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METODOS DE ELEMENTOS FINITOS MIXTOS PARA ELASTICIDAD INCOMPRESIBLE NO LINEAL

Tesis para optar al grado de
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Resumen

En esta tesis desarrollamos nuevos métodos de elementos finitos mixtos para modelar una clase de problemas en elasticidad incompresible no lineal sobre dominios Lipschitz en el plano. Para este estudio consideramos dos problemas modelos, a saber,

- *un problema de transmisión exterior*, definido por el acoplamiento de un cierto material elástico incompresible no lineal en un dominio acotado con un material elástico incompresible lineal en el dominio complementario no acotado,

- *un problema de valores de contorno con condiciones de frontera mixtas*, definido por un material elástico incompresible no lineal sobre un dominio acotado.

Para el análisis del problema de transmisión utilizamos el método de *Dirichlet-to-Neumann*, que consiste en transformar el problema exterior en un problema de valores de contorno sobre un dominio acotado, utilizando una frontera artificial apropiada sobre la cual el dato de Neumann se define en función del dato de Dirichlet. Esta función, llamada de *Dirichlet-to-Neumann* (DtN), es una condición de frontera no local exacta que se expresa en términos de una serie de Fourier infinita. Este enfoque nos permite definir una formulación variacional mixta, donde el desplazamiento y la presión hidroestática son las incógnitas.

Para el segundo problema, el enfoque variacional está basado en el principio de Hu-Washizu, el cual se caracteriza por el hecho de que, además de las variables *desplazamiento* y *esfuerzo*, se agrega la *deformación* como una tercera incógnita. Incorporando de manera débil la simetría del tensor *esfuerzo*, y la traza del vector *desplazamiento* sobre la frontera de Neumann, ambas como incógnitas auxiliares, obtenemos una nueva formulación variacional mixta para este problema, que tiene una estructura de punto silla doble.

Para las formulaciones mixtas de ambos problemas probamos que los esquemas de Galerkin asociados están bien propuestos, así como también proporcionamos las respectivas razones de convergencia que resultan ser optimales en el tamaño de la malla. Además, realizamos un análisis de error a posteriori para cada formulación, de donde obtenemos estimadores confiables para el cálculo adaptivo de las respectivas soluciones discretas.

Finalmente, para el esquema de Galerkin de punto silla doble, proporcionamos varios resultados numéricos que ilustran el buen comportamiento de los algoritmos de refinamiento de mallas propuestos.

Abstract

In this thesis we develop new mixed finite element methods for the modelling of a class of nonlinear problems in incompressible elasticity on Lipschitz domains in the plane. We consider the following model problems:

- *an exterior transmission problem*, defined by the coupling of a certain nonlinear incompressible elastic material in a bounded domain with a linear incompressible elastic material in the unbounded exterior domain.
- *a boundary value problem with mixed boundary conditions*, defined by a nonlinear incompressible elastic material on a bounded domain.

For the analysis of the transmission problem we use the *Dirichlet-to-Neumann* method, which consists in transforming the exterior problem in a boundary value problem on a bounded domain, by introducing a suitable artificial boundary on which the Neumann data is defined in terms of the Dirichlet data. This function, named *Dirichlet-to-Neumann* (DtN) mapping, is an exact nonlocal artificial boundary condition that is expressed in terms of an infinite Fourier series. This approach allows us to define a mixed variational formulation in which the displacement and the hydrostatic pressure are the unknowns.

For the second problem, the variational approach is based on the Hu-Washizu principle, which is characterized by the fact that, besides the displacement and the stress, it includes the deformation as a third unknown. In addition, the symmetry of the stress tensor is imposed weakly, and the trace of the displacement on the Neumann boundary is also incorporated as an additional unknow, which finally yields a two-fold saddle point operator equation as the resulting variational formulation.

We prove that the associated Galerkin schemes of both mixed formulations are well-posed, provide the corresponding optimal rates of convergence, and derive re-

liable a-posteriori estimates for the adaptive computation of the respective discrete solutions.

Finally, several numerical results illustrating the good performance of the adaptive algorithm for the two-fold saddle point Galerkin scheme are presented.

Contents

Chapter 1

Introducción

1.1 Motivación

El tratamiento numérico de modelos matemáticos (lineales o no lineales) que describen conductas elásticas es de gran importancia en las ciencias y la industria. Cada construcción, estructura, vehículo y máquina, tiene que ser testeada a través de simulaciones numéricas para verificar si ellos pueden soportar varios tipos de fuerzas externas y de tracción a las cuales serán sometidas. Tales simulaciones son también necesarias para encontrar criterios para las posibles fallas de un material, una construcción o una máquina.

La herramienta principal para este tratamiento es el **método de elementos finitos**, el cual nos lleva a aproximaciones calculables de la solución real en espacios de dimensión finita.

Muchas de las aplicaciones de estos métodos están basadas en formulaciones variacionales primales. En este enfoque, todas las cantidades de interés se formulan en términos de las variables originales y se resuelve el problema para estas incógnitas. Las otras cantidades físicas se recuperan vía un pos-procesamiento. Este enfoque de elementos finitos es muy útil en aquellos casos en los cuales las variables originales (primales) son las de mayor importancia.

Sin embargo, en el estudio de problemas de elasticidad lineal o no lineal, el interés principal podría concentrarse en la variable dual que describe los *esfuerzos* dentro de un material, además de la variable primal que describe las deformaciones del mismo.

Por otro lado, es bien conocido que si el material es elástico lineal e incompresible, no es posible determinar el esfuerzo sólo a partir del desplazamiento, sin tener que obligadamente encontrarnos con el llamado bloqueo numérico (*locking*) en nuestro esquema numérico. Por lo tanto, se requiere introducir una incógnita adicional en el problema que caracterize la condición de incompresibilidad del material.

Un enfoque que permite formular tales problemas con dos o más variables de interés es el que dá la formulación variacional mixta, cuyos esquemas de Galerkin asociados se llaman métodos de elementos finitos mixtos. En estos enfoques las variables adicionales son aproximadas directamente en subespacios de elementos finitos mixtos, lo cual evita el pos-procesamiento al cual nos referimos anteriormente.

