = 0$ for all $k \in \{j, \dots, j+\delta\}$, so by definition of the source term $\tilde{S}_{\mathrm{O}}(\tilde{\boldsymbol{u}}^{\Delta}, \boldsymbol{u}^{\Delta})_j^n$ (6.3.13) we obtain

$$\begin{aligned}
\tilde{S}_{\mathrm{O}}(\tilde{\boldsymbol{u}}^{\Delta}, \boldsymbol{u}^{\Delta})_j^n &= \begin{cases} K_1(\rho_{\max} - \tilde{u}_{2,j})\tilde{u}_{1,j}, & \text{if, } v_2(\tilde{R}_1(\tilde{u}_1^{\Delta})_j) < v_2(\tilde{u}_{1,j}), \\ & \tilde{R}_2(u_1^{\Delta})_j = 0 \text{ and } \tilde{R}_2(u_2^{\Delta})_j = 0 \\ 0, & \text{otherwise,} \end{cases} \\
&= \begin{cases} 0, & \text{if, } \ v_2(\tilde{R}_1(\tilde{u}_1^{\Delta})_j) < v_2(\tilde{u}_{1,j}), \quad \tilde{R}_2(u_1^{\Delta})_j = 0 \\ & \text{and } \ \tilde{R}_2(u_2^{\Delta})_j = 0 \\ 0, & \text{otherwise,} \end{cases}
\end{aligned}$$

and regarding to the term (6.3.14) we have

$$
\begin{aligned}
\tilde{S}_{\mathrm{R}}(\tilde{\boldsymbol{u}}^{\Delta}, \boldsymbol{u}^{\Delta})_j &= \begin{cases} K_2(\rho_{\max} - \tilde{u}_{1,j})\tilde{u}_{2,j}, & \text{if, } v_2(\tilde{R}_1(\tilde{u}_1^{\Delta})_j) > v_1(\tilde{u}_{1,j}), \\ & \text{or } \tilde{R}_2(\rho_1^{\Delta})_j^n > 0 \text{ or } \tilde{R}_2(\rho_2^{\Delta})_j^n > 0 \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} K_2(\rho_{\max} - \tilde{\rho}_{1,j}^n)\tilde{\rho}_{2,j}^n, & \text{if, } v_2(\tilde{R}_1(\tilde{\rho}_1^{\Delta})_j^n) > v_1(\tilde{\rho}_{1,j}^n), \\ & \text{or } \tilde{R}_2(\rho_1^{\Delta})_j^n > 0 \text{ or } \tilde{R}_2(\rho_2^{\Delta})_j^n > 0 \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}
$$

Thus, we have gotten $\tilde{S}_{\mathrm{O}}\left(\tilde{\boldsymbol{u}}, \boldsymbol{u}\right)_j = 0$ and $\tilde{S}_{\mathrm{R}}\left(\tilde{\boldsymbol{u}}, \boldsymbol{u}\right)_j = 0$, i.e., $\tilde{S}_j^{n+1/2} = 0$ and this implies that, for (6.3.20) in Algorithm (4.3.1) we obtain

$$\tilde{\boldsymbol{u}}_j^{n+1} = \tilde{\boldsymbol{u}}_j^{n+1/2} + \Delta t[-\tilde{S}_j^{n+1/2}, \tilde{S}_j^{n+1/2}] = 0.$$

2. The proof of this property is similar to that of the previous item, but taking into account that in this case $u_{1,k} = 0$ and $u_{2,k} = 0$ for all $k \in \{j, \ldots, j+\delta\}$.

□

Remark 6.4. *Properties proved in Lemma 6.3 tell us that in a same cell, overtaking of two different classes of vehicles traveling in opposite direction on a same lane is not allowed at the same time. Implicitly this means that two vehicles traveling in opposite direction on a same lane can not occupy the same cell.*

In order to prove the existence of solutions of model (6.2.1), in the next lemmas we will show some properties of the approximate solutions computed by Algorithm 6.3.1. We start proved positivity of approximate solutions.

Lemma 6.5 **(Positivity).** *Let Assumptions 6.2.1 hold. Then under following CFL condition*

$$\Delta t \leq \min\left\{\frac{\Delta x}{\mathcal{C} + \mathcal{D}}, \frac{1}{K\rho_{\max}}\right\}, \tag{6.3.21}$$

the approximate solutions computed by means Algorithm (6.3.1) *satisfies*

$$0 \leq \boldsymbol{\rho}_j^{n+1}, \tilde{\boldsymbol{\rho}}_j^{n+1} \leq \rho_{\max},$$

for all $j \in \mathbb{Z}$. Here,

$$\mathcal{C} = \max\{\|v_1\|_{\mathbf{L}^\infty([0,1])}, \|v_2\|_{\mathbf{L}^\infty([0,1])}\},$$

$$\mathcal{D} = \max\{\rho_{\max}\|v_1'\|_{\mathbf{L}^\infty([0,1])}, \rho_{\max}\|v_2'\|_{\mathbf{L}^\infty([0,1])}\},$$

and $K = \max\{K_1, K_2\}$.

Proof. We assume that $0 \leq \boldsymbol{\rho}_j^n, \tilde{\boldsymbol{\rho}}_j^n \leq \rho_{\max}$ for all $j \in \mathbb{Z}$, then first for the convective part, for $i = 1, 2$, we have

$$\begin{aligned} \rho_{i,j}^{n+1/2} &= \left(1 - \lambda v_1(\rho_{i,j+1}^n + (\rho_{\max} - \rho_{i,j+1}^n)\chi_\varepsilon(\tilde{R}_{[i],j+1/2}^n))\right)\rho_{i,j}^n \\ &\quad + \rho_{i,j-1}^n v_1(\rho_{i,j-1}^n + (\rho_{\max} - \rho_{i,j-1}^n)\chi_\varepsilon(\tilde{R}_{[i],j-1/2}^n)) \\ &\geq 0, \end{aligned}$$

and similarly we can obtain $\tilde{\rho}_{i,j}^{n+1/2} \geq 0$. Now, in order to simplify the notation we will denote $F(u, w, \tilde{R}) = uv_i(w + (\rho_{max} - w)\chi_\varepsilon(\tilde{R}))$, for $i = 1, 2$ and observe that $\partial_1 F = v_i(w + (\rho_{max} - w)\chi_\varepsilon(\tilde{R})) \geq 0$, $\partial_2 F = uv_i'(w + (\rho_{max} - w)\chi_\varepsilon(\tilde{R}))(1 - \chi_\varepsilon(\tilde{R})) \leq 0$ and $\partial_3 F = uv_i'(w + (\rho_{max} - w)\chi_\varepsilon(\tilde{R}))(\rho_{max} - w)\chi_\varepsilon'(\tilde{R}) \leq 0$. With this notation we can write $\rho_{i,j}^{n+1/2}$, $i = 1, 2$ in the scheme (6.3.15) as follows

$$\rho_{i,j}^{n+1/2} = \rho_{i,j}^n - \lambda\left[F(\rho_{i,j}^n, \rho_{i,j+1}^n, \tilde{R}_{[i],j+1/2}) - F(\rho_{i,j-1}^n, \rho_{i,j}^n, \tilde{R}_{[i],j-1/2})\right], \tag{6.3.22}$$

now observe that by the CFL condition (6.3.21), we get

$$\begin{aligned} \rho_{i,j}^{n+1/2} &\leq \rho_{i,j}^n + \lambda\left[F(\rho_{\max}, \rho_{i,j}^n, \tilde{R}_{[i],j-1/2}) - F(\rho_{\max}, \rho_{\max}, \tilde{R}_{[i],j-1/2})\right] \\ &= \rho_{i,j}^n + \lambda\left[-\partial_2 F(\nu_{j+1/2}^n)(\rho_{\max} - \rho_{i,j}^n)\right] \\ &= \left(1 - \lambda(-\partial_2 F(\nu_{j+1/2}^n))\right)\rho_{i,j}^n - \lambda\partial_2 F(\nu_{j+1/2}^n)\rho_{\max} \\ &\leq \rho_{\max}, \end{aligned}$$

where $\nu_{j+1/2}^n \in (\rho_{i,j}^n, \rho_{\max})$. In the same way we can compute

$$\begin{aligned} \tilde{\rho}_{i,j}^{n+1/2} &\leq \left(1 - \lambda(-\partial_2 F(\tilde{\nu}_{j+1/2}^n))\right)\tilde{\rho}_{i,j}^n - \lambda\partial_2 F(\tilde{\nu}_{j+1/2}^n)\rho_{\max} \\ &\leq \rho_{\max}. \end{aligned}$$

Now for reactive terms (6.3.19) we have the following estimates,

- If $S_{\mathrm{O}} > 0$,

$$\begin{aligned} \rho_{1,j}^{n+1} &= \rho_{1,j}^{n+1/2} - \Delta t K_1(\rho_{\max} - \rho_{2,j}^{n+1/2})\rho_{1,j}^{n+1/2} \\ &= \left(1 - \Delta t K_1(\rho_{\max} - \rho_{2,j}^{n+1/2})\right)\rho_{1,j}^{n+1/2} \\ &\geq 0, \end{aligned}$$

  and also

$$\begin{aligned} \rho_{1,j}^{n+1} &= \rho_{1,j}^{n+1/2} - \Delta t K_1(\rho_{\max} - \rho_{2,j}^{n+1/2})\rho_{1,j}^{n+1/2} \\ &= \left(1 - \Delta t K_1(\rho_{\max} - \rho_{2,j}^{n+1/2})\right)\rho_{1,j}^{n+1/2} \end{aligned}$$

$$
\begin{aligned}
&\leq \rho_{1,j}^{n+1/2} \\
&\leq \rho_{\max}.
\end{aligned}
$$

In the same way,

$$
\begin{aligned}
\rho_{2,j}^{n+1} &= \rho_{2,j}^{n+1/2} + \Delta t K_1(\rho_{\max} - \rho_{2,j}^{n+1/2})\rho_{1,j}^{n+1/2} \\
&= \left(1 - \Delta t K_1 \rho_1^{n+1/2}\right) \rho_{2,j}^{n+1/2} + \Delta t K_1 \rho_{\max} \rho_{1,j}^{n+1/2} \\
&\geq 0,
\end{aligned}
$$

and

$$
\begin{aligned}
\rho_{2,j}^{n+1} &= \rho_{2,j}^{n+1/2} + \Delta t K_1(\rho_{\max} - \rho_{2,j}^{n+1/2})\rho_{1,j}^{n+1/2} \\
&= \left(1 - \Delta t K_1 \rho_1^{n+1/2}\right) \rho_{2,j}^{n+1/2} + \Delta t K_1 \rho_{\max} \rho_{1,j}^{n+1/2} \\
&\leq \left(1 - \Delta t K_1 \rho_1^{n+1/2}\right) \rho_{\max} + \Delta t K_1 \rho_{\max} \rho_{1,j}^{n+1/2} \\
&\leq \rho_{\max}.
\end{aligned}
$$

- If $S_{\mathrm{R}} > 0$,

$$
\begin{aligned}
\rho_{1,j}^{n+1} &= \rho_{1,j}^{n+1/2} + \Delta t K_2(\rho_{\max} - \rho_{1,j}^{n+1/2})\rho_{2,j}^{n+1/2} \\
&= \left(1 - \Delta t K_2 \rho_2^{n+1/2}\right) \rho_{1,j}^{n+1/2} + \Delta t K_2 \rho_{\max} \rho_{2,j}^{n+1/2} \\
&\geq 0,
\end{aligned}
$$

and also,

$$
\begin{aligned}
\rho_{1,j}^{n+1} &= \rho_{1,j}^{n+1/2} + \Delta t K_2(\rho_{\max} - \rho_{1,j}^{n+1/2})\rho_{2,j}^{n+1/2} \\
&= \left(1 - \Delta t K_2 \rho_2^{n+1/2}\right) \rho_{1,j}^{n+1/2} + \Delta t K_2 \rho_{\max} \rho_{2,j}^{n+1/2} \\
&\leq \left(1 - \Delta t K_2 \rho_2^{n+1/2}\right) \rho_{\max} + \Delta t K_2 \rho_{\max} \rho_{2,j}^{n+1/2} \\
&\leq \rho_{\max}
\end{aligned}
$$

In the same way,

$$
\begin{aligned}
\rho_{2,j}^{n+1} &= \rho_{2,j}^{n+1/2} - \Delta t K_2(\rho_{\max} - \rho_{1,j}^{n+1/2})\rho_{2,j}^{n+1/2} \\
&= \left(1 - \Delta t K_2(\rho_{\max} - \rho_{1,j}^{n+1/2})\right) \rho_{2,j}^{n+1/2} \\
&\geq 0,
\end{aligned}
$$

$$
\begin{aligned}
\rho_{2,j}^{n+1} &= \left(1 - \Delta t K_2(\rho_{\max} - \rho_{1,j}^{n+1/2})\right) \rho_{2,j}^{n+1/2} \\
&\leq \rho_{2,j}^{n+1/2} \\
&\leq \rho_{\max}.
\end{aligned}
$$

Following a similar procedure we get $0 \leq \tilde{\boldsymbol{\rho}}_{\boldsymbol{j}}^{\boldsymbol{n+1}} \leq \rho_{\max}$.

□

Lemma 6.6 ($\mathbf{L^1}$**−bounds).** *Let* $\rho_i^0, \tilde{\rho}_i^0 \in \mathbf{L^1}(\mathbb{R};[0,1])$*, for* $i = 1, 2$ *and let the Assumption* (6.2.1) *holds. Under CFL condition* (6.3.21)*, the approximate solutions* $\boldsymbol{\rho^\Delta}, \boldsymbol{\tilde{\rho}^\Delta}$ *constructed by means algorithm* (6.3.1) *satisfies*

$$\left\|\boldsymbol{\rho^\Delta}\right\|_{\mathbf{L^1}(\mathbb{R})} := \left\|\rho_1^\Delta(t)\right\|_{\mathbf{L^1}(\mathbb{R})} + \left\|\rho_2^\Delta(t)\right\|_{\mathbf{L^1}(\mathbb{R})} = \left\|\rho_1^0\right\|_{\mathbf{L^1}(\mathbb{R})} + \left\|\rho_2^0\right\|_{\mathbf{L^1}(\mathbb{R})}.$$

Proof. The proof is done by induction. Observe that

$$\begin{aligned}\left\|\rho_1^{n+1/2}\right\|_{\mathbf{L^1}(\mathbb{R})} &= \left\|\rho_1^n\right\|_{\mathbf{L^1}(\mathbb{R})} = \left\|\rho_1^0\right\|_{\mathbf{L^1}(\mathbb{R})},\\ \left\|\rho_2^{n+1/2}\right\|_{\mathbf{L^1}(\mathbb{R})} &= \left\|\rho_2^n\right\|_{\mathbf{L^1}(\mathbb{R})} = \left\|\rho_2^0\right\|_{\mathbf{L^1}(\mathbb{R})},\end{aligned}$$

then, since $\rho_{1,j}^{n+1/2} \geq 0$ and $\rho_{2,j}^{n+1/2} \geq 0$ we get

$$\left\|\rho_1^{n+1/2}\right\|_{\mathbf{L^1}(\mathbb{R})} + \left\|\rho_2^{n+1/2}\right\|_{\mathbf{L^1}(\mathbb{R})} = \left\|\rho_1^0\right\|_{\mathbf{L^1}(\mathbb{R})} + \left\|\rho_2^0\right\|_{\mathbf{L^1}(\mathbb{R})}.$$

Now, we consider the reactive terms (6.3.19). Note the fact that when we compute $\rho_{1,j}^{n+1} + \rho_{2,j}^{n+1}$ the source terms sum up to 0, for which we get

$$\left\|\rho_1^{n+1}\right\|_{\mathbf{L^1}(\mathbb{R})} + \left\|\rho_2^{n+1}\right\|_{\mathbf{L^1}(\mathbb{R})} = \left\|\rho_1^{n+1/2}\right\|_{\mathbf{L^1}(\mathbb{R})} + \left\|\rho_2^{n+1/2}\right\|_{\mathbf{L^1}(\mathbb{R})} = \left\|\rho_1^0\right\|_{\mathbf{L^1}(\mathbb{R})} + \left\|\rho_2^0\right\|_{\mathbf{L^1}(\mathbb{R})}.$$

In the same way we get

$$\left\|\tilde{\rho}_1^{n+1/2}\right\|_{\mathbf{L^1}(\mathbb{R})} + \left\|\tilde{\rho}_2^{n+1/2}\right\|_{\mathbf{L^1}(\mathbb{R})} = \left\|\tilde{\rho}_1^n\right\|_{\mathbf{L^1}(\mathbb{R})} + \left\|\tilde{\rho}_2^n\right\|_{\mathbf{L^1}(\mathbb{R})} = \left\|\tilde{\rho}_1^0\right\|_{\mathbf{L^1}(\mathbb{R})} + \left\|\tilde{\rho}_2^0\right\|_{\mathbf{L^1}(\mathbb{R})},$$

and

$$\left\|\tilde{\rho}_1^{n+1/2}\right\|_{\mathbf{L^1}(\mathbb{R})} + \left\|\tilde{\rho}_2^{n+1/2}\right\|_{\mathbf{L^1}(\mathbb{R})} = \left\|\tilde{\rho}_1^n\right\|_{\mathbf{L^1}(\mathbb{R})} + \left\|\tilde{\rho}_2^n\right\|_{\mathbf{L^1}(\mathbb{R})} = \left\|\tilde{\rho}_1^0\right\|_{\mathbf{L^1}(\mathbb{R})} + \left\|\tilde{\rho}_2^0\right\|_{\mathbf{L^1}(\mathbb{R})},$$

□

Lemma 6.7 ($\mathbf{L}^\infty$**−bound).** *Let* $\boldsymbol{\rho_j^0}, \boldsymbol{\tilde{\rho}_j^0} \in \mathbf{L}^\infty(\mathbb{R}, \mathbb{R}^+)$*. Let the Assumptions 6.2.1 and the CFL condition* (6.3.21) *hold. Then, for all* $T > 0$*, there exist positive constants* $\mathcal{A}$ *and* $\mathcal{B}$ *such that* $\boldsymbol{\rho^\Delta}, \boldsymbol{\tilde{\rho}^\Delta}$ *constructed through Algorithm 6.3.1 satisfies*

$$\left\|\boldsymbol{\rho^n}\right\|_{\mathbf{L}^\infty} \leq \mathrm{e}^{T(\mathcal{A}+\mathcal{B})} \left\|\boldsymbol{\rho^0}\right\|_{\mathbf{L}^\infty},$$

Here we denote $\left\|\boldsymbol{\rho^n}\right\| = \left\|(\rho_i^n, \tilde{\rho}_{[i]}^n)\right\| = \max_{i,j}\{|\rho_{i,j}|, \left|\tilde{\rho}_{[i],j}\right|\}$.

Proof. In order to prove the estimate in the $\mathbf{L}^\infty$ norm we will work with the convective terms written like in (6.3.22) and we will follow closely [27]. First we add and subtract the term $\lambda F(\rho_{i,j}^n, \rho_{i,j}^n, \tilde{R}_{[i],j+1/2})$ in (6.3.22), considering

$$\xi_{j+1/2}^n = (u_{j+1/2}^n, w_{j+1/2}^n, \tilde{\sigma}_j^n) \in \mathcal{I}\left(\left(\rho_j^n, \rho_{j+1}^n, \tilde{R}_{[i],j+1/2}^n\right), \left(\rho_{j-1}^n, \rho_j^n, \tilde{R}_{[i],j-1/2}^n\right)\right),$$

and denoting $\hat{\rho} := \max\{\rho^n_{i,j-1}, \rho^n_{i,j}, \rho^n_{i,j+1}, \tilde{\rho}^n_{[i],j-1}, \tilde{\rho}^n_{[i],j}, \tilde{\rho}^n_{[i],j+1}\}$, we get

$$\begin{aligned}
\rho^{n+1/2}_{i,j} &= \left(1+\lambda\left(\partial_2 F(\xi_{j+1/2}) - \partial_1 F(\xi^n_{j-1/2})\right)\right)\rho^n_{i,j} - \lambda\partial_2 F(\xi_{j+1/2})\rho^n_{i,j+1} \\
&\quad +\lambda\partial_1 F(\xi^n_{j-1/2})\rho^n_{i,j-1} - \partial_3 F(\xi_{j+1/2})\left(\tilde{R}_{[i],j+1/2} - \tilde{R}_{[i],j-1/2}\right) \\
&\leq \hat{\rho} - \partial_3 F(\xi_{j+1/2})\left|\tilde{R}_{[i],j+1/2} - \tilde{R}_{[i],j-1/2}\right| \qquad (6.3.23)
\end{aligned}$$

and likewise,

$$\begin{aligned}
\tilde{\rho}^{n+1/2}_{[i],j} &= \left(1+\lambda\left(\partial_2 F(\tilde{\xi}_{j-1/2}) - \partial_1 F(\tilde{\xi}^n_{j+1/2})\right)\right)\tilde{\rho}^n_{[i],j} + \lambda\partial_1 F(\tilde{\xi}_{j+1/2})\tilde{\rho}^n_{[i],j+1} \\
&\quad -\lambda\partial_2 F(\tilde{\xi}^n_{j-1/2})\tilde{\rho}^n_{[i],j-1} + \partial_3 F(\tilde{\xi}_{j-1/2})\left(R_{i,j+1/2} - R_{i,j-1/2}\right) \\
&\leq \hat{\rho} - \partial_3 F(\tilde{\xi}_{j-1/2})\left|R_{i,j+1/2} - R_{i,j-1/2}\right| \qquad (6.3.24)
\end{aligned}$$

The next step is to compute the differences of convolution terms in the last term in (6.3.23) and (6.3.24), we estimate it as follows

$$\begin{aligned}
\left|\tilde{R}^n_{[i],j+1/2} - \tilde{R}^n_{[i],j-1/2}\right| &= \left|\sum_{k=0}^{N-1}\omega^k_\eta\tilde{\rho}^n_{[i],j+k+1} - \sum_{k=0}^{N-1}\omega^k_\eta\tilde{\rho}^n_{[i],j+k}\right| \\
&= \left|\sum_{k=1}^{N}\left(\omega^{k-1}_\eta - \omega^k_\eta\right)\tilde{\rho}^n_{[i],j+k} - \omega^0_\eta\tilde{\rho}^n_{[i],j}\right| \\
&\leq \sum_{k=1}^{N}\left(\omega^{k-1}_\eta - \omega^k_\eta\right)\tilde{\rho}^n_{[i],j+k} + \omega^0_\eta\tilde{\rho}^n_{[i],j} \\
&\leq 2\omega^0_\eta\left\|\boldsymbol{\rho^n}\right\|_{\mathbf{L}^\infty} \qquad (6.3.25) \\
&\leq 2\Delta x\omega_\eta(0)\left\|\boldsymbol{\rho^n}\right\|_{\mathbf{L}^\infty}, \qquad (6.3.26)
\end{aligned}$$

because $\omega^0_\eta \leq \Delta x\omega_\eta(0)$. In the same way we can compute

$$\left|R^n_{i,j+1/2} - R^n_{i,j-1/2}\right| \leq 2\Delta x\hat{\omega}_\eta(0)\left\|\boldsymbol{\rho^n}\right\|_{\mathbf{L}^\infty}, \qquad (6.3.27)$$

then by replacing (6.3.25), (6.3.27) in (6.3.23) and (6.3.24) respectively we get the following estimate for $\mathbf{L}^\infty-$norm

$$\left\|\boldsymbol{\rho^{n+1/2}}\right\|_{\mathbf{L}^\infty} \leq (1+\Delta t\mathcal{A})\left\|\boldsymbol{\rho^n}\right\|_{\mathbf{L}^\infty}, \qquad (6.3.28)$$

where $\mathcal{A} = 2\|\partial_3 F\|_{\mathbf{L}^\infty}W_0$, and $W_0 = \omega_\eta(0) + \hat{\omega}_\eta(0)$. Next, we proceed to bound the solution at time $n+1$. For this end, we need to consider the following cases:

Case 1: This case implies that $\boldsymbol{\rho_j^{n+1/2}} = 0$ and then $\boldsymbol{\rho}_j^{n+1} = 0$. In this case we have the following results:

$$\tilde{\rho}^{n+1}_{1,j} = \tilde{\rho}^{n+1/2}_{1,j} - \Delta t\tilde{S}_{\mathrm{O}}\left(\tilde{\rho}^{n+1/2}_{1,j}, \tilde{\rho}^{n+1/2}_{2,j}, \rho^{n+1/2}_{1,j}, \rho^{n+1/2}_{2,j}\right)$$

	$S_{\mathrm{O}}(\rho_1,\rho_2,\tilde{\rho}_1,\tilde{\rho}_2)>0$	$S_{\mathrm{O}}(\rho_1,\rho_2,\tilde{\rho}_1,\tilde{\rho}_2)=0$
$\tilde{S}_{\mathrm{O}}(\tilde{\rho}_1,\tilde{\rho}_2,\rho_1,\rho_2)>0$	X	Case 1
$\tilde{S}_{\mathrm{O}}(\tilde{\rho}_1,\tilde{\rho}_2,\rho_1,\rho_2)=0$	Case 2	Case 3

Table 6.3.1: Cases considered to prove $\mathbf{L}^\infty-$ bound of approximate solutions to problem (6.2.1) at time $n+1$

$$\leq \quad \tilde{\rho}_{1,j}^{n+1/2},$$

thus,

$$\left\|\boldsymbol{\rho^{n+1}}\right\|_{\mathbf{L}^\infty} \leq \left\|\boldsymbol{\rho^{n+1/2}}\right\|_{\mathbf{L}^\infty}.$$

Likewise,

$$\begin{aligned}
\tilde{\rho}_{2,j}^{n+1} &= \tilde{\rho}_{2,j}^{n+1/2} + \Delta t \tilde{S}_{\mathrm{O}}\left(\tilde{\rho}_{1,j}^{n+1/2}, \tilde{\rho}_{2,j}^{n+1/2}, \rho_{1,j}^{n+1/2}, \rho_{2,j}^{n+1/2}\right) \\
&= \tilde{\rho}_{2,j}^{n+1/2} + \Delta t \Bigg(\tilde{S}_{\mathrm{O}}\left(\tilde{\rho}_{1,j}^{n+1/2}, \tilde{\rho}_{2,j}^{n+1/2}, \rho_{1,j}^{n+1/2}, \rho_{2,j}^{n+1/2}\right) \\
&\quad - \tilde{S}_{\mathrm{O}}\left(0, \tilde{\rho}_{2,j}^{n+1/2}, \rho_{1,j}^{n+1/2}, \rho_{2,j}^{n+1/2}\right) \Bigg) \\
&= \tilde{\rho}_{2,j}^{n+1/2} + \Delta t \partial_1 \tilde{S}_{\mathrm{O}}(\zeta_2) \tilde{\rho}_{1,j}^{n+1/2},
\end{aligned}$$

then

$$\left\|\boldsymbol{\rho^{n+1}}\right\|_{\mathbf{L}^\infty} \leq \left(1 + \Delta t \left\|\partial_1 \tilde{S}_{\mathrm{O}}\right\|_{\mathbf{L}^\infty}\right) \left\|\boldsymbol{\rho^{n+1/2}}\right\|_{\mathbf{L}^\infty}.$$

Case 2: This case implies that $S_{\mathrm{R}} = 0$ and by Lemma 6.3 $\boldsymbol{\tilde{\rho}_j^{n+1/2}} = 0$ and therefore $\tilde{S}_{\mathrm{R}}$ $\boldsymbol{\tilde{\rho}_j^{n+1}} = 0$. Then

$$\begin{aligned}
\rho_{1,j}^{n+1} &= \rho_{1,j}^{n+1/2} - \Delta t S_{\mathrm{O}}\left(\rho_{1,j}^{n+1/2}, \rho_{2,j}^{n+1/2}, \tilde{\rho}_{1,j}^{n+1/2}, \tilde{\rho}_{2,j}^{n+1/2}\right) \\
&\leq \rho_{1,j}^{n+1/2},
\end{aligned}$$

thus,

$$\left\|\boldsymbol{\rho^{n+1}}\right\|_{\mathbf{L}^\infty} \leq \left\|\boldsymbol{\rho^{n+1/2}}\right\|_{\mathbf{L}^\infty}.$$