Métodos mixtos para el problema de elasticidad lineal se deducen por ejemplo, a partir del principio de Hellinger-Reissner, en el cual la solución $(\boldsymbol{\sigma}, \mathbf{u})$ está caracterizada como un punto silla del funcional respectivo, como también a partir del principio de Hu-Washizu, que se caracteriza porque además de las variables $\boldsymbol{\sigma}$ y $\mathbf{u}$ se agrega como incógnita auxiliar el tensor de *deformación* $\mathbf{e}(\mathbf{u})$, con lo cual la solución $(\boldsymbol{\sigma}, \mathbf{u}, \mathbf{e}(\mathbf{u}))$ resulta ser un punto silla del funcional correspondiente. Ahora, si además el material es casi incompresible o incompresible, se dispone del enfoque PEERS, método basado en una modificación del principio de Hellinger-Reissner, cuya principal característica es que la condición de simetría del tensor esfuerzo se impone débilmente a través de una incógnita auxiliar en la nueva formulación.

Otros métodos mixtos que modelan la incompresibilidad del material y donde el interés principal es la variable *desplazamiento*, son los llamados tipo Stokes o métodos mixtos-primales, donde se busca la solución $(\mathbf{u}, p)$, con la variable p actuando como una presión hidroestática y que fundamentalmente permite caracterizar la condición de incompresibilidad del material.

La literatura sobre métodos mixtos para problemas no lineales no es muy abundante en comparación con aquella para problemas lineales. Una idea comúnmente utilizada para estudiar ecuaciones elípticas no lineales está basada en la inversión, gracias al teorema de la función implícita, de la ecuación constitutiva correspondiente. Pero, cuando la ecuación constitutiva no es invertible de manera explícita, se sugiere la incorporación de variables auxiliares para obviar esta inversión. Más precisamente, se introduce el gradiente (en teoría del potencial y conducción de calor) o el tensor de deformación (en elasticidad) como una incógnita adicional, con lo cual la formulación débil que se origina se puede escribir como un sistema de ecuaciones de operadores con estructura de punto silla doble (también llamadas formulaciones mixtas dual-dual).

Muchas de las formulaciones variacionales mixtas usadas para los casos incompresibles y casi incompresible, son de tipo Stokes. Solo algunas referencias consideran el llamado esquema mixto-dual, el cual está caracterizado por la utilización de los espacios $[L^2(\Omega)]^2$ y $H(\mathbf{div};\Omega)$ para las variables desplazamiento y tensor de esfuerzo respectivamente.

Por otro lado, para problemas no lineales usualmente no es posible obtener algún tipo de información a priori respecto de la solución que nos permita construir una malla conveniente para el esquema de elementos finitos. Por lo tanto, con la intención de obtener un buen comportamiento en el proceso de convergencia es que se requiere aplicar un algoritmo adecuado de refinamiento, el cual usualmente se basa en estimaciones de error a posteriori. Al respecto, la ausencia de formulaciones mixtas-duales se observa también en los trabajos concernientes a análisis de error a posteriori, y en el caso de medios incompresibles con ecuación constitutiva no lineal.

Análogamente en relación a materiales incompresibles sobre dominios exteriores no acotados, existen sólo algunos trabajos.

Finalmente, como podemos observar, varias son las motivaciones que justifican el desarrollo de esta tesis.

1.2 Discusión bibliográfica

Las primeras aplicaciones de elementos finitos mixtos datan desde la mitad de los sesenta cuando fueron utilizados para modelar flexiones de placa, independientemente, por Herrmann [59] y Hellan [58]. Más tarde, Dunham y Pister [31] usaron el principio variacional de Hellinger-Reissner para desarrollar elementos finitos mixtos para problemas de elasticidad plana.

Por otra parte, en las últimas dos décadas la combinación de métodos de elementos finitos (FEM) con métodos de ecuaciones integrales de frontera (BEM) para resolver problemas de transmisión interior y exterior ha experimentado un desarrollo creciente (ver, por ejemplo, [21], [28], [30], [34], [41], [45], [49], [52], [60], [62], [65], [69] y las referencias incluídas en ellas). Sin embargo, un procedimiento alternativo para problemas de transmisión exterior no lineal-lineal consiste en emplear aplicaciones Dirichlet-to-Neumann (DtN) en lugar de BEM. Esto significa que uno primero introduce un círculo suficientemente grande Γ (en $\mathbb{R}^2$) o una esfera (en $\mathbb{R}^3$), de modo que el dominio lineal es dividido en una región anular acotada y una no acotada. Luego, uno deriva una fórmula explícita para el dato de Neumann sobre Γ en términos del dato de Dirichlet sobre la misma curva, que es conocido como aplicación de Dirichlet-to-Neumann. Esto ha sido hecho para varios operadores elípticos, incluyendo el sistema Lamé para elasticidad, usando desarrollos en serie de Fourier (ver, por ejemplo [32], [51], [55], [57]). Además, la aplicación obtenida en [57] junto con el enfoque de elemento finito mixto de [49] (ver también [41]) fué utilizado en [34] para estudiar la solubilidad débil de un problema exterior de interface hiperelástica. El mismo problema ha sido resuelto antes en [7] usando métodos de elementos finitos de desplazamiento.