Likewise, we have

$$\begin{aligned}
\rho_{2,j}^{n+1} &= \rho_{2,j}^{n+1} + \Delta t S_{\mathrm{O}}\left(\rho_{1,j}^{n+1/2}, \rho_{2,j}^{n+1/2}, \tilde{\rho}_{1,j}^{n+1/2}, \tilde{\rho}_{2,j}^{n+1/2}\right) \\
&= \rho_{2,j}^{n+1} + \Delta t \Bigg(S_{\mathrm{O}}\left(\rho_{1,j}^{n+1/2}, \rho_{2,j}^{n+1/2}, \tilde{\rho}_{1,j}^{n+1/2}, \tilde{\rho}_{2,j}^{n+1/2}\right)
\end{aligned}$$

$$
\begin{aligned}
& -S_{\mathrm{O}}\left(0, \rho_{2,j}^{n+1/2}, \tilde{\rho}_{1,j}^{n+1/2}, \tilde{\rho}_{2,j}^{n+1/2}\right)\Big) \\
= \; & \rho_{2,j}^{n+1} + \Delta t \partial_1 S_{\mathrm{O}}(\zeta_1) \rho_{1,j}^{n+1/2},
\end{aligned}
$$

so we have

$$
\rho_{2,j}^{n+1} + \tilde{\rho}_{1,j}^{n+1} \;=\; \left(\rho_{2,j}^{n+1} + \tilde{\rho}_{1,j}^{n+1/2}\right) + \Delta t \partial_1 S_{\mathrm{O}}(\zeta_1) \left(\rho_{1,j}^{n+1/2} + \tilde{\rho}_{2,j}^{n+1/2}\right),
$$

and it implies

$$
\left\|\boldsymbol{\rho}^{\boldsymbol{n+1}}\right\|_{\mathbf{L}^\infty} \;\leq\; \left(1 + \Delta t \|\partial_1 S_{\mathrm{O}}\|_{\mathbf{L}^\infty}\right) \left\|\boldsymbol{\rho}^{\boldsymbol{n+1/2}}\right\|_{\mathbf{L}^\infty}.
$$

Case 3: In this case we have the following estimates

$$
\rho_{1,j}^{n+1} = \rho_{1,j}^{n+1/2} + \Delta t \partial_2 S_{\mathrm{R}}(\zeta_3) \rho_{2,j}^{n+1/2},
$$

and

$$
\tilde{\rho}_{2,j}^{n+1} \leq \tilde{\rho}_{2,j}^{n+1/2},
$$

then

$$
\left\|\boldsymbol{\rho}^{\boldsymbol{n+1}}\right\|_{\mathbf{L}^\infty} \leq \left\|\boldsymbol{\rho}^{\boldsymbol{n+1/2}}\right\|_{\mathbf{L}^\infty} + \Delta t \|\partial_2 S_{\mathrm{R}}\|_{\mathbf{L}^\infty} \left\|\boldsymbol{\rho}^{\boldsymbol{n+1/2}}\right\|_{\mathbf{L}^\infty}.
$$

Likewise for the other lane we have

$$
\left\|\boldsymbol{\rho}^{\boldsymbol{n+1}}\right\|_{\mathbf{L}^\infty} \leq \left\|\boldsymbol{\rho}^{\boldsymbol{n+1/2}}\right\|_{\mathbf{L}^\infty} + \Delta t \left\|\partial_2 \tilde{S}_{\mathrm{R}}\right\|_{\mathbf{L}^\infty} \left\|\boldsymbol{\rho}^{\boldsymbol{n+1/2}}\right\|_{\mathbf{L}^\infty}.
$$

So we get

$$
\left\|\boldsymbol{\rho}^{\boldsymbol{n+1}}\right\|_{\mathbf{L}^\infty} \leq (1 + \Delta t \mathcal{B}_{\mathrm{R}}) \left\|\boldsymbol{\rho}^{\boldsymbol{n+1/2}}\right\|_{\mathbf{L}^\infty},
$$

where $\mathcal{B}_{\mathrm{R}} = \max\left\{\|\partial_2 S_{\mathrm{R}}\|_{\mathbf{L}^\infty}, \left\|\partial_2 \tilde{S}_{\mathrm{R}}\right\|_{\mathbf{L}^\infty}\right\}$.

Now if we consider $\mathcal{B} = \max\left\{\left\|\partial_1 \tilde{S}_{\mathrm{O}}\right\|, \|\partial_1 S_{\mathrm{O}}\|, \mathcal{B}_{\mathrm{R}}\right\}$ we can bound the $\mathbf{L}^\infty-$ norm of the solution at time $n+1$ in all the considered cases, as follow

$$
\left\|\boldsymbol{\rho}^{\boldsymbol{n+1}}\right\|_{\mathbf{L}^\infty} \leq (1 + \Delta t \mathcal{B}) \left\|\boldsymbol{\rho}^{\boldsymbol{n+1/2}}\right\|_{\mathbf{L}^\infty}. \tag{6.3.29}
$$

By replacing (6.3.28) in (6.3.29)

$$
\left\|\boldsymbol{\rho}^{\boldsymbol{n+1}}\right\|_{\mathbf{L}^\infty} \leq e^{\Delta t(\mathcal{A}+\mathcal{B})} \left\|\boldsymbol{\rho}^{\boldsymbol{n}}\right\|_{\mathbf{L}^\infty}.
$$

Then, applying an iterative procedure we finally get the desired result.

Remark 6.8. *By virtue of Lemma 6.3 and Remark 6.4 it is not needed to consider the case marked with X in Table 6.3.1.*

□

Next lemma is an important property that allows us to prove the **BV** estimates later.

Lemma 6.9 (Lipschitz continuity of the source terms). *The maps S_{O}, S_{R}, $\tilde{S}_{\mathrm{O}}$ and $\tilde{S}_{\mathrm{R}}$ defined in* (6.1.2), (6.1.3), (6.1.4) *and* (6.1.5), *respectively, are Lipschitz continuous in first and second argument with Lipschitz constant $K = K_1 + K_2$.*

Proof. In order to prove this lemma, we need to consider the following cases:
Case 4. This case implies that $S_{\mathrm{R}}(\rho_1, \rho_2, \tilde{\rho}_1, \tilde{\rho}_2) = 0$ and $S_{\mathrm{R}}(u_1, u_2, \tilde{u}_1, \tilde{u}_2) = 0$, then

	$S_{\mathrm{O}}(\rho_1, \rho_2, \tilde{\rho}_1, \tilde{\rho}_2) > 0$	$S_{\mathrm{O}}(\rho_1, \rho_2, \tilde{\rho}_1, \tilde{\rho}_2) = 0$
$S_{\mathrm{O}}(u_1, u_2, \tilde{u}_1, \tilde{u}_2) > 0$	Case 4	Case 5
$S_{\mathrm{O}}(u_1, u_2, \tilde{u}_1, \tilde{u}_2) = 0$	Case 6	Case 7

Table 6.3.2: Cases considered to prove Lipschitz continuity of source terms

$$\begin{aligned}
& S_{\mathrm{O}}(\rho_1, \rho_2, \tilde{\rho}_1, \tilde{\rho}_2) - S_{\mathrm{O}}(u_1, u_2, \tilde{u}_1, \tilde{u}_2) \\
&= K_1\left((\rho_{\max} - \rho_2)\rho_1 - (\rho_{\max} - u_2)u_1\right) \\
&= K_1\left((\rho_{\max} - \rho_2)\rho_1 \pm (\rho_{\max} - \rho_2)u_1 - (\rho_{\max} - u_2)u_1\right) \\
&= K_1\left((\rho_{\max} - \rho_2)(\rho_1 - u_1) + (u_2 - \rho_2)u_1\right) \\
&\leq K_1\left((\rho_1 - u_1) + (u_2 - \rho_2)\right),
\end{aligned}$$

then, taking absolute value in the above equality, we get

$$\left|S_{\mathrm{O}}(\rho_1, \rho_2, \tilde{\rho}_1, \tilde{\rho}_2) - S_{\mathrm{O}}(u_1, u_2, \tilde{u}_1, \tilde{u}_2)\right| \leq K_1\left(|\rho_1 - u_1| + |\rho_2 - u_2|\right).$$

Next, we will turn to Case 6, the Case 5 is similar.
Case 6. Regarding the Case 13, it implies $S_{\mathrm{R}}(\rho_1, \rho_2, \tilde{\rho}_1, \tilde{\rho}_2) = 0$ and $S_{\mathrm{R}}(u_1, u_2, \tilde{u}_1, \tilde{u}_2) > 0$,
then we have the following estimates

$$\begin{aligned}
& \left|S_{\mathrm{O}}(\rho_1, \rho_2, \tilde{\rho}_1, \tilde{\rho}_2) - S_{\mathrm{R}}(u_1, u_2, \tilde{u}_1, \tilde{u}_2)\right| \\
& \left|S_{\mathrm{O}}(\rho_1, \rho_2, \tilde{\rho}_1, \tilde{\rho}_2) - S_{\mathrm{O}}(u_1, u_2, \tilde{u}_1, \tilde{u}_2) + S_{\mathrm{R}}(\rho_1, \rho_2, \tilde{\rho}_1, \tilde{\rho}_2) - S_{\mathrm{R}}(u_1, u_2, \tilde{u}_1, \tilde{u}_2)\right| \\
&\leq \left|S_{\mathrm{O}}(\rho_1, \rho_2, \tilde{\rho}_1, \tilde{\rho}_2) - S_{\mathrm{O}}(u_1, u_2, \tilde{u}_1, \tilde{u}_2)\right| + \left|S_{\mathrm{R}}(\rho_1, \rho_2, \tilde{\rho}_1, \tilde{\rho}_2) - S_{\mathrm{R}}(u_1, u_2, \tilde{u}_1, \tilde{u}_2)\right| \\
&\leq K_1\left(|\rho_1 - u_1| + |\rho_2 - u_2|\right) + K_2\left(|\rho_1 - u_1| + |\rho_2 - u_2|\right) \\
&= K\left(|\rho_1 - u_1| + |\rho_2 - u_2|\right).
\end{aligned}$$

Case 7. This case implies that $S_{\mathrm{R}}(\rho_1, \rho_2, \tilde{\rho}_1, \tilde{\rho}_2) > 0$ and $S_{\mathrm{R}}(u_1, u_2, \tilde{u}_1, \tilde{u}_2) > 0$, then we get

$$S_{\mathrm{R}}(\rho_1, \rho_2, \tilde{\rho}_1, \tilde{\rho}_2) - S_{\mathrm{R}}(u_1, u_2, \tilde{u}_1, \tilde{u}_2) \quad = \quad K_2\left((\rho_{\max} - \rho_1)\rho_2 - (\rho_{\max} - u_1)u_2\right)$$

$$
\begin{aligned}
&= K_2\left(\rho_2(u_1-\rho_1)+(\rho_2-u_2)(\rho_{\max}-u_1)\right)\\
&\le K_2\left((u_1-\rho_1)+(\rho_2-u_2)\right),
\end{aligned}
$$

then,

$$
\left|S_{\mathrm{R}}\left(\rho_1,\rho_2,\tilde\rho_1,\tilde\rho_2\right)-S_{\mathrm{R}}\left(u_1,u_2,\tilde u_1,\tilde u_2\right)\right|\le K_2\left(|\rho_1-u_1|+|\rho_2-u_2|\right).
$$

Considering similar cases we can prove the Lipschitz continuity of $\tilde S_{\mathrm{O}}$ and $\tilde S_{\mathrm{R}}$. □

6.3.2 BV estimates

Proposition 6.9.1 (BV estimates in space). *Let $\boldsymbol{\rho^0_j},\boldsymbol{\tilde\rho^0_j}\in(\mathbf{L}^\infty\cap\mathbf{BV})(\mathbb{R};\mathbb{R}^+)$. Let the Assumptions 6.2.1 and CFL condition* (6.3.21) *hold. Then, for all $T>0$ there exist a positive constant $\mathcal{H}$ such that $\boldsymbol{\rho^\Delta}$, $\boldsymbol{\tilde\rho^\Delta}$ constructed through Algorithm 6.3.1 satisfies the following estimate: for all $n=0,\dots,N_T$,*

$$
\sum_{i=1}^{2}\left(\mathrm{TV}(\rho_i^n)+\mathrm{TV}(\tilde\rho_{[i]}^n)\right)\le e^{T(2K+\mathcal{H})}\left(\sum_{i=1}^{2}\left(\mathrm{TV}(\rho_i^0)+\mathrm{TV}(\tilde\rho_{[i]}^0)\right)+2T\mathcal{G}\right)\tag{6.3.30}
$$

Proof. Taking into account (6.3.22) we can write

$$
\rho_{i,j+1}^{n+1/2}=\rho_{i,j+1}^n-\lambda\left[F(\rho_{i,j+1}^n,\rho_{i,j+2}^n,\tilde R_{[i],j+3/2})-F(\rho_{i,j}^n,\rho_{i,j+1}^n,\tilde R_{[i],j+1/2})\right].
$$