Ahora, aproximaciones de elementos finitos de materiales incompresibles sobre dominios acotados han sido estudiados usando métodos mixtos penalizados en varios trabajos (ver, por ejemplo[20], [33] y [74]). En lo que respecta a materiales incompresibles sobre dominios exteriores no acotados, sólo trabajos recientes pueden ser mencionados. En particular, el acoplamiento de FEM y BEM se aplica en [24] para resolver un problema de transmisión exterior no lineal-lineal en elasticidad incompresible. Aquí se asume que el material no lineal acotado es gobernado por un operador uniformemente monótono, mientras que en la region exterior no acotada se considera elasticidad lineal. La formulación variacional resultante, obtenida usando FEM y BEM en los dominios no lineal y lineal, respectivamente, es una forma mixta tipo Stokes. Existencia y unicidad de soluciones de las formulaciones continua y discreta, junto con convergencia casi-optimal y un estimador de error a posteriori de tipo residual explícito, son las principales contribuciones en [24]. Por otro lado, las técnicas de [56], [57] para derivar aplicaciones de Dirichlet-to-Neumann sobre fronteras artificiales se extienden en [55] a materiales incompressibles no lineales sobre problemas exteriores en el plano. Luego, siguiendo el enfoque usual, ellos reducen el problema original en un problema sobre un dominio acotado, que es resuelto numéricamente usando también un método de elementos finitos mixto tipo Stokes.

Actualmente se sabe que la posibilidad de calcular los esfuerzos $\boldsymbol{\sigma}$ de manera más exacta que los desplazamientos $\mathbf{u}$ constituye la principal ventaja de usar métodos de elementos finitos mixto-dual en elasticidad (ver, por ejemplo [9], [10], y Capítulos IV y VII in [20]). Sin embargo, este enfoque no ha sido totalmente investigado en los casos incompresible y casi incompresible, por lo cual muchas de las formulaciones variacionales empleadas son de tipo Stokes o mixtas-primales (ver, por ejemplo Capítulo VI en [20] y las referencias incluídas en ellas). La ausencia de formulaciones mixtas-dual para elasticidad incompresible ha sido notado en los trabajos relativos a análisis de error a posteriori y ecuaciones constitutivas no lineales (ver, por ejemplo [22], [23], [24], y [27]).

Más recientemente, en [48] se introduce y analiza un método de elementos finitos mixto-dual para el material incompresible no lineal estudiado en [27] con condiciones de Dirichlet pura. El enfoque en [48] sigue trabajos previos sobre métodos mixto-dual para problemas de valores de contorno no lineal (ver, por ejemplo [15], [39], [41], y [46]) y considera el tensor de deformación y la rotación como incógnitas adicionales. Esto lleva a una ecuación de operadores de punto silla doble como la formulación variacional correspondiente, y por lo tanto la leve generalización de la teoría de Babuška-Brezzi desarrollada en [36] se aplica para probar que los esquemas continuo y discreto están bien propuestos. Es importante notar que en esta formulación mixta-dual la condición de frontera de Dirichlet es natural, y por tanto se requiere una ecuación adicional para establecer la existencia y unicidad de soluciones para ambas formulaciones. La ecuación usada en [48], que con frecuencia aparece en la literatura (ver, por ejemplo [9], [10], and [20]), está dada por $\int_\Omega \operatorname{tr}(\boldsymbol{\sigma}) = 0$, donde $\operatorname{tr}(\boldsymbol{\sigma})$ representa la traza del tensor $\boldsymbol{\sigma}$. En esta forma, mostramos en [48] que un enriquecimiento apropiado del subespacio de elementos finitos PEERS garantiza que el esquema de Galerkin está bien propuesto. Además, también se desarrolla en [48] un análisis de error a posteriori basado en problemas locales, que combina el clásico enfoque de Bank-Weiser de [14] con la técnica de [22], [23], y [24].

En relación a análisis de error a posteriori, la literatura para problemas lineales y no lineales es extensa (ver, por ejemplo [5, 71] y las referencias incluídas en ellas). Para formulaciones mixtas, el trabajo de Verfüth [71] proporciona un estimador de error a posteriori de tipo residual explícito para el problema de Stokes. Luego, Alonso en [6] utiliza espacios de Raviart-Thomas y de Brezzi-Douglas-Marini para obtener estimadores basados en evaluaciones residuales y en la solución de problemas locales, para una clase de ecuaciones diferenciales parciales elípticas de segundo orden. También, en relación con elementos de Raviart-Thomas están las contribuciones de [19, 26] donde se obtienen estimadores de error residual confiables y eficientes. A su vez, en [1] se propone una generalización del estimador de error residual jerárquico de Bank y Smith de [13]. También, el clásico estimador de Bank-Weiser de [14] y los resultados de [3, 4], fueron ampliados en [22] para elasticidad con grandes deformaciones, incluyendo el caso incompresible. Este método, que involucra la solución de problemas de Neumann locales, fué también aplicado en [23] para obtener estimadores de error residual implícitos para el acoplamiento de elementos finitos y elementos de frontera.

1.3 Objetivos y resultados principales

Los objetivos globales de esta tesis son aplicar métodos de elementos finitos mixtos y obtener estimadores de error a posteriori para dos clases de problemas en elasticidad bidimensional en medios incompresibles con ecuación constitutiva no lineal

Con el objeto de explicitar de mejor forma las principales contribuciones de esta tesis, a continuación se describen los problemas modelos que estudiamos. En esta dirección, nuestro primer modelo es el siguiente problema de transmisión exterior: Sea Ω_0 un dominio acotado y simplemente conexo de $\mathbb{R}^2$ con frontera Γ_0 Lipschitz-continua. Además, sea Ω_1 la región anular acotada por Γ_0 y otra curva Lipschitz-continua cerrada Γ_1 cuya región interior contiene a Ω_0. Denotamos $\Omega_e := \mathbb{R}^2 - (\bar{\Omega}_0 \cup \bar{\Omega}_1)$. Entonces, dado $\mathbf{f}_1 \in [L^2(\Omega_1)]^2$, se desea encontrar $\mathbf{u} \in [H^1(\Omega_1)]^2 \cap [H^1_{loc}(\Omega_e)]^2$ y $p \in L^2(\mathbb{R}^2 - \bar{\Omega}_0)$ tal que

$$\mathbf{u} = \mathbf{0} \quad \text{sobre} \quad \Gamma_0\,, \qquad -\mathbf{div}\,\boldsymbol{\sigma}(\mathbf{u},p) = \mathbf{f}_1 \quad \text{en} \quad \Omega_1\,,$$

$$\operatorname{div}\mathbf{u} = 0 \quad \text{en} \quad \Omega_1 \cup \Omega_e\,, \qquad -\mathbf{div}\,\boldsymbol{\sigma}(\mathbf{u},p) = \mathbf{0} \quad \text{en} \quad \Omega_e\,,$$

$$\boldsymbol{\sigma}(\mathbf{u},p) := \tilde{\lambda}(\operatorname{dev}(\mathbf{e}(\mathbf{u})))\,\operatorname{tr}(\mathbf{e}(\mathbf{u}))\,\mathbf{I} + \tilde{\mu}(\operatorname{dev}(\mathbf{e}(\mathbf{u})))\,\mathbf{e}(\mathbf{u}) - p\,\mathbf{I} \quad \text{en} \quad \Omega_1\,,$$

$$\boldsymbol{\sigma}(\mathbf{u},p) := 2\,\mu\,\mathbf{e}(\mathbf{u}) - p\,\mathbf{I} \quad \text{en} \quad \Omega_e\,,$$

$$\mathbf{u} \text{ es acotado y } p \to 0 \text{ cuando } ||x|| \to +\infty\,,$$

$$\lim_{\substack{x \to x_0 \\ x \in \Omega_1}} \boldsymbol{\sigma}(\mathbf{u}(x), p(x))\, \boldsymbol{\nu}(x_0) = \lim_{\substack{x \to x_0 \\ x \in \Omega_e}} \boldsymbol{\sigma}(\mathbf{u}(x), p(x))\, \boldsymbol{\nu}(x_0)\,, \quad \forall\, x_0 \in \Gamma_1,$$

donde $\tilde{\lambda}$, $\tilde{\mu}$: $\mathbb{R}^+ \to \mathbb{R}$ son funciones de Lamé no lineales, $\mu > 0$ es una constante de Lamé y $\boldsymbol{\nu}$ denota el vector normal unitario exterior a Γ_1.

Los resultados principales obtenidos en relación con los objetivos globales son:

- Una formulación variacional mixta tipo Stokes no lineal, donde el operador que define la nolinealidad depende de la aplicación de Dirichlet-to-Neumann de [55]. Reemplazando la aplicación de Dirichlet-to-Neumann por un truncamiento de la serie de Fourier de [55], obtenemos también una formulación continua-aproximada.

- Resultados de existencia y unicidad de soluciones para ambas formulaciones continuas, para lo cual se utilizan la monotonía fuerte y la Lipschitz continuidad de un cierto operador no lineal, así como resultados de regularidad clásicos. También, obtenemos estimaciones de error a priori entre las soluciones de estas dos formulaciones.

- Existencia y unicidad de soluciones para el esquema de Galerkin asociado a la formulación mixta aproximada, y estimaciones de Cea para las soluciones de los esquemas continuo, continuo-aproximado y de Galerkin.

- Ejemplos de subespacios de elementos finitos que satisfacen las condiciones de compatibilidad discreta.

- Un estimador de error a posteriori confiable de caracter residual explícito.

Todos estos resultados se encuentran desarrollados en el artículo [37]:

G.N. Gatica, L.F. Gatica, E.P. Stephan, *A FEM-DtN formulation for a nonlinear exterior problem in incompressible elasticity*. Mathematical Methods in the Applied Sciences, vol. 26, 2, pp. 151-170,(2003).

Nuestro segundo modelo está dado por el siguiente problema de valores de contorno: Sea Ω un dominio acotado y simplemente conexo en $\mathbb{R}^2$ con frontera poligonal Γ, y tal que todos sus ángulos interiores se encuentran en $(0, 2\pi)$. También, sean Γ_D

y Γ_N subconjuntos abiertos disjuntos de Γ, con $|\Gamma_D|, |\Gamma_N| \neq 0$, tal que $\Gamma = \bar{\Gamma}_D \cup \bar{\Gamma}_N$. Si $\boldsymbol{\sigma}(\mathbf{u}, p)$, $\mathbf{e}(\mathbf{u})$, y $\mathbf{I} \in \mathbb{R}^{2\times 2}$ denotan el tensor de Cauchy, el tensor de tensión de pequeñas deformaciones, y la matriz identidad, respectivamente, entonces dado $\mathbf{f} \in [L^2(\Omega)]^2$ y $\mathbf{g} \in [H^{-1/2}(\Gamma_N)]^2$, se desea encontrar $(\boldsymbol{\sigma}, \mathbf{u}, p)$ en espacios apropiados tal que

$$\boldsymbol{\sigma} = \mathcal{N}(\mathbf{e}(\mathbf{u})) + p\,\mathbf{I} \quad \text{en} \quad \Omega, \quad \mathbf{div}\,\boldsymbol{\sigma} = -\mathbf{f} \quad \text{en} \quad \Omega, \quad \operatorname{div}\mathbf{u} = 0 \quad \text{en} \quad \Omega,$$
$$\mathbf{u} = \mathbf{0} \quad \text{sobre} \quad \Gamma_D, \quad \text{y} \quad \boldsymbol{\sigma\nu} = \mathbf{g} \quad \text{sobre} \quad \Gamma_N\,,$$

donde $\boldsymbol{\nu}$ denota el vector normal unitario exterior a Γ_N, **div** denota el operador de divergencia usual div actuando sobre cada fila del tensor correspondiente, y el espacio Sobolev $[H^{-1/2}(\Gamma_N)]^2$ es el dual de

$$[H_{00}^{1/2}(\Gamma_N)]^2 := \{\mathbf{v}|_{\Gamma_N} \;:\; \mathbf{v} \in [H^1(\Omega)]^2, \quad \mathbf{v} = \mathbf{0} \quad \text{on } \Gamma_D\}.$$

Para este problema los resultados obtenidos en relación con nuestros objetivos globales son los siguientes:

- Una formulación variacional mixta-dual y el esquema de elemento finito mixto asociado. Se prueba su equivalencia con una ecuación de operadores de punto silla triple, lo que simplifica posteriormente el análisis del problema de existencia y unicidad de soluciones.