Setting $\Delta_{i,j+1/2}^{n+1/2}=\rho_{i,j+1}^{n+1/2}-\rho_{i,j}^{n+1/2}$, for all $i=1,2$ we compute the following estimates

$$
\begin{aligned}
\Delta_{i,j+1/2}^{n+1/2} &= \left[1-\lambda\left(\partial_1F(\xi_{j+1/2}^n)-\partial_2F(\xi_{j-1/2}^n)\right)\right]\Delta_{j+1/2}^n\\
&\quad-\lambda\partial_2F(\xi_{j+1/2}^n)\Delta_{i,j+3/2}^n+\lambda\partial_1F(\xi_{j-1/2}^n)\Delta_{j-1/2}^n\\
&\quad-\lambda\partial_3F(\xi_{j+1/2}^n)\left(\tilde R_{[i],j+3/2}^n-\tilde R_{[i],j+1/2}^n\right)\\
&\quad+\lambda\partial_3F(\xi_{j-1/2}^n)\left(\tilde R_{[i],j+1/2}^n-\tilde R_{[i],j-1/2}^n\right)\\
&= \left[1-\lambda\left(\partial_1F(\xi_{j+1/2}^n)-\partial_2F(\xi_{j-1/2}^n)\right)\right]\Delta_{j+1/2}^n\\
&\quad-\lambda\partial_2F(\xi_{j+1/2}^n)\Delta_{i,j+3/2}^n+\lambda\partial_1F(\xi_{j-1/2}^n)\Delta_{j-1/2}^n\\
&\quad+\lambda\partial_3F(\xi_{j-1/2}^n)\left[\left(\tilde R_{[i],j+1/2}^n-\tilde R_{[i],j-1/2}^n\right)-\left(\tilde R_{[i],j+3/2}^n-\tilde R_{[i],j+1/2}^n\right)\right]\\
&\quad+\lambda\left[\partial_3F(\xi_{j-1/2}^n)-\partial_3F(\xi_{j+1/2}^n)\right]\left(\tilde R_{[i],j+1/2}^n-\tilde R_{[i],j-1/2}^n\right)
\end{aligned}
$$

where $\xi_{j+1/2}^n=\left(u_{j+1/2}^n,w_{j+1/2}^n,\tilde\sigma_{j+1/2}^n\right)\in\mathcal{I}\left((\rho_j^n,\rho_{j+1}^n,\tilde R_{[i],j+1/2}^n),(\rho_{j-1}^n,\rho_j^n,\tilde R_{[i],j-1/2}^n)\right)$. Observe that the first term in the last equality is positive because of (CFL) condition (6.3.21), so taking absolute value in the above equality we get

$$\left|\Delta_{i,j+1/2}^{n+1/2}\right| \tag{6.3.31}$$
$$= \left[1 - \lambda \left(\partial_1 F(\xi_{j+1/2}^n) - \partial_2 F(\xi_{j-1/2}^n)\right)\right] \left|\Delta_{j+1/2}^n\right|$$
$$-\lambda \partial_2 F(\xi_{j+1/2}^n) \left|\Delta_{i,j+3/2}^n\right| + \lambda \partial_1 F(\xi_{j-1/2}^n) \left|\Delta_{j-1/2}^n\right|$$
$$+\lambda \|\partial_3 F\|_{\mathbf{L}^\infty} \left|\left(\tilde{R}_{[i],j+3/2}^n - \tilde{R}_{[i],j+1/2}^n\right) - \left(\tilde{R}_{[i],j+1/2}^n - \tilde{R}_{[i],j-1/2}^n\right)\right| \tag{6.3.32}$$
$$+\lambda \left|\partial_3 F(\xi_{j-1/2}^n) - \partial_3 F(\xi_{j+1/2}^n)\right| \left|\tilde{R}_{[i],j+1/2}^n - \tilde{R}_{[i],j-1/2}^n\right|. \tag{6.3.33}$$

Next, the term (6.3.32) can be estimated as follow

$$\left|\left(\tilde{R}_{[i],j+3/2}^n - \tilde{R}_{[i],j+1/2}^n\right) - \left(\tilde{R}_{[i],j+1/2}^n - \tilde{R}_{[i],j-1/2}^n\right)\right|$$
$$= \left|\Delta x \left[\left(\sum_{k=0}^{N-1} \omega_\eta^k \tilde{\rho}_{j+k+2}^n - \sum_{k=0}^{N-1} \omega_\eta^k \tilde{\rho}_{j+k+1}^n\right) - \left(\sum_{k=0}^{N-1} \omega_\eta^k \tilde{\rho}_{j+k+1}^n - \sum_{k=0}^{N-1} \omega_\eta^k \tilde{\rho}_{j+k}^n\right)\right]\right|$$
$$= \left|\sum_{k=0}^{N-1} \omega_\eta^k \left(\tilde{\rho}_{j+k+2}^n - \tilde{\rho}_{j+k+1}^n\right) - \sum_{k=0}^{N-1} \omega_\eta^k \left(\tilde{\rho}_{j+k+1}^n - \tilde{\rho}_{j+k}^n\right)\right|$$
$$= \left|\sum_{k=1}^{N} \omega_\eta^{k-1} \left(\tilde{\rho}_{j+k+1}^n - \tilde{\rho}_{j+k}^n\right) - \sum_{k=0}^{N-1} \omega_\eta^k \left(\tilde{\rho}_{j+k+1}^n - \tilde{\rho}_{j+k}^n\right)\right|$$
$$= \left|\sum_{k=1}^{N} \left(\omega_\eta^{k-1} - \omega_\eta^k\right) \left(\tilde{\rho}_{j+k+1}^n - \tilde{\rho}_{j+k}^n\right) - \omega_\eta^0 \left(\tilde{\rho}_{j+1}^n - \tilde{\rho}_j^n\right)\right|$$
$$\leq \sum_{k=1}^{N} \left(\omega_\eta^{k-1} - \omega_\eta^k\right) \left|\tilde{\rho}_{j+k+1}^n - \tilde{\rho}_{j+k}^n\right| + \omega_\eta^0 \left|\tilde{\rho}_{j+1}^n - \tilde{\rho}_j^n\right|,$$

and summing over all $j \in \mathbb{Z}$ we get

$$\sum_{j\in\mathbb{Z}} \left|\left(\tilde{R}_{[i],j+3/2}^n - \tilde{R}_{[i],j+1/2}^n\right) - \left(\tilde{R}_{[i],j+1/2}^n - \tilde{R}_{[i],j-1/2}^n\right)\right|$$
$$\leq \Delta x \left[\sum_{k=1}^{N} \left(\omega_\eta^{k-1} - \omega_\eta^k\right) \mathrm{TV}\left(\tilde{\rho}_{[i]}^n\right) + \omega_\eta^0 \mathrm{TV}\left(\tilde{\rho}_{[i]}^n\right)\right]$$
$$= 2\Delta x \omega_\eta(0) \mathrm{TV}\left(\tilde{\rho}_{[i]}^n\right).$$

Now, for (6.3.33) we have the following estimates

$$\left|\partial_3 F(\xi_{j-1/2}^n) - \partial_3 F(\xi_{j+1/2}^n)\right| \leq \|\nabla \partial_3 F\|_{\mathbf{L}^\infty} \left|\xi_{j+1/2}^n - \xi_{j-1/2}^n\right|,$$

and by the choice of $\xi_{j+1/2}^n$, the term $\left|\xi_{j+1/2}^n - \xi_{j-1/2}^n\right|$ can be decomposed as follows

$$
\begin{aligned}
&\left|\xi^n_{j+1/2}-\xi^n_{j-1/2}\right| \\
&= \left|\theta\rho^n_{j+1}+(1-\theta)\rho^n_j-\mu\rho^n_j+(1-\mu)\rho^n_{j-1}\right| && (6.3.34)\\
&\quad+\left|\vartheta\rho^n_{j+2}+(1-\vartheta)\rho^n_{j+1}-\iota\rho^n_{j+1}+(1-\iota)\rho^n_j\right| && (6.3.35)\\
&\quad+\left|\alpha\tilde{R}^n_{[i],j+3/2}+(1-\alpha)\tilde{R}^n_{[i],j+1/2}-\beta\tilde{R}^n_{[i],j+1/2}-(1-\beta)\tilde{R}^n_{[i],j-1/2}\right|, && (6.3.36)
\end{aligned}
$$

so, for (6.3.34) we obtain

$$
\begin{aligned}
\left|\theta\rho^n_{j+1}+(1-\theta)\rho^n_j-\mu\rho^n_j+(1-\mu)\rho^n_{j-1}\right| &= \left|\theta(\rho^n_{j+1}-\rho^n_j)+(1-\mu)(\rho^n_j-\rho^n_{j-1})\right|\\
&\le \left|\rho^n_{j+1}-\rho^n_j\right|+\left|\rho^n_j-\rho^n_{j-1}\right|.
\end{aligned}
$$

Similarly, for (6.3.35) we have

$$
\left|\vartheta\rho^n_{j+2}+(1-\vartheta)\rho^n_{j+1}-\iota\rho^n_{j+1}+(1-\iota)\rho^n_j\right| \le \left|\rho^n_{j+2}-\rho^n_{j+1}\right|+\left|\rho^n_{j+1}-\rho^n_j\right|,
$$

and finally, for (6.3.36) we get

$$
\begin{aligned}
&\left|\alpha\tilde{R}^n_{[i],j+3/2}+(1-\alpha)\tilde{R}^n_{[i],j+1/2}-\beta\tilde{R}^n_{[i],j+1/2}-(1-\beta)\tilde{R}^n_{[i],j-1/2}\right|\\
=&\ \Bigg|\sum_{k=1}^{N}\left\{\alpha\left(\omega^{k-1}_\eta-\omega^k_\eta\right)+(1-\beta)\left(\omega^k_\eta-\omega^{k+1}_\eta\right)\right\}\tilde{\rho}^n_{j+k+1}\\
&\quad-\left(\alpha\omega^0_\eta+(1-\beta)\omega^1_\eta\right)\tilde{\rho}^n_{j+1}+(1-\beta)\omega^0_\eta\left(\tilde{\rho}^n_{j+1}-\tilde{\rho}^n_j\right)\Bigg|\\
\le&\ \sum_{k=1}^{N}\left\{\alpha\left(\omega^{k-1}_\eta-\omega^k_\eta\right)+(1-\beta)\left(\omega^k_\eta-\omega^{k+1}_\eta\right)\right\}\tilde{\rho}^n_{j+k+1}\\
&\quad+\left(\alpha\omega^0_\eta+(1-\beta)\omega^1_\eta\right)\tilde{\rho}^n_{j+1}+(1-\beta)\omega^0_\eta\left|\tilde{\rho}^n_{j+1}-\tilde{\rho}^n_j\right|,
\end{aligned}
$$

thus,

$$
\begin{aligned}
&\sum_{j\in\mathbb{Z}}\left|\alpha\tilde{R}^n_{[i],j+3/2}+(1-\alpha)\tilde{R}^n_{[i],j+1/2}-\beta\tilde{R}^n_{[i],j+1/2}-(1-\beta)\tilde{R}^n_{[i],j-1/2}\right|\\
\le&\ \sum_{j\in\mathbb{Z}}\tilde{\rho}^n_j\left(\sum_{k=1}^{N}\left(\omega^{k-1}_\eta-\omega^{k+1}_\eta\right)\right)+2\omega^0_\eta\sum_{j\in\mathbb{Z}}\tilde{\rho}^n_j+\omega^0_\eta\sum_{j\in\mathbb{Z}}\left|\tilde{\rho}^n_{j+1}-\tilde{\rho}^n_j\right|\\
\le&\ 4\omega_\eta(0)\|\tilde{\rho}^n_{[i]}\|_{\mathbf{L^1}}+\Delta x\omega_\eta(0)\mathrm{TV}\left(\tilde{\rho}^n_{[i]}\right).
\end{aligned}
$$

Therefore, taking into account all the above estimates we get

$$
\sum_{j\in\mathbb{Z}}\left|\Delta^{n+1/2}_{i,j+1/2}\right| \le (1+\Delta t\mathcal{H}_1)\,\mathrm{TV}\left(\rho^n_i\right)+\Delta t\mathcal{H}_2\mathrm{TV}(\tilde{\rho}^n_{[i]})+\Delta t\mathcal{H}_3,
$$

where

$$\begin{aligned}
\mathcal{H}_1 &= 8\omega_\eta^0 \|\boldsymbol{\rho^n}\|_{\mathbf{L}^\infty} \|\nabla\partial_3 F\|_{\mathbf{L}^\infty},\\
\mathcal{H}_2 &= 2\left(\|\partial_3 F\|_{\mathbf{L}^\infty}\omega_\eta(0) + \Delta x(\omega_\eta(0))^2\|\boldsymbol{\rho^n}\|_{\mathbf{L}^\infty}\|\nabla\partial_3 F\|_{\mathbf{L}^\infty}\right),\\
\mathcal{H}_3 &= 8(\omega_\eta(0))^2\|\tilde\rho^n_{[i]}\|_{\mathbf{L}^1}\|\boldsymbol{\rho^n}\|_{\mathbf{L}^\infty}\|\nabla\partial_3 F\|_{\mathbf{L}^\infty}.
\end{aligned}$$

Likewise, we can estimate

$$\sum_{j\in\mathbb{Z}} \left|\tilde\Delta^{n+1/2}_{[i],j+1/2}\right| \le \left(1+\Delta t\tilde{\mathcal{H}}_1\right)\mathrm{TV}\left(\tilde{\rho_{[i]}}^n\right) + \Delta t\tilde{\mathcal{H}}_2\mathrm{TV}(\rho_i^n) + \Delta t\tilde{\mathcal{H}}_3,$$

$$\begin{aligned}
\tilde{\mathcal{H}}_1 &= 8\hat\omega_\eta(0)\|\boldsymbol{\rho^n}\|_{\mathbf{L}^\infty}\|\nabla\partial_3 F\|_{\mathbf{L}^\infty},\\
\tilde{\mathcal{H}}_2 &= 2\left(\|\partial_3 F\|_{\mathbf{L}^\infty}\hat\omega_\eta(0) + \Delta x(\hat\omega_\eta(0))^2\|\boldsymbol{\rho^n}\|_{\mathbf{L}^\infty}\|\nabla\partial_3 F\|_{\mathbf{L}^\infty}\right),\\
\tilde{\mathcal{H}}_3 &= 8(\hat\omega_\eta(0))^2\|\rho^n_i\|_{\mathbf{L}^1}\|\boldsymbol{\rho^n}\|_{\mathbf{L}^\infty}\|\nabla\partial_3 F\|_{\mathbf{L}^\infty}.
\end{aligned}$$

Thus we get, for $i=1,2$

$$\begin{aligned}
&\sum_{j\in\mathbb{Z}}\left(\left|\Delta^{n+1/2}_{i,j+1/2}\right| + \left|\tilde\Delta^{n+1/2}_{[i],j+1/2}\right|\right)\\
&= (1+\Delta t\mathcal{H}_1)\,\mathrm{TV}(\rho_i^n) + \Delta t\mathcal{H}_2\mathrm{TV}(\tilde\rho^n_{[i]}) + \Delta t\mathcal{H}_3\\
&\quad + \left(1+\Delta t\tilde{\mathcal{H}}_1\right)\mathrm{TV}\left(\tilde{\rho_{[i]}}^n\right) + \Delta t\tilde{\mathcal{H}}_2\mathrm{TV}(\rho_i^n) + \Delta t\tilde{\mathcal{H}}_3\\
&= \left(1+\Delta t\left(\mathcal{H}_1+\tilde{\mathcal{H}}_2\right)\right)\mathrm{TV}(\rho_i^n) + \left(1+\Delta t\left(\mathcal{H}_2+\tilde{\mathcal{H}}_1\right)\right)\mathrm{TV}\left(\tilde{\rho_{[i]}}^n\right)\\
&\quad + \Delta t\left(\mathcal{H}_3+\tilde{\mathcal{H}}_3\right)\\
&\le (1+\Delta t\mathcal{H})\left(\mathrm{TV}(\rho_i^n) + \mathrm{TV}\left(\tilde\rho^n_{[i]}\right)\right) + \Delta t\mathcal{G},
\end{aligned}$$

where $\mathcal{H} = \max\{\mathcal{H}_1+\tilde{\mathcal{H}}_2, \mathcal{H}_2+\tilde{\mathcal{H}}_1\}$ and $\mathcal{G} = \mathcal{H}_3+\tilde{\mathcal{H}}_3$. If we sum the two lanes, we get

$$\begin{aligned}
&\sum_{i=1}^{2}\left(\mathrm{TV}(\rho_i^{n+1/2}) + \mathrm{TV}(\tilde\rho^{n+1/2}_{[i]})\right)\\
&\le (1+\Delta t\mathcal{H})\sum_{i=1}^{2}\left(\mathrm{TV}(\rho_i^n) + \mathrm{TV}\left(\tilde\rho^n_{[i]}\right)\right) + 2\Delta t\mathcal{G}
\end{aligned}\tag{6.3.37}$$