- Una leve extensión de la teoría de Babuška-Brezzi para la clase anterior de ecuaciones de operadores tanto a nivel continuo como discreto, el cual está basado en el trabajo [36]. Probamos que las respectivas formulaciones mixtas están bien propuestas y mostramos tasas de error optimal de orden $O(h)$.

- Un análisis de error a posteriori, de donde obtenemos un estimador confiable basado en problemas locales. Damos varios ejemplos numéricos que ilustran la buena realización del algoritmo resultante y su capacidad para localizar singularidades.

Estos resultados forman parte del artículo [38]:

G.N. Gatica, L.F. Gatica, E.P. Stephan, *A dual-mixed finite element method for nonlinear incompressible elasticity with mixed boundary conditions.* Preprint 2005-06, Departamento de Ingeniería Matemática, Universidad de Concepción,(2005), submitted.

1.4 Organización de la tesis

Esta tesis está compuesta de 5 capítulos.

Después de la presente introducción, en el Capítulo 2 presentamos el problema de transmisión no lineal-lineal y su transformación en un problema de valores de contorno en un dominio acotado, el cual obtenemos usando un acoplamiento entre el método de elementos finitos y una aplicación DtN. También, usamos la representación en serie de [55] para la aplicación DtN y definimos el problema aproximado asociado. Obtenemos las formulaciones variacionales para ambos problemas, que resultan ser mixtas de tipo Stokes, y los correspondientes resultados de existencia y unicidad. Estudiamos la aproximación de Galerkin y el análisis de error correspondiente. Damos ejemplos de subespacios de elementos finitos que satisfacen las condiciones inf-sup discretas. Desarrollamos un análisis de error a posteriori y obtenemos un estimador confiable de tipo residual implícito para esta formulación.

En el Capítulo 3 presentamos el problema de valores de contorno con condiciones de frontera mixtas en elasticidad no lineal. Introducimos la formulación variacional mixta-dual y establecemos su equivalencia con una ecuación de operadores de punto silla triple. Damos una leve extensión de la teoría de Babuška-Brezzi a esta clase de problemas y probamos resultados de existencia y unicidad de soluciones para la formulación continua. Hacemos suposiciones apropiadas para el tamaño de la malla y aplicamos nuevamente la teoría abstracta de Babuška-Brezzi para probar que el esquema de Galerkin asociado está bien propuesto. También, desarrollamos un análisis de error a posteriori y obtenemos estimadores confiables basados en problemas locales, uno implícito y globalmente cuasi-eficiente y otro totalmente explícito, para la formulación mixta-dual.

En el Capítulo 4 desarrollamos varios ejemplos numéricos que muestran la efectividad del algoritmo adaptativo para calcular las soluciones del esquema discreto correspondiente al problema estudiado en el Capítulo 3.

Finalmente, las conclusiones principales sobre el trabajo desarrollado en esta tesis las resumimos en el Capítulo 5, donde además presentamos direcciones para el desarrollo de futuras investigaciones.

Chapter 2

A FEM-DtN formulation for a nonlinear exterior problem in incompressible elasticity

In this Chapter we combine the usual finite element method (FEM) with a Dirichlet-to-Neumann (DtN) mapping, derived in terms of an infinite Fourier series, to study the solvability and Galerkin approximations of an exterior transmission problem arising in nonlinear incompressible 2d-elasticity. We show that the variational formulation can be written in a Stokes-type mixed form with a linear constraint and a nonlinear main operator. Then, we provide the uniqueness of solution for the continuous and discrete formulations, and derive a Cea type estimate for the associated error. In particular, our error analysis considers the practical case in which the DtN mapping is approximated by the corresponding finite Fourier series. Finally, an a-posteriori error estimate, well suited for adaptive computations, is also given.

2.1 Introduction

The purpose of this Chapter is to study the solvability and Galerkin approximations of the model problem in [24], by using the Dirichlet-to-Neumann mapping from [55] instead of BEM. In this way, since no singular boundary integral operators are needed, the present formulation becomes simpler than the one from [24]. However, because of the introduction of the auxiliary circle, the computational domain gets larger than the original one. This means that both formulations are comparable and therefore either one could be applied. The rest of the Chapter is organized as follows. In Section 2.2, we first introduce the nonlinear exterior transmission problem and then transform it, using the Dirichlet-to-Neumann mapping, into a boundary value problem in a bounded domain. Also, we replace the DtN mapping by a finite Fourier series and define the associated approximated problem. The variational formulations for both problems, which become of the Stokes-type mixed form, and the corresponding solvability results are provided in Section 2.3. Finally, the Galerkin approximations and the error analysis are studied in Section 2.4. Here we define specific finite element subspaces satisfying the discrete compatibility conditions. Finally, in Section 2.5 we follow the analysis from [24] and derive an a-posteriori error estimate, of explicit residual type, for our approximated formulation.

2.2 The nonlinear exterior transmission problem

Let Ω_0 be a bounded and simply connected domain in $\mathbb{R}^2$ with Lipschitz-continuous boundary Γ_0, and let Ω_1 be the annular domain bounded by Γ_0 and another Lipschitz-continuous curve Γ_1 whose interior region contains Ω_0. Then, we denote $\Omega_e := \mathbb{R}^2 - (\bar{\Omega}_0 \cup \bar{\Omega}_1)$. Our goal is to determine the displacement $\mathbf{u} := (u_1, u_2)^T$ and the hydrostatic pressure p of an incompressible material occupying the region $\mathbb{R}^2 - \bar{\Omega}_0$, under the action of some external forces, which is hyperelastic in Ω_1 and linear elastic in Ω_e.