Let us compute now, for the reactive part (6.3.19)

$$\Delta^{n+1}_{i,j+1/2} = \Delta^{n+1/2}_{i,j+1/2} - \Delta t\left(S_{\mathrm{O,j+1}} - S_{\mathrm{O,j}} - (S_{\mathrm{R,j+1}} - S_{\mathrm{R,j}})\right),$$

now applying absolute value and Lipschitz continuity of the source terms given in Lemma 6.9 and summing over all $j\in\mathbb{Z}$ we get. for $i=1,2$

$$\sum_{j\in\mathbb{Z}}\left|\Delta^{n+1}_{i,j+1/2}\right| \le (1+\Delta tK)\sum_{j\in\mathbb{Z}}\left|\Delta^{n+1/2}_{i,j+1/2}\right| + \Delta tK\sum_{j\in\mathbb{Z}}\left|\Delta^{n+1/2}_{[i],j+1/2}\right|,$$

In a similar way for the other species we get,

$$\sum_{j\in\mathbb{Z}} \left|\tilde{\Delta}^{n+1}_{[i],j+1/2}\right| \leq (1+\Delta t K)\sum_{j\in\mathbb{Z}} \left|\tilde{\Delta}^{n+1/2}_{[i],j+1/2}\right| + \Delta t K \sum_{j\in\mathbb{Z}} \left|\tilde{\Delta}^{n+1/2}_{i,j+1/2}\right|,$$

then

$$\begin{aligned}\sum_{j\in\mathbb{Z}} \left(\left|\Delta^{n+1}_{i,j+1/2}\right| + \left|\tilde{\Delta}^{n+1}_{[i],j+1/2}\right|\right) \leq\ & (1+\Delta t K)\sum_{j\in\mathbb{Z}} \left(\left|\Delta^{n+1/2}_{i,j+1/2}\right| + \left|\tilde{\Delta}^{n+1/2}_{[i],j+1/2}\right|\right) \\ & +\Delta t K \sum_{j\in\mathbb{Z}} \left(\left|\Delta^{n+1/2}_{[i],j+1/2}\right| + \left|\tilde{\Delta}^{n+1/2}_{i,j+1/2}\right|\right),\end{aligned}$$

summing the two lanes and by (6.3.37) we get

$$\begin{aligned}&\sum_{i=1}^{2}\left(\mathrm{TV}(\rho_i^{n+1}) + \mathrm{TV}(\tilde{\rho}_{[i]}^{n+1})\right) \\ &\leq (1+2\Delta t K)\sum_{i=1}^{2}\left(\mathrm{TV}(\rho_i^{n+1/2}) + \mathrm{TV}(\tilde{\rho}_{[i]}^{n+1/2})\right) \\ &\leq (1+2\Delta t K)\left((1+\Delta t H)\sum_{i=1}^{2}\left(\mathrm{TV}(\rho_i^{n}) + \mathrm{TV}(\tilde{\rho}_{[i]}^{n})\right) + 2\Delta t \mathcal{G}\right) \\ &= (1+2\Delta t K)(1+\Delta t H)\sum_{i=1}^{2}\left(\mathrm{TV}(\rho_i^{n}) + \mathrm{TV}(\tilde{\rho}_{[i]}^{n})\right) \\ &\quad +(1+2\Delta t K)2\Delta t\mathcal{G} \\ &\leq e^{\Delta t(2K+\mathcal{H})}\left(\sum_{i=1}^{2}\left(\mathrm{TV}(\rho_i^{n}) + \mathrm{TV}(\tilde{\rho}_{[i]}^{n})\right) + 2\Delta t\mathcal{G}\right),\end{aligned}$$

i.e.,

$$\sum_{i=1}^{2}\left(\mathrm{TV}(\rho_i^{n+1}) + \mathrm{TV}(\tilde{\rho}_{[i]}^{n+1})\right) \leq e^{\Delta t(2K+\mathcal{H})}\left(\sum_{i=1}^{2}\left(\mathrm{TV}(\rho_i^{n}) + \mathrm{TV}(\tilde{\rho}_{[i]}^{n})\right) + 2\Delta t\mathcal{G}\right).$$

Then applying an iterative process we get

$$\sum_{i=1}^{2}\left(\mathrm{TV}(\rho_i^{n}) + \mathrm{TV}(\tilde{\rho}_{[i]}^{n})\right) \leq e^{T(2K+\mathcal{H})}\left(\sum_{i=1}^{2}\left(\mathrm{TV}(\rho_i^{0}) + \mathrm{TV}(\tilde{\rho}_{[i]}^{0})\right) + 2T\mathcal{G}\right).$$

□

Corollary 6.10 (BV estimate in space and time). *Let* $\boldsymbol{\rho_j^0}, \boldsymbol{\tilde{\rho}_j^0} \in (\mathbf{L}^\infty \cap \mathbf{BV})(\mathbb{R};\mathbb{R}^+)$. *Let the Assumption 6.2.1 and CFL condition* (6.3.21) *hold. Then, for all* $T > 0$,

$\boldsymbol{\rho}^{\boldsymbol{\Delta}}$, $\tilde{\boldsymbol{\rho}}^{\boldsymbol{\Delta}}$ *constructed through Algorithm 6.3.1 satisfies the following estimate: for all* $n = 1, \ldots, N_T$,

$$\sum_{n=0}^{N_T-1} \sum_{i=1}^{2} \sum_{j\in\mathbb{Z}} \Delta t \left|\rho_{i,j+1}^{n} - \rho_{i,j}^{n}\right| + (T - N_T\Delta t) \sum_{i=1}^{2} \sum_{j\in\mathbb{Z}} \left|\rho_{i,j+1}^{N_T} - \rho_{i,j}^{N_T}\right|$$
$$+ \sum_{n=0}^{N_T-1} \sum_{i=1}^{2} \sum_{j\in\mathbb{Z}} \Delta x \left|\rho_{i,j}^{n+1} - \rho_{i,j}^{n}\right|$$
$$\leq T e^{T(2K\mathcal{H})} (1 + 2\mathcal{L}) \left(\sum_{i=1}^{2} \left(\mathrm{TV}(\rho_i^0) + \mathrm{TV}(\tilde{\rho}_{[i]}^0) \right) + 2T\mathcal{G} \right)$$
$$+ 2T \max\left\{ \left\|\partial_1 S_{\mathrm{O}}\right\|_{\mathbf{L}^\infty}, \left\|\partial_2 S_{\mathrm{R}}\right\|_{\mathbf{L}^\infty} \right\} \left(\left\|\rho_i^0\right\|_{\mathbf{L}^1(\mathbb{R})} + \left\|\rho_{[i]}^0\right\|_{\mathbf{L}^1(\mathbb{R})} \right).$$

Proof. By means **BV** estimate in space (6.3.30), we have

$$\sum_{n=0}^{N_T-1} \sum_{i=1}^{2} \sum_{j\in\mathbb{Z}} \Delta t \left|\rho_{i,j+1}^{n} - \rho_{i,j}^{n}\right| + (T - N_T\Delta t) \sum_{i=1}^{2} \sum_{j\in\mathbb{Z}} \left|\rho_{i,j+1}^{N_T} - \rho_{i,j}^{N_T}\right|$$
$$\leq e^{T(2K+\mathcal{H})} \left(\sum_{i=1}^{2} \left(\mathrm{TV}(\rho_i^0) + \mathrm{TV}(\tilde{\rho}_{[i]}^0) \right) + 2T\mathcal{G} \right). \tag{6.3.38}$$

On the other hand, observe that

$$\left|\rho_{i,j}^{n+1} - \rho_{i,j}^{n}\right| \leq \left|\rho_{i,j}^{n+1} - \rho_{i,j}^{n+1/2}\right| + \left|\rho_{i,j}^{n+1/2} - \rho_{i,j}^{n}\right|. \tag{6.3.39}$$

Now we estimate every term on right hand side of the inequality (6.3.39). By (6.3.19) in **Algorithm 6.3.1** we have

$$\begin{aligned}
\left|\rho_{i,j}^{n+1} - \rho_{i,j}^{n+1/2}\right| &\leq \Delta t \left|S_{\mathrm{O}}(\rho_1^n, \rho_2^n, \tilde{\rho}_1^n, \tilde{\rho}_2^n) - S_{\mathrm{R}}(\rho_1^n, \rho_2^n, \tilde{\rho}_1^n, \tilde{\rho}_2^n)\right| \\
&\leq \Delta t \left|\partial_1 S_{\mathrm{O}}(\rho_1^n, \rho_2^n, \tilde{\rho}_1^n, \tilde{\rho}_2^n)\rho_1^n + \partial_2 S_{\mathrm{R}}(\rho_1^n, \rho_2^n, \tilde{\rho}_1^n, \tilde{\rho}_2^n)\rho_2^n\right| \\
&\leq \Delta t \max\left\{ \left\|\partial_1 S_{\mathrm{O}}\right\|_{\mathbf{L}^\infty}, \left\|\partial_2 S_{\mathrm{R}}\right\| \right\}_{\mathbf{L}^\infty} \left(\rho_{i,j}^n + \rho_{[i],j}^n \right),
\end{aligned}$$

then, multiplying by Δx and summing over all $j \in \mathbb{Z}$,

$$\Delta x \sum_{j\in\mathbb{Z}} \left|\rho_{i,j}^{n+1} - \rho_{i,j}^{n+1/2}\right| \tag{6.3.40}$$
$$\leq \Delta t \max\left\{ \left\|\partial_1 S_{\mathrm{O}}\right\|_{\mathbf{L}^\infty}, \left\|\partial_2 S_{\mathrm{R}}\right\| \right\}_{\mathbf{L}^\infty} \left(\left\|\rho_i^n\right\|_{\mathbf{L}^1(\mathbb{R})} + \left\|\rho_{[i]}^n\right\|_{\mathbf{L}^1(\mathbb{R})} \right). \tag{6.3.41}$$

Now we analyze the second term on the right hand side of (6.3.39). Since the numerical flux defined in (6.3.2) is Lipschitz continuous in all their arguments with Lipschitz constant $\mathcal{L} = \max\{\left\|\partial_1 F\right\|, \left\|\partial_2 F\right\|, \left\|\partial_3 F\right\|\}$, we get

$$\left|\rho_{i,j}^{n+1/2} - \rho_{i,j}^{n}\right| = \left|\lambda \left(F(\rho_j^n), \rho_{j+1}^n, \tilde{R}_{[i],j+1/2}^n) - F(\rho_j^n), \rho_{j+1}^n, \tilde{R}_{[i],j+1/2}^n) \right)\right|$$

$$\leq \quad \lambda\mathcal{L}\left(\left|\rho^n_{i,j}-\rho^n_{i,j-1}\right|+\left|\rho^n_{i,j+1}-\rho^n_{i,j}\right|+\left|\tilde{R}^n_{[i],j+1/2}-\tilde{R}^n_{[i],j-1/2}\right|\right).$$

then multiplying by Δx and summing over all $j \in \mathbb{Z}$ we have

$$\Delta x \sum_{j\in\mathbb{Z}}\left|\rho^{n+1/2}_{i,j}-\rho^n_{i,j}\right|$$
$$\leq \Delta t\mathcal{L}\left(\sum_{j\in\mathbb{Z}}\left|\rho^n_{i,j}-\rho^n_{i,j-1}\right|+\sum_{j\in\mathbb{Z}}\left|\rho^n_{i,j+1}-\rho^n_{i,j}\right|+\sum_{j\in\mathbb{Z}}\left|\tilde{R}^n_{i,j+1/2}-\tilde{R}^n_{i,j+1/2}\right|\right),$$

and noticing that

$$\sum_{j\in\mathbb{Z}}\left|\tilde{R}^n_{i,j+1/2}-\tilde{R}^n_{i,j+1/2}\right|\leq \mathrm{TV}(\tilde{\rho}^n_{[i]}),$$

we get the following estimate

$$\Delta x \sum_{j\in\mathbb{Z}}\left|\rho^{n+1/2}_{i,j}-\rho^n_{i,j}\right| \quad\leq\quad 2\Delta t\mathcal{L}\left(\mathrm{TV}(\rho^n_i)+\mathrm{TV}(\tilde{\rho}^n_{[i]})\right). \tag{6.3.42}$$

Collecting together (6.3.40), (6.3.42) and summing for $i = 1, 2$ we obtain

$$\Delta x \sum_{i=1}^{2}\sum_{j\in\mathbb{Z}}\left|\rho^{n+1}_{i,j}-\rho^n_{i,j}\right|$$
$$\leq 2\Delta t \max\left\{\left\|\partial_1 S_{\mathrm{O}}\right\|_{\mathbf{L}^\infty}, \left\|\partial_2 S_{\mathrm{R}}\right\|_{\mathbf{L}^\infty}\right\}\left(\left\|\rho^n_i\right\|_{\mathbf{L^1}(\mathbb{R})}+\left\|\rho^n_{[i]}\right\|_{\mathbf{L^1}(\mathbb{R})}\right)$$
$$+2\Delta t\mathcal{L}\sum_{i=1}^{2}\left(\mathrm{TV}(\rho^n_i)+\mathrm{TV}(\tilde{\rho}^n_{[i]})\right),$$

by using Lemma 6.6 and Proposition 6.3.30 we get

$$\Delta x \sum_{i=1}^{2}\sum_{j\in\mathbb{Z}}\left|\rho^{n+1}_{i,j}-\rho^n_{i,j}\right|$$
$$\leq 2\Delta t \max\left\{\left\|\partial_1 S_{\mathrm{O}}\right\|_{\mathbf{L}^\infty}, \left\|\partial_2 S_{\mathrm{R}}\right\|_{\mathbf{L}^\infty}\right\}\left(\left\|\rho^0_i\right\|_{\mathbf{L^1}(\mathbb{R})}+\left\|\rho^0_{[i]}\right\|_{\mathbf{L^1}(\mathbb{R})}\right)$$
$$+2\Delta t\mathcal{L}e^{T(2K+\mathcal{H})}\left(\sum_{i=1}^{2}\left(\mathrm{TV}(\rho^0_i)+\mathrm{TV}(\tilde{\rho}^0_{[i]})\right)+2T\mathcal{G}\right). \tag{6.3.43}$$

Finally, collecting together (4.3.13), (6.3.43) and summing for n from 0 until $N_T - 1$ we get the following **BV** bound in space and time

$$\sum_{n=0}^{N_T-1}\sum_{i=1}^{2}\sum_{j\in\mathbb{Z}}\Delta t\left|\rho^n_{i,j+1}-\rho^n_{i,j}\right|+(T-N_T\Delta t)\sum_{i=1}^{2}\sum_{j\in\mathbb{Z}}\left|\rho^{N_T}_{i,j+1}-\rho^{N_T}_{i,j}\right|$$

$$+\Delta x \sum_{i=1}^{2} \sum_{j\in\mathbb{Z}} \left|\rho_{i,j}^{n+1} - \rho_{i,j}^{n}\right|$$

$$\leq T e^{T(2K\mathcal{H})} (1+2\mathcal{L}) \left(\sum_{i=1}^{2} \left(\mathrm{TV}(\rho_i^0) + \mathrm{TV}(\tilde{\rho}_{[i]}^0) \right) + 2T\mathcal{G} \right)$$

$$+2T \max\left\{ \|\partial_1 S_{\mathrm{O}}\|_{\mathbf{L}^\infty}, \|\partial_2 S_{\mathrm{R}}\|_{\mathbf{L}^\infty} \right\} \left(\left\|\rho_i^0\right\|_{\mathbf{L^1}(\mathbb{R})} + \left\|\rho_{[i]}^0\right\|_{\mathbf{L^1}(\mathbb{R})} \right). \quad (6.3.44)$$

□

6.3.3 Proof of Theorem 6.2

The convergence of the approximate solutions constructed by **Algorithm 6.3.1** towards the weak solution can be proven by applying Helly's compactness theorem. The latter can be applied due to Lemma 6.5, Lemma 6.7 and Corollary 6.10 and states that there exists sub-sequences of approximate solutions $\boldsymbol{\rho^\Delta}$ and $\boldsymbol{\tilde{\rho}^\Delta}$ that converge in $\mathbf{L^1}$ to functions $\boldsymbol{\rho}, \boldsymbol{\tilde{\rho}} \in \mathbf{L}^\infty([0,T]\times\mathbb{R};\mathbb{R}^+)$, respectively.

Now we need to prove that this limit function is indeed a weak solution to (6.2.1), in the sense of Definition 6.1.

Lemma 6.11. *Let $\boldsymbol{\rho_j^0}, \boldsymbol{\tilde{\rho}_j^0} \in BV(\mathbb{R};\mathbb{R}^+)$, and the Assumptions 6.2.1 and the CFL condition* (6.3.21) *be in effect. Then the piecewise constant approximate solutions $\boldsymbol{\rho^\Delta}$, $\boldsymbol{\tilde{\rho}^\Delta}$ resulting from the **Algorithm 6.3.1** converge, as $\Delta x \to 0$, towards an weak solution of* (6.2.1).

Proof. Let $\varphi \in C_{\mathrm{c}}^1([0,T];\mathbb{R}^+)$ for some $T>0$. Multiplying first (6.3.15) by $\Delta x \varphi(t^n, x_j)$ and summing over $j \in \mathbb{Z}$ and over $n = 0, \dots, N_T$ yields

$$\underbrace{\Delta x \sum_{j\in\mathbb{Z}} \sum_{n=0}^{N_T} \left(\boldsymbol{\rho_j^{n+1/2}} - \boldsymbol{\rho_j^n} \right) \varphi(t^n, x_j)}_{I_1} + \underbrace{\Delta t \sum_{j\in\mathbb{Z}} \sum_{n=0}^{N_T} \left(\boldsymbol{F_{j+1/2}^n} - \boldsymbol{F_{j-1/2}^n} \right) \varphi(t^n, x_j)}_{I_2} = 0.$$

We first consider I_1.

$$I_1 = -\Delta t \Delta x \sum_{j\in\mathbb{Z}} \sum_{n=0}^{N_T} \boldsymbol{\rho_j^{n+1/2}} \frac{\left(\varphi(t^{n+1/2}, x_j) - \varphi(t^n, x_j)\right)}{\Delta t} - \Delta t \sum_{j\in\mathbb{Z}} \boldsymbol{\rho_j^0} \varphi(0, x_j),$$

and by the the Dominate Convergence Theorem, we get for $i = 1, 2$.

$$I_1 \to -\int_0^T \int_{\mathbb{R}} \boldsymbol{\rho_i}(t,x) \partial_x \varphi(t,x) \mathrm{d}x \mathrm{d}t - \int_{\mathbb{R}} \boldsymbol{\rho_j^0} \varphi(0,x) \mathrm{d}x.$$

We now study I_2. This term can be rewrite as

$$I_2 \quad = \quad -\Delta t \Delta x \sum_{j\in\mathbb{Z}} \sum_{n=0}^{N_T} \boldsymbol{F^n_{j+1/2}} \left(\frac{\varphi(t^n, x_{j+1}) - \varphi(t^n, x_j)}{\Delta x} \right),$$

and again by Dominate Convergence Theorem we get

$$I_2 \to -\int_0^t \int_{\mathbb{R}} \boldsymbol{F} \partial_x \varphi(t,x) \mathrm{d}x \mathrm{d}t,$$

where $\boldsymbol{F} = \rho_i v_i(\rho_i + (\rho_{\max} - \rho_i)\chi_\varepsilon(\tilde{\rho}_{[i]} * \omega_\eta))$ thus

$$I_1 + I_2 \to -\int_0^T \int_{\mathbb{R}} \boldsymbol{\rho_i}(t,x) \partial_x \varphi(t,x) \mathrm{d}x \mathrm{d}t - \int_{\mathbb{R}} \boldsymbol{\rho^0_j} \varphi(0,x) \mathrm{d}x - \int_0^t \int_{\mathbb{R}} \boldsymbol{F} \partial_x \varphi(t,x) \mathrm{d}x \mathrm{d}t.$$

Now, the next step in the proof is to multiply (6.3.19) by $\Delta x \varphi(t^n, x_j)$ and summing over $j \in \mathbb{Z}$ and over $n = 0, \ldots, N_T$ yields

$$\Delta x \sum_{j\in\mathbb{Z}} \sum_{n=0}^{N_T} \left(\boldsymbol{\rho^{n+1}_j} - \boldsymbol{\rho^{n+1/2}_j} \right) \varphi(t^n, x_j)$$
$$-\Delta x \Delta t \sum_{j\in\mathbb{Z}} \sum_{n=0}^{N_T} \left(-\boldsymbol{S^{n+1/2}_j}, \boldsymbol{S^{n+1/2}_j} \right) \varphi(t^n, x_j) = 0,$$

replacing $\boldsymbol{\rho^{n+1/2}_j}$ of (6.3.15) in above equality we get

$$I_1 + I_2 + I_3 = 0,$$

where

$$I_3 = -\Delta x \Delta t \sum_{j\in\mathbb{Z}} \sum_{n=0}^{N_T} \left(-\boldsymbol{S^{n+1/2}_j}, \boldsymbol{S^{n+1/2}_j} \right) \varphi(t^n, x_j),$$

of which we can deduce that

$$I_3 \to -\int_0^T \int_{\mathbb{R}} \left(-\left(\boldsymbol{S_O} - \boldsymbol{S_R} \right), \boldsymbol{S_O} - \boldsymbol{S_R} \right) \varphi(t,x).$$

Therefore,

$$\int_0^T \int_{\mathbb{R}} \boldsymbol{\rho_i}(t,x) \partial_x \varphi(t,x) \mathrm{d}x \mathrm{d}t + \int_0^t \int_{\mathbb{R}} \boldsymbol{F} \partial_x \varphi(t,x) \mathrm{d}x \mathrm{d}t$$
$$+ \int_0^T \int_{\mathbb{R}} \left(-\left(\boldsymbol{S_O} - \boldsymbol{S_R} \right), \boldsymbol{S_O} - \boldsymbol{S_R} \right) \varphi(t,x)$$
$$+ \int_{\mathbb{R}} \boldsymbol{\rho^0_j} \varphi(0,x) \mathrm{d}x = 0.$$

□

6.4 Numerical examples

In the following numerical tests we will solve (6.2.1) numerically for $x \in [0, 5]$ and we set Δt satisfying CFL condition (6.3.21). In each numerical example we consider $\eta = 0.1$, $\delta = 0.5$ and the same maximum velocity for each class of vehicles, namely $V_{\max} = 1$, we also consider the following regularization for the indicator function

$$\chi_\varepsilon(x) = \begin{cases} 0, & \text{if} \quad x < 0 \\ \exp(-50(\frac{x-\varepsilon}{\varepsilon})^2), & \text{if} \quad 0 \le x \le \varepsilon \\ 1, & \text{if} \quad x > \varepsilon, \end{cases}$$

with $\varepsilon = 0.1$.

6.4.1 Example 12.

This example is intended to describe the behaviour of vehicles on a two-lanes and two-way road in which the lane 1 is occupied for vehicles traveling from left to right and there are no vehicles traveling in opposite direction, i.e we have a empty road with presence only of vehicles ρ_1, more specifically we consider $\rho_1^0 = \begin{cases} 0.5 & \text{if} \quad 0.2 < x < 0.6 \\ 0.9 & \text{if} \quad 1 < x < 2 \end{cases}$ and $\rho_2^0 = \tilde{\rho}_1^0 = \tilde{\rho}_2^0 = 0$. In Figure 2.6.1 we show the evolution of $\boldsymbol{\rho}^{\boldsymbol{\Delta}}(\cdot, t)$ for $t \in [0, 2.5]$ with $\Delta x = 1/160$. This initial condition indicate that initially there are two queues of vehicles on the Lane 1, the first one with a medium density and second one with high density concentration; the first queue is shorter than second one. As time progresses we can observe that vehicles at the head of the first queue advance forming a rarefaction wave and then a few number of vehicles overtake using the Lane 2, also, we can see a few number of vehicles in the back of first queue overtaking to the Lane 2. Regarding to second queue, we can observe that between $x = 1$ and $x = 1.5$ a high concentration of vehicles overtaking using the Lane 2, after a while that vehicles return to Lane 1, and between $t = 0.6$ and $t = 1.8$, approximately, we see the formation of a shock due to the difference of densities concentration of vehicles and finally, the vehicles at the head of second queue form a rarefaction wave, it means that vehicles advance normally on the Lane 1 without to do overtaking.

6.4.2 Example 13

Main aim of this example is show that model proposed (6.2.1) doesn't allow crashes between vehicles. For this, we consider $\rho_1^0 = 0.9$ for $x \in [0.5, 1.5]$ and $\tilde{\rho}_2^0 = 0.9$ for $x \in [2.5, 3]$. This numerical example show a extreme case in which the specie ρ_1 travel on lane 1 and there is a lane invasion by the specie traveling in opposite direction, i.e., there is presence of $\tilde{\rho}_2$ closely to ρ_1 on the Lane 1. In Figure 6.4.2 we can see the evolution of the behavior of vehicles for $t \in [0, 2.5]$ with $\Delta x = 1/160$. The initial condition tell us that initially there are a big concentration of vehicles of both classes ρ_1 and $\tilde{\rho}_2$. As time progresses, vehicles of the class $\tilde{\rho}_2$ return quickly to their preferred lane, $\tilde{\rho}_1$, while vehicles

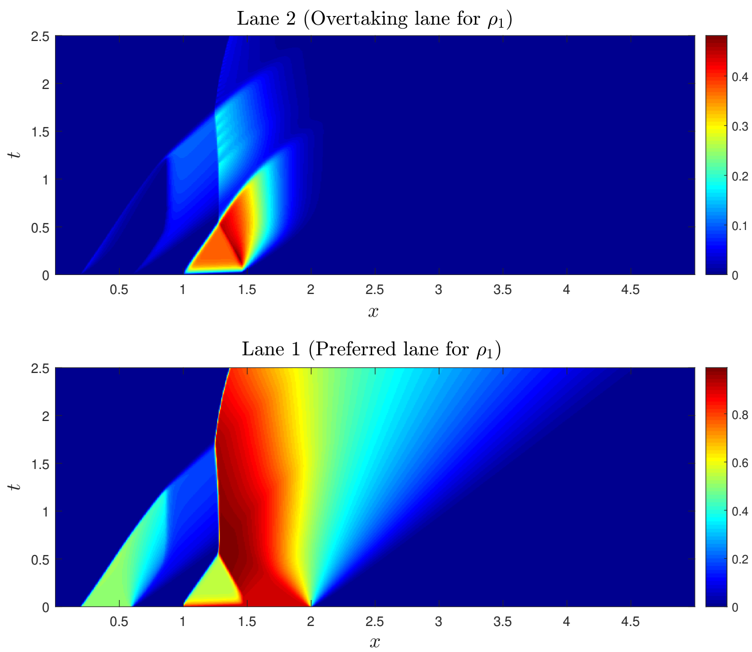

Figure 6.4.1: Numerical approximation of solutions in Example 12 at time $t \in [0, 2.5]$. Bottom: Vehicles of class ρ_1 traveling from left to right on Lane 1. Top: vehicles of class $\tilde{\rho}_2$ using lane 2 for overtaking.

of the class ρ_1 at back of queue, namely $x \in [0.5, 1]$, do overtaking using the Lane 2, while at head of queue, vehicles ρ_1 advance normally forming a rarefaction wave. After, on Lane 2, with respect to class $\tilde{\rho}_1$, we can see the formation of a shock at $t = 0.75$ and $x = 1.625$, approximately, it happens because the vehicles that have been overtaking, ρ_2, and the vehicles of $\tilde{\rho}_1$ that returned from lane 1 are facing each other; at that moment, the vehicles of the $\tilde{\rho}_1$ class wait for those coming from the opposite direction to return to their preferred lane, so that they can continue moving forward. Now, regarding the class ρ_2, from $x = 1$ we can see a decrease in density for all t, it means that after overtaking vehicles return to their preferred lane. In Figure 6.4.3 we see the behaviour of approximate solutions in three different times, $t = 0$, $t = 0.3$ and $t = 1$.