In what follows, $\mathbb{R}^{2\times 2}$ is the space of square matrices of order 2 with real entries, $\mathbf{I} := (\delta_{ij})$ is the identity matrix of $\mathbb{R}^{2\times 2}$, and given $\boldsymbol{\tau} := (\tau_{ij})$, $\boldsymbol{\sigma} := (\sigma_{ij}) \in \mathbb{R}^{2\times 2}$, we use the notations

$$\mathrm{tr}\,(\boldsymbol{\tau}) := \sum_{i=1}^{2} \tau_{ii} \quad , \quad \boldsymbol{\sigma} : \boldsymbol{\tau} := \sum_{i,j=1}^{2} \sigma_{ij}\, \tau_{ij}\,.$$

Now, as a description of the hyperelasticity we assume the validity of the Hencky-Mises stress-strain relation as discussed in [63] (see, also, [64] and [72]). In other words, if $\boldsymbol{\sigma}(\mathbf{u}, p) := (\sigma_{ij}(\mathbf{u}, p)) \in \mathbb{R}^{2\times 2}$ denotes the Cauchy stress tensor and $\mathbf{e}(\mathbf{u}) := (e_{ij}(\mathbf{u})) \in \mathbb{R}^{2\times 2}$ is the strain tensor of small deformations, with $e_{ij}(\mathbf{u}) :=$

$\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$, then the constitutive equation in Ω_1 is given by

$$\sigma_{ij}(\mathbf{u}, p) \;=\; \frac{\partial}{\partial e_{ij}}\,\Psi(\mathbf{e}(\mathbf{u})) \;-\; p\,\delta_{ij} \quad \forall\, i,\, j \in \{1, 2\}\,, \tag{2.2.1}$$

where $\Psi : \mathbb{R}^{2\times 2} \to \mathbb{R}$ is the stored energy function

$$\Psi(\boldsymbol{\tau}) = \frac{1}{2}\,\kappa\,(\mathrm{tr}\,(\boldsymbol{\tau}))^2 \,+\, \bar{\mu}\,\Phi(\mathrm{dev}(\boldsymbol{\tau})) \qquad \forall\, \boldsymbol{\tau} \in \mathbb{R}^{2\times 2}\,. \tag{2.2.2}$$

Here, κ and $\bar{\mu}$ are positive constants ($\bar{\mu}$ is called the ground state shear modulus), $\Phi : [0, +\infty) \to \mathbb{R}$ is a function of class C^2 and $\mathrm{dev} : \mathbb{R}^{2\times 2} \to \mathbb{R}^+$ is defined by $\mathrm{dev}(\boldsymbol{\tau}) := \left(\boldsymbol{\tau} - \frac{1}{2}\mathrm{tr}(\boldsymbol{\tau})\mathbf{I}\right) : \left(\boldsymbol{\tau} - \frac{1}{2}\mathrm{tr}(\boldsymbol{\tau})\mathbf{I}\right)$. In addition, we assume $\Phi'(0) = 1$ and that there exist constants C_1, C_2 and C_3 such that

$$\begin{gathered} 0 < C_1 \le \bar{\mu}\,\Phi'(t) < \kappa \quad , \quad |t\,\Phi''(t)| \le C_2 \quad \text{and} \\ \Phi'(t) + 2t\Phi''(t) \ge C_3 > 0 \end{gathered} \tag{2.2.3}$$

for all $t \in [0, +\infty)$.

Hence, by computing the right hand side of (2.2.1) according to (2.2.2), the constitutive equation in Ω_1 becomes

$$\boldsymbol{\sigma}(\mathbf{u}, p) := \tilde{\lambda}(\mathrm{dev}(\mathbf{e}(\mathbf{u})))\,\mathrm{tr}(\mathbf{e}(\mathbf{u}))\,\mathbf{I} + \tilde{\mu}(\mathrm{dev}(\mathbf{e}(\mathbf{u})))\,\mathbf{e}(\mathbf{u}) - p\,\mathbf{I}\,, \tag{2.2.4}$$

where $\tilde{\lambda}$, $\tilde{\mu} : \mathbb{R}^+ \to \mathbb{R}$ are the nonlinear Lamé functions defined by

$$\tilde{\mu}(t) := 2\,\bar{\mu}\,\Phi'(t) \quad \text{and} \quad \tilde{\lambda}(t) := \kappa - \frac{1}{2}\,\tilde{\mu}(t) \qquad \forall\, t \in \mathbb{R}^+\,.$$

Further, by using the assumptions from (2.2.3), we deduce that there exist positive constants μ_0, μ_1 and μ_2 such that for all $t \in \mathbb{R}^+$

$$0 < \mu_0 \le \tilde{\mu}(t) < 2\,\kappa \quad \text{and} \quad 0 < \mu_1 \le \tilde{\mu}(t) + 2t\tilde{\mu}'(t) \le \mu_2\,.$$

On the other hand, for the linear elastic material in Ω_e, the corresponding constitutive equation is given by (see, e.g. [68])

$$\boldsymbol{\sigma}(\mathbf{u}, p) := 2\,\mu\,\mathbf{e}(\mathbf{u}) \,-\, p\,\mathbf{I}\,, \tag{2.2.5}$$

where $\mu > 0$ is the Lamé constant.

Next, let us adopt the notation $[H^1(\Omega_1)]^2 \cap [H^1_{loc}(\Omega_e)]^2$ to denote the space of functions $\mathbf{v} := (v_1, v_2)^T$ defined in $\Omega_1 \cup \Gamma_1 \cup \Omega_e$ such that $\mathbf{v}|_{\Omega_1} \in [H^1(\Omega_1)]^2$ and $\mathbf{v}|_{\Omega_e} \in [H^1_{loc}(\Omega_e)]^2$. Then given $\mathbf{f}_1 \in [L^2(\Omega_1)]^2$, our nonlinear exterior transmission problem reads as follows: *Find a vector field* $\mathbf{u} \in [H^1(\Omega_1)]^2 \cap [H^1_{loc}(\Omega_e)]^2$ *and a scalar field* $p \in L^2(\mathbb{R}^2 - \bar{\Omega}_0)$ *such that*

$$\mathbf{u} = 0 \quad \text{on} \quad \Gamma_0\,, \quad -\,\mathbf{div}\,\boldsymbol{\sigma}(\mathbf{u},p) = \mathbf{f}_1 \quad \text{in} \quad \Omega_1\,,$$