6.4.3 Example 14

With this example we want to illustrate the convergence of approximate solutions. For this end, we consider the same parameters of Example 13 at time $t = 2.5$. In Figure 6.4.4 we can see several approximate solutions computing by means Algorithm 6.3.1 for $\Delta x = 1/20, 1/40, 1/80, 1/160$ and reference solution computed for $\Delta x = 1/640$. As expected,

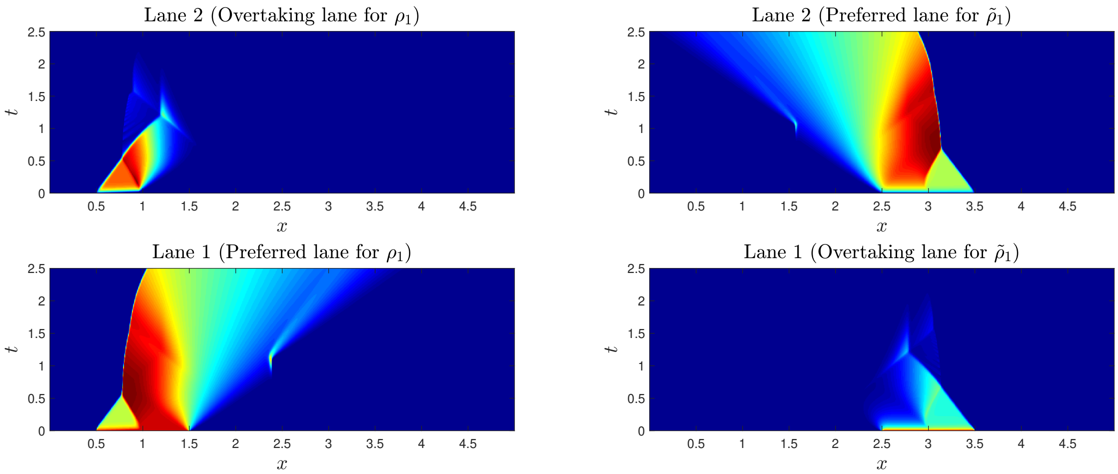

Figure 6.4.2: Example 13. Bottom left: Vehicles of class ρ_1 traveling from left to right on Lane 1. Bottom right: Vehicles of class $\tilde{\rho}_2$ traveling from right to left on Lane 1. Top left: vehicles of class ρ_2 traveling from left to right, using lane 2 for overtaking. Top right: vehicles of class $\tilde{\rho}_1$ traveling from right to left, using lane 2 for returning, $t \in [0, 2.5]$.

as Δx goes to 0 the numerical solutions better approximate the reference solution, what can be checked in Table 6.4.1, in which the total error and the Experimental Order of Convergence (E.O.C.) in $\mathbf{L^1}$ norm are shown.

Table 6.4.1: Example 14: Total L^1-error $e_{\Delta x}(u)$ for numerical solutions computed by means of HW-type numerical scheme in Algorithm 6.3.1 with $\Delta x = 1/M$, $M = 20, 40, 80, 160$.

$T = 0.25$		
$1/\Delta x$	Total $e_{\Delta x}$	E.O.C.
20	0.274	—
40	0.1438	1.0134
80	0.0736	1.0932
160	0.0418	0.8012

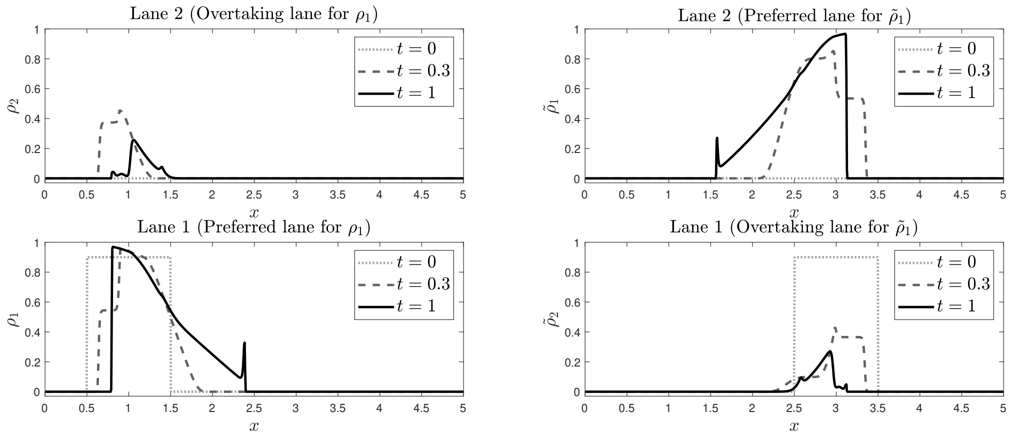

Figure 6.4.3: Example 13: numerical solutions of system (6.2.1) at $t = 0$, $t = 0.3$ and $t = 1$. Bottom left: Density ρ_1 of vehicles traveling from left to right on Lane 1. At $t = 0$ we see the initial queue, at $t = 0.3$ a small shock with negative velocity is formed on the back of queue and a rarefaction wave on their head; at $t = 1$ vehicles warn of the proximity of those coming from the opposite direction, this makes them slow down and the density begins to increase. Top left: Density ρ_2 of vehicles traveling from left to right. At $t = 0$ we have the initial condition, at $t = 0.3$ we observe an increasing of density, because vehicles of ρ_1 are using lane 2 for overtaking and at $t = 1$ we observe a decreasing of density because vehicles are returning to their preferred lane. Bottom right: Density $\tilde{\rho}_2$ of vehicles traveling from right to left on the lane 1. At $t = 0$ we observe the initial condition, at $t = 0.3$ it is observed that a big density of vehicles return to their preferred lane, lane 2, and at $t = 1$ density continues decreasing because vehicles continue returning to lane 2. Top right: Density $\tilde{\rho}_1$ of vehicles traveling from right to left. At $t = 0$ we see $\tilde{\rho}_1 = 0$, at $t = 0.3$ we observe an increasing of density, because vehicles of $\tilde{\rho}_2$ are returning and then we observe the same pattern as in bottom left, even for $t = 1$.

6.5 Conclusions of Chapter 6

In this chapter we introduced a system of nonlocal balance laws which describes vehicular traffic flow in a two way and two lane road. System allows vehicles to overtake to adjacent lane and return to their preferred lane, namely, lane to their right. We distinguish four classes of vehicles, labeled ρ_1, ρ_2, $\tilde{\rho}_1$, $\tilde{\rho}_2$, according to the direction of travel and the lane that they use. We provided compactness estimates that allowed us to apply the Helly's Compactness Theorem to prove existence and convergence of entropy solutions. We show some numerical experiments in which some features of the model are displayed, namely, no crashes between vehicles, the overtaking and returning maneuvers, and also that two vehicles of different classes traveling in opposite direction can not

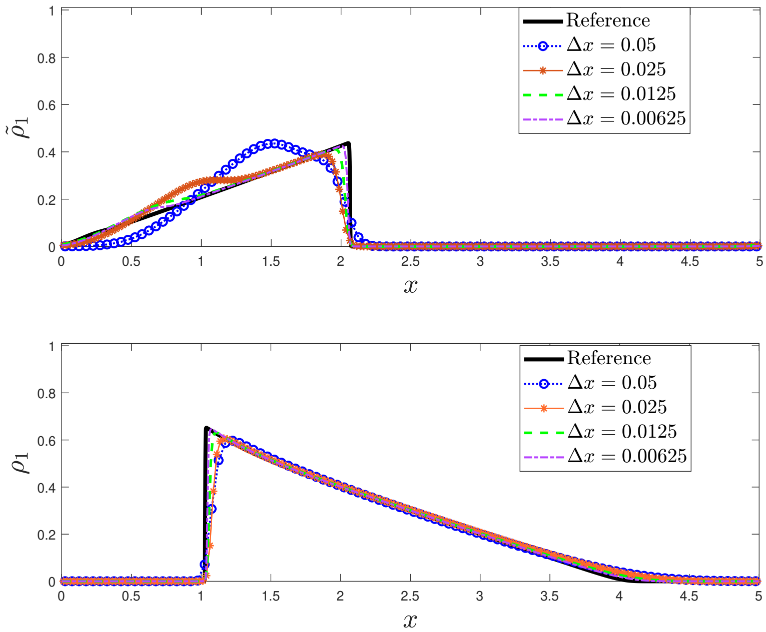

Figure 6.4.4: Example 14. Bottom: Convergence of sequence of approximate solutions ρ_1^Δ to reference solution. Top: Convergence of sequence of approximate solutions $\tilde{\rho}_1^\Delta$ to reference solution.

occupy the same cell. We was not able to prove a maximum principle for the proposed numerical scheme, but in all the numerical examples shown we can observe that the sum of vehicles density traveling in opposite direction in a same lane is always less than 1.

Chapter 7

Conclusions and future work

In this chapter we present a discussion of the main results of this thesis and a description of the future work to develop.

7.1 Conclusions

The main results of the thesis are related to the modeling, analysis and numerical approximation of nonlocal balance laws that model sedimentation and vehicular traffic. In this thesis we provided new models that allow to describe vehicular traffic flow on roads with different realistic features, e.g., roads with rough conditions, roads with on-off ramps and two way two lane roads. Likewise, this work also provides an approach for a rigorous treatment of boundary conditions in the case of a spatially one-dimensional nonlocal IBVP which models, for instance, a batch sedimentation process in a closed column, moreover we developed new numerical schemes that are more accurate and less diffusive in comparison to schemes based on the Lax-Friedrichs flux. We studied well-posedness of each introduced model and developed numerical simulations in order to show the behaviour of solutions. Here we present a summary with the main contributions and conclusions of the thesis.

The simulation model proposed in [55] can be understood as a simple method for approximating solutions of scalar conservation laws whose flux is of density times velocity type, where the density and velocity factors are evaluated on neighboring cells. The resulting scheme converges to the unique entropy solution of the underlying problem. The same idea was applied in the Chapter 2 of this thesis in order to devise a numerical scheme for a class of one-dimensional scalar conservation laws with nonlocal flux and initial and boundary conditions. The new developed numerical scheme takes advantage of the form in which the flow is written and results less diffusive than schemes based on the Lax-Friedrichs flux. Numerical experiments provided evidence of performance of the numerical scheme.

In Chapter 3 we proved the well-posedness for a class of space-discontinuous scalar conservation laws with non-local flux arising in traffic modeling. We approximate the problem by means of a HW-type numerical scheme and we show convergence to a entropy weak solution of nonlocal conservation law considered in this chapter. We also establish $\mathbf{L^1}$ stability, and thus uniqueness for weak solutions satisfying the entropy condition.

In Chapter 4 we propose a new traffic flow model with nonlocal flux, which through the source and sink terms includes the effects of the inflow and output flow over the on- and off- ramps, respectively. The source term include a nonlocal term which models the fact that drivers on the on-ramp can see what happens behind and in front of them on the main road. We proved well-posedness of the proposed model and present some numerical experiments in order to show the effect of including ramps in a road. A limit model as the kernel support tends to zero is numerically investigated.

Chapter 5 was focused on to prove the stability of entropy weak solutions of a nonlocal balance law that models vehicular traffic flow on a road with on- and off-ramps, with respect to a function kernel present in a source term. We get an estimate of the dependence of the solution with respect to the initial datum, the on-ramp rate, the off-ramp rate and the mentioned kernel function. We also shown a sample numerical experiment in order to model an optimization problem in traffic flow with on-ramps.

Chapter 6, on the basis of assumptions about of behaviour of drivers concerning acceleration or slow down and lane changing maneuvers, a macroscopic traffic flow model for two lanes and two way road is constructed. The model allows maneuvers of lane changes, i.e. overtaking and returning between the two lanes, due to vehicle interactions, such maneuvers are modeled through source and sink terms. The resulting model is a system of balance law with nonlocal flux functions, in which each equation is a generalization of nonlocal LWR models proposed in [14, 51], but unlike those works, the model developed here considers velocity functions that depends not only on density of the class of vehicles traveling in their preferred lane but also of density of vehicles traveling in opposite direction on each lane. We proved the existence of weak solutions of the system and for that we use the HW-type numerical scheme in order to approximating solutions and proved that the limit of the discrete solutions constitutes a weak solution. In addition, we show numerical simulations in order to describe different traffic situations.

7.2 Future works

The methods developed and the results obtained in this thesis have motivated several ongoing and future projects. Some of them are described below:

1. **Vehicular traffic.** In the frame of this topic it has been proposed:

    - **Multiclass and multilane traffic flow with on- and off-ramps.** It is well known that on-ramp merging has a great impact on traffic efficiency, if one concentrates on highway networks the reduction of the capacity is often due to on- and off-ramps, for this reason is important to construct more generalized macroscopic models in order to describe the dynamics of multiclass traffic flow on a multilane highways including on- and off-ramps. In this sense, the non-local models play a key role regarding lane changes and vehicular speed control. Taking into account this, It is natural propose to extend to its multilane and multiclass version the non-local model with on- and off-ramps (4.2.1) introduced in the Chapter 4 and study their well-posedness. A representation of such a model is shown in Figure 7.2.1.

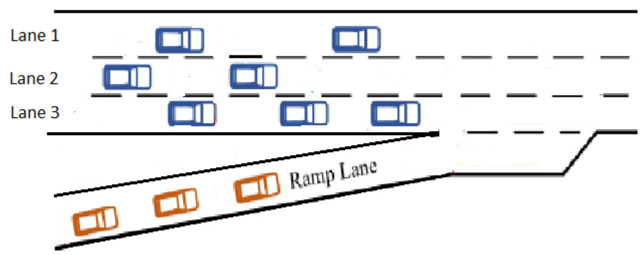

Figure 7.2.1: Section of a highway with three lanes an on-ramp.