$$\mathbf{div}\,\mathbf{u} = 0 \quad \text{in} \quad \Omega_1 \cup \Omega_e\,, \quad -\,\mathbf{div}\,\boldsymbol{\sigma}(\mathbf{u},p) = 0 \quad \text{in} \quad \Omega_e\,, \tag{2.2.6}$$

$$\mathbf{u} \text{ is bounded and } p \to 0 \text{ as } ||x|| \to +\infty\,,$$

and the tractions are continuous across Γ_1, that is for all $x_0 \in \Gamma_1$

$$\lim_{\substack{x \to x_0 \\ x \in \Omega_1}} \boldsymbol{\sigma}(\mathbf{u}(x), p(x))\,\boldsymbol{\nu}(x_0) = \lim_{\substack{x \to x_0 \\ x \in \Omega_e}} \boldsymbol{\sigma}(\mathbf{u}(x), p(x))\,\boldsymbol{\nu}(x_0)\,,$$

where $\boldsymbol{\nu} := (\nu_1, \nu_2)^T$ is the unit outward normal to Γ_1.

Now, let $L^2(\Omega_1)^{2\times 2}$ be the space of matrices of order 2 with entries in $L^2(\Omega_1)$. Then we denote the corresponding subspace of symmetric matrices by $L^2(\Omega_1)^{2\times 2}_s$ and let $\mathcal{A} : L^2(\Omega_1)^{2\times 2}_s \to L^2(\Omega_1)^{2\times 2}_s$ be the nonlinear operator such that

$$\mathcal{A}(\mathbf{e}(\mathbf{v})) := \tilde{\lambda}(\mathrm{dev}(\mathbf{e}(\mathbf{v})))\,\mathrm{tr}(\mathbf{e}(\mathbf{v}))\,\mathbf{I} + \tilde{\mu}(\mathrm{dev}(\mathbf{e}(\mathbf{v})))\,\mathbf{e}(\mathbf{v}) \tag{2.2.7}$$

for all $\mathbf{v} \in [H^1(\Omega_1)]^2$. Then, (2.2.4) reads

$$\boldsymbol{\sigma}(\mathbf{u},p) = \mathcal{A}(\mathbf{e}(\mathbf{u})) - p\,\mathbf{I} \quad \text{in} \quad \Omega_1\,, \tag{2.2.8}$$

and according to this equation and (2.2.5), the second and fourth equations in (2.2.6) become, respectively,

$$-\,\mathbf{div}\,\mathcal{A}(\mathbf{e}(\mathbf{u})) + \nabla p = \mathbf{f}_1 \quad \text{in} \quad \Omega_1$$

and

$$-\,2\mu\,\mathbf{div}\,\mathbf{e}(\mathbf{u}) + \nabla p = 0 \quad \text{in} \quad \Omega_e\,.$$

In order to reformulate (2.2.6) in a more suitable way, we now let Γ be a circle centered at the origin with radius r, such that the interior region of Γ contains $\bar{\Omega}_0 \cup \bar{\Omega}_1$. Then, we let Ω_2 be the annular domain bounded by Γ_1 and Γ, and denote

$\Omega_r := \mathbb{R}^2 - \bar{\Omega}_0 \cup \bar{\Omega}$, with $\Omega := \Omega_1 \cup \Gamma_1 \cup \Omega_2$. Thus, (2.2.6) can be separated into a problem in Ω and a problem in Ω_r, together with appropriate transmission conditions on the circle Γ, that is

$$\boldsymbol{\sigma}(\mathbf{u},p)\boldsymbol{\nu} \;=\; \boldsymbol{\sigma}(\mathbf{u}_r,p_r)\boldsymbol{\nu} \quad \text{and} \quad \mathbf{u} \;=\; \mathbf{u}_r \quad \text{on} \quad \Gamma\,,$$

where $(\mathbf{u}_r,p_r) := (\mathbf{u},p)|_{\Omega_r}$ and $\boldsymbol{\nu}$ denotes the unit outward normal to Γ. See the geometry of the problem in Figure 2.1 below.

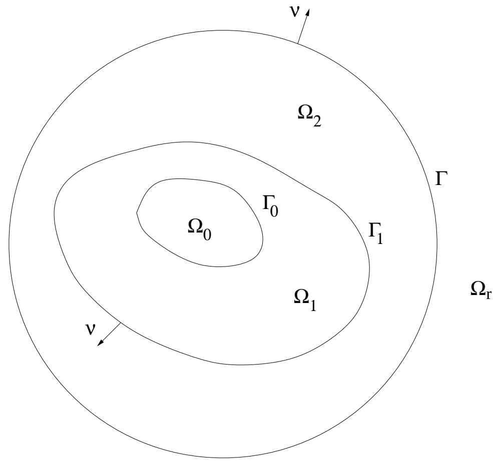

Figure 2.1. Set up of the domain and artificial boundary.

Now, as shown in [55], $(\mathbf{u}_r,p_r)$ can be obtained analytically in terms of $\mathbf{u}|_\Gamma$, by using Fourier series developments. In particular, according to the analysis in Section 2 of [55], we find that

$$\boldsymbol{\sigma}(\mathbf{u}_r,p_r)\,\boldsymbol{\nu} \;=\; Q(\mathbf{u}) := (Q_1(\mathbf{u}),Q_2(\mathbf{u}))^T\,, \tag{2.2.9}$$

where the Dirichlet-to-Neumann mapping Q is defined by

$$Q_j(\mathbf{u})(r,\theta) := \frac{2\mu}{\pi r}\sum_{n=1}^{\infty}\frac{\partial}{\partial\theta}\int_0^{2\pi}\frac{\cos n(\varphi-\theta)}{n}\,\frac{\partial u_j}{\partial\varphi}\,(r,\varphi)\,d\varphi \;\;\forall\, j\in\{1,2\},\,\forall\,\theta\in[0,2\pi],$$

with $\mathbf{u}|_\Gamma := (u_1(r,\varphi),u_2(r,\varphi))^T$ for all $\varphi\in[0,2\pi]$.