- **Second-order schemes.** We would like to consider a second-order version of the model (4.2.1) as in [82].
- **Traffic flow management by means of calibrating models with real data.** Due to the realistic features introduced in the traffic models studied here and the rapidly growing amount of data obtained by GPS, navigating tools like Google Maps or Waze and mobile phones within vehicles, it is necessary to consider the objective of developing a data-driven modeling approach for accurate calibration and simulation of vehicular traffic in real-life transportation networks, with applications in real-time decision support systems and urban planning. Moreover, by comparing model predictions with data and changing the values of the model parameters to obtain a maximum fit, a model can be calibrated which is a prerequisite for any meaningful application

2. **Sedimentation processes.** It has been proposed

- **Non-local balance laws with memory.** In sedimentation processes one of the forces that may act on a given particle is the Basset force, which addresses the temporal delay in the development of the boundary layer surrounding the particle as a consequence of changes in the relative velocity. This force is usually called the "history" force. This force is represented by an integral whose accurate numerical evaluation is rather difficult. Taking into account this, I am interested in develop approximate solutions to the non-local initial value problem

$$\gamma u_t + \beta * u_t + f(u)_x + g(y)_y = \Delta A(u), \quad (x,y,t) \in \Pi_T := \mathbb{R}^2 \times (0,T) \tag{7.2.1}$$
$$u(x,y,0) = u_0(x,y) \quad (x,y) \in \mathbb{R}^2, \tag{7.2.2}$$

We assume that $A(u) = \int_0^u a(s)ds, \quad a(u) \geq 0$. Moreover, γ is a parameter that may assume the values $\gamma = 0$ or $\gamma = 1$, which also includes all cases $0 < \gamma < 1$ by suitable scaling. the memory term $\beta * u_t$ denotes the Volterra convolution integral

$$\beta * u_t = (\beta * u)_t(x,y,t) = \int_0^t \beta(t-s)u_t(x,y,s)ds.$$

Here we assume that for all $t > 0$, $\beta \in L^1_{loc}(\mathbb{R}^+)$, $\beta(t) \geq 0$ is a non-increasing function. Of particular interest is the choice

$$\beta(t) = \frac{1}{\Gamma(1-\alpha)} t^{-\alpha}, \quad \alpha \in (0,1),$$

for which $\beta * u_t$ corresponds to the Caputo fractional derivative of order α. In order to carry out the proposed study I will rely on [69], where a numerical scheme was considered to approach an one dimensional conservation laws with memory represented by a Volterra term with a smooth decreasing but possibly unbounded kernel. Likewise, another important work to base the study of the proposed problem is [41], where the authors consider consistent, conservative-form, monotone difference schemes for a strongly degenerate nonlinear convection-diffusion equation in one space dimension in which solutions can be discontinuous and, in general, are not uniquely determined by their data, and they prove that the difference scheme converge to the unique **BV** entropy weak solution of the problem.

3. **Pedestrian flow.** We aim to investigate how to extend the numerical scheme developed in this thesis in applications such as crowd dynamics models where the function $V(x,t)$ is a vector field containing the preference directions and nonlocal corrections terms.

Appendix

Appendix A: Technical estimates

In this appendix we show the computations for some estimates used in the proofs of above sections. First, we prove estimates (2.4.11) in the proof of Lemma 2.8. We denote $y_k := (k-1/2)\Delta x$, for $k \in \mathbb{Z}$ and then we compute

$$
\begin{aligned}
&\left|V^n_{j+3/2} - V^n_{j+1/2}\right| \\
&= \left| \frac{\Delta x}{W_{j+3/2}} \sum_{k=1}^{M} \omega^{k-j-1} v(\rho^n_k) - \frac{\Delta x}{W_{j+1/2}} \sum_{k=1}^{M} \omega^{k-j} v(\rho^n_k) \right| \\
&= \left| \frac{\Delta x}{W_{j+3/2}} \sum_{k=1}^{M} (\omega^{k-j-1} - \omega^{k-j}) v(\rho^n_k) + \Delta x \left(\frac{1}{W_{j+3/2}} - \frac{1}{W_{j+1/2}} \right) \sum_{k=1}^{M} \omega^{k-j} v(\rho^n_k) \right| \\
&\leq \left| \frac{\Delta x}{W_{j+3/2}} \sum_{k=1}^{M} (\omega^{k-j-1} - \omega^{k-j}) v(\rho^n_k) \right| \\
&\quad + \left| \frac{\Delta x^2}{W_{j+3/2} W_{j+1/2}} \left(\sum_{k=1}^{M} \omega^{k-j} v(\rho^n_k) \right) \sum_{k=1}^{M} \left(\omega^{k-j} - \omega^{k-j-1} \right) \right| \\
&\leq \frac{\Delta x \|v\|_\infty}{|W_{j+3/2}|} \sum_{k=1}^{M} \left| \omega^{k-j-1} - \omega^{k-j} \right| \\
&\quad + \frac{\Delta x \|v\|_\infty}{W_{j+3/2} W_{j+1/2}} \left| \Delta x \left(\sum_{k=1}^{M} \omega^{k-j} \right) \sum_{k=1}^{M} \left(\omega^{k-j} - \omega^{k-j-1} \right) \right| \\
&\leq \frac{\Delta x}{K_\omega} \|v\|_\infty \left(\sum_{k=1}^{M} \left| \int_{y_{k-j-1}}^{y_{k-j}} \omega'(y) \mathrm{d}y \right| + \sum_{k=1}^{M} \left| \omega^{k-j} - \omega^{k-j-1} \right| \right) \leq \mathcal{L} \Delta x
\end{aligned}
$$

with $\mathcal{L} = 2K_\omega^{-1} \|v\|_\infty \|\omega'\|_{L^1(\mathbb{R})}$. Now following closely [50], we compute

$$
\begin{aligned}
&\left|V^n_{j+3/2} - 2V^n_{j+1/2} + V^n_{j-1/2}\right| \\
&= \left| \Delta x \sum_{k=1}^{M} v(\rho^n_k) \left(\frac{\omega^{k-j-1}}{W_{j+3/2}} - \frac{\omega^{k-j}}{W_{j+1/2}} - \frac{\omega^{k-j}}{W_{j+1/2}} + \frac{\omega^{k-j+1}}{W_{j-1/2}} \right) \right|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \left| \frac{\Delta x^3}{W_{j+3/2}W_{j+1/2}} \left(\left(\sum_{k=1}^{M} v(\rho_k^n) \int_{\xi_{k-j-1/2}}^{\xi_{k-j+1/2}} \omega''(y)\,\mathrm{d}y \right) \sum_{l=1}^{M} \omega^{l-j} \right.\right. \\
&\qquad \left.\left. - \left(\sum_{k=1}^{M} v(\rho_k^n)\omega^{k-j} \right) \sum_{l=1}^{M} \int_{\xi_{l-j-1/2}}^{\xi_{l-j+1/2}} \omega''(y)\,\mathrm{d}y \right) \right| \\
&\quad + \left| \frac{2\Delta x^4 \|\omega'\|_{L^1}}{W_{j+3/2}W_{j+1/2}W_{j-1/2}} \left(\left(\sum_{k=1}^{M} v(\rho_k^n)\omega^{k-j} \right) \sum_{l=1}^{M} \omega'(\xi_{l-j+1/2}) \right.\right. \\
&\qquad \left.\left. - \left(\sum_{k=1}^{M} v(\rho_k^n)\omega'(\xi_{k-j+1/2}) \right) \sum_{l=1}^{M} \omega^{l-j} \right) \right| \\
&\leq \left| \frac{\Delta x^2 \|v\|_\infty}{W_{j+3/2}W_{j+1/2}} \left(\sum_{k=1}^{M} \int_{\xi_{k-j-1/2}}^{\xi_{k-j+1/2}} \omega''(y)\,\mathrm{d}y \right) \Delta x \sum_{l=1}^{M} \omega^{l-j} \right| \\
&\quad + \left| \frac{\Delta x^2 \|v\|_\infty}{W_{j+3/2}W_{j+1/2}} \left(\Delta x \sum_{k=1}^{M} \omega^{k-j} \right) \sum_{l=1}^{M} \int_{\xi_{l-j-1/2}}^{\xi_{l-j+1/2}} \omega''(y)\,\mathrm{d}y \right| \\
&\quad + \left| \frac{2\Delta x^2 \|\omega'\|_{L^1(I)} \|v\|_\infty}{W_{j+3/2}W_{j+1/2}W_{j-1/2}} \left(\Delta x \sum_{k=1}^{M} \omega^{k-j} \right) \Delta x \sum_{l=1}^{M} \omega'(\xi_{l-j+1/2}) \right| \\
&\quad + \left| \frac{2\Delta x^2 \|\omega'\|_{L^1(I)} \|v\|_\infty}{W_{j+3/2}W_{j+1/2}W_{j-1/2}} \left(\Delta x \sum_{k=1}^{M} \omega'(\xi_{k-j+1/2}) \right) \Delta x \sum_{l=1}^{M} \omega^{l-j} \right| \\
&\leq \Delta x^2 \|v\|_\infty \left(\left| \frac{1}{W_{j+3/2}} \sum_{k=1}^{M} \int_{\xi_{k-j-1/2}}^{\xi_{k-j+1/2}} \omega''(y)\,\mathrm{d}y \right| \right. \\
&\qquad + \left| \frac{1}{W_{j+3/2}} \sum_{l=1}^{M} \int_{\xi_{l-j-1/2}}^{\xi_{l-j+1/2}} \omega''(y)\,\mathrm{d}y \right| \\
&\qquad + \left| \frac{2\|\omega'\|_{L^1(I)}}{W_{j+3/2}W_{j-1/2}} \Delta x \sum_{l=1}^{M} \omega'(\xi_{l-j+1/2}) \right| \\
&\qquad \left. + \left| \frac{2\|\omega'\|_{L^1(I)}}{W_{j+3/2}W_{j-1/2}} \Delta x \sum_{k=1}^{M} \omega'(\xi_{k-j+1/2}) \right| \right) \\
&\leq \frac{\Delta x^2 \|v\|_\infty}{K_\omega} \left(\left| \sum_{k=1}^{M} \int_{\xi_{k-j-1/2}}^{\xi_{k-j+1/2}} \omega''(y)\mathrm{d}y \right| + \left| \sum_{l=1}^{M} \int_{\xi_{l-j-1/2}}^{\xi_{l-j+1/2}} \omega''(y)\mathrm{d}y \right| \right) \\
&\quad + 2K_\omega^{-2} \Delta x^2 \|\omega'\|_{L^1(I)}^2 \|v\|_\infty + 2K_\omega^{-2} \Delta x^2 \|\omega'\|_{L^1(I)}^2 \|v\|_\infty \leq \Delta x^2 \mathcal{W},
\end{aligned}
$$

where we set $\mathcal{W} := 2K_\omega^{-1}\|v\|_\infty \|\omega''\|_{L^1(\mathbb{R})} + 4K_\omega^{-2}\|\omega'\|_{L^1(I)}^2 \|v\|_\infty$ and $\xi_{k-j+1/2} \in]y_{k-j-1}, y_{k-j}[$. This concludes the proof of estimates (2.4.11) in the proof of Lemma 2.8. Next, we establish estimates (2.3.1) to (2.3.4) in Section 2.5. To this end we calculate

$$
\begin{aligned}
&\big|\partial_x V(x,t)\big| \\
&= \left| -\frac{W'(x)}{(W(x))^2}\int_a^b v(\rho(y,t))\omega(y-x)\,\mathrm{d}y - \frac{1}{W(x)}\int_a^b v(\rho(y,t))\omega'(y-x)\,\mathrm{d}y \right| \\
&\le 2K_\omega^{-1}\|v\|_\infty\|\omega'\|_{L^1(I)},
\end{aligned}
$$

$$
\begin{aligned}
\big|\partial_{xx}^2 V\big| &= \bigg|\frac{-(W(x))^2W''(x)+2(W'(x))W(x)}{(W(x))^4}\int_a^b v(\rho(y,t))\omega(y-x)\,\mathrm{d}y \\
&\quad + \frac{W'(x)}{(W(x))^2}\int_a^b v(\rho(y,t))\omega'(y-x)\,\mathrm{d}y \\
&\quad + \frac{W'(x)}{(W(x))^2}\int_a^b v(\rho(y,t))\omega'(y-x)\,\mathrm{d}y \\
&\quad + \frac{1}{W(x)}\int_a^b v(\rho(y,t))\omega''(y-x)\,\mathrm{d}y\bigg| \\
&\le K_\omega^{-2}\|v\|_\infty\|\omega''\|_{L^1(I)}\|\omega\|_{L^1(I)} + 2K_\omega^{-3}\|v\|_\infty\|\omega'\|_{L^1(I)}\|\omega\|_{L^1(I)} \\
&\quad + 2K_\omega^{-2}\|v\|_\infty\|\omega'\|_{L^1(I)} + K_\omega^{-1}\|v\|_\infty\|\omega''\|_{L^1(I)},
\end{aligned}
$$

$$
\begin{aligned}
\big|V(x,t)-U(x,t)\big| &= \left|\frac{1}{W(x)}\int_a^b \big(v(\rho(y,t))-v(\sigma(y,t))\big)\omega(y-x)\,\mathrm{d}y\right| \\
&= \left|\frac{1}{W(x)}\int_a^b v'(\xi)(\rho-\sigma)(y,t)\omega(y-x)\,\mathrm{d}y\right| \\
&\le \frac{\|\omega\|_{\infty(\mathbb{R})}\|v'\|_\infty}{K_\omega}\int_a^b |\rho(y,t)-\sigma(y,t)|\,\mathrm{d}y,
\end{aligned}
$$

$$
\begin{aligned}
\big|\partial_x V(x,t)-\partial_x U(x,t)\big| &= \bigg| -\frac{W'(x)}{(W(x))^2}\int_a^b \big(v(\rho(y,t))-v(\sigma(y,t))\big)\omega(y-x)\,\mathrm{d}y \\
&\quad - \frac{1}{W(x)}\int_a^b \big(v(\rho(y,t))-v(\sigma(y,t))\big)\omega'(y-x)\,\mathrm{d}y\bigg| \\
&= \bigg| -\frac{W'(x)}{(W(x))^2}\int_a^b v'(\xi)(\rho-\sigma)(y,t)\omega(y-x)\,\mathrm{d}y \\
&\quad - \frac{1}{W(x)}\int_a^b v'(\xi)(\rho-\sigma)(y,t)\omega'(y-x)\,\mathrm{d}y\bigg| \\
&\le \mathcal{M}\int_a^b |\rho(y,t)-\sigma(y,t)|\,\mathrm{d}y,
\end{aligned}
$$

with $\mathcal{M}$ as in (2.3.4).

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