In this way, by using (2.2.9) we are lead to a boundary value problem in Ω with an implicit Neumann boundary condition on the circle Γ. Also, given $N\in\mathbf{N}$, we may consider $Q^N(\mathbf{u}) := (Q_1^N(\mathbf{u}),Q_2^N(\mathbf{u}))^T$ instead of Q in (2.2.9), where

$$Q_j^N(\mathbf{u})(r,\theta) := \frac{2\mu}{\pi r}\sum_{n=1}^{N}\frac{\partial}{\partial\theta}\int_0^{2\pi}\frac{\cos n(\varphi-\theta)}{n}\,\frac{\partial u_j}{\partial\varphi}(r,\varphi)\,d\varphi\,,$$

and obtain an approximate boundary value problem in Ω whose solution is denoted by $(\mathbf{u}^N, p^N)$. More specifically, this problem reads: *Find* $(\mathbf{u}^N, p^N) \in [H^1(\Omega)]^2 \times L^2(\Omega)$ *such that*

$$\begin{aligned}
\mathbf{u}^N &= \mathbf{0} \quad \text{on} \quad \Gamma_0 \,, \\
-\mathbf{div}\, \boldsymbol{\sigma}(\mathbf{u}^N, p^N) &= \begin{cases} \mathbf{f}_1 & \text{in} \quad \Omega_1 \\ \mathbf{0} & \text{in} \quad \Omega_2 \end{cases} \,, \\
\boldsymbol{\sigma}(\mathbf{u}^N, p^N) &= \begin{cases} \mathcal{A}(\mathbf{e}(\mathbf{u}^N)) - p^N \mathbf{I} & \text{in} \quad \Omega_1 \\ 2\,\mu\, \mathbf{e}(\mathbf{u}^N) - p^N \mathbf{I} & \text{in} \quad \Omega_2 \end{cases} \,, \\
\operatorname{div} \mathbf{u}^N &= 0 \quad \text{in} \quad \Omega \,, \\
\boldsymbol{\sigma}(\mathbf{u}^N, p^N)\, \boldsymbol{\nu} &= Q^N(\mathbf{u}^N) \quad \text{on} \quad \Gamma \,.
\end{aligned} \tag{2.2.10}$$

In the following section we introduce the variational formulations of these boundary value problems in Ω, show that they are well posed and estimate the corresponding error between their solutions $(\mathbf{u}, p)$ and $(\mathbf{u}^N, p^N)$.

2.3 The variational formulations

Because of the Dirichlet boundary conditions on Γ_0 we define the space

$$H^1_{\Gamma_0}(\Omega) := \{\, v \in H^1(\Omega) : \quad v = 0 \quad \text{on} \quad \Gamma_0 \,\} \,,$$

and then put

$$X := [H^1_{\Gamma_0}(\Omega)]^2 \quad , \quad M := L^2(\Omega) \,,$$

provided with the norms of $[H^1(\Omega)]^2$ and $L^2(\Omega)$, respectively.

Then, following the usual integration by parts procedure, we arrive at the following Stokes-type mixed variational formulation of the boundary value problem in Ω: *Find* $(\mathbf{u}, p) \in X \times M$ *such that*

$$\begin{aligned}
[\mathbf{A}(\mathbf{u}), \mathbf{v}] + [\mathbf{B}(\mathbf{v}), p] &= [\mathbf{F}, \mathbf{v}] \quad \forall\, \mathbf{v} \in X \,, \\
[\mathbf{B}(\mathbf{u}), q] &= 0 \quad \forall\, q \in M \,,
\end{aligned} \tag{2.3.1}$$

where the nonlinear operator $\mathbf{A} : X \to X'$, the linear operator $\mathbf{B} : X \to M'$ and the functional $\mathbf{F} \in X'$ are defined, respectively, by

$$[\mathbf{A}(\mathbf{w}), \mathbf{v}] := \int_{\Omega_1} \mathcal{A}(\mathbf{e}(\mathbf{w})) : \mathbf{e}(\mathbf{v})\, dx + 2\mu \int_{\Omega_2} \mathbf{e}(\mathbf{w}) : \mathbf{e}(\mathbf{v})\, dx - \langle Q(\mathbf{w}), \mathbf{v} \rangle \,, \tag{2.3.2}$$

$$[\mathbf{B}(\mathbf{v}), q] \; := \; - \int_{\Omega} q \, \mathbf{div} \, \mathbf{v} \, dx \quad \text{and} \quad [\mathbf{F}, \mathbf{v}] \; := \; \int_{\Omega_1} \mathbf{f}_1 \cdot \mathbf{v} \, dx \tag{2.3.3}$$

for all $\mathbf{w}$, $\mathbf{v} \in X$ and for all $q \in M$.

Hereafter, $[\,\cdot,\cdot\,]$ denotes the duality pairing induced by the corresponding operators, and $\langle\,\cdot,\cdot\,\rangle$ stands for the duality pairing of $[H^{-1/2}(\Gamma)]^2$ and $[H^{1/2}(\Gamma)]^2$ with respect to the $[L^2(\Gamma)]^2$-inner product. Equivalently, we may also write

$$\langle Q(\mathbf{w}), \mathbf{v} \rangle \; = \; - \frac{2\mu}{\pi} \sum_{j=1}^{2} \sum_{n=1}^{\infty} \int_0^{2\pi} \int_0^{2\pi} \frac{\cos\, n(\varphi - \theta)}{n} \frac{\partial w_j}{\partial \varphi}(r, \varphi) \frac{\partial v_j}{\partial \theta}(r, \theta) \, d\varphi \, d\theta$$

for all $\mathbf{w} := (w_1, w_2)^T$, $\mathbf{v} := (v_1, v_2)^T \in X$.

Similarly, the variational formulation of the approximate boundary value problem (2.2.10) reads: *Find* $(\mathbf{u}^N, p^N) \in X \times M$ *such that*

$$\begin{aligned} [\mathbf{A}^N(\mathbf{u}^N), \mathbf{v}] + [\mathbf{B}(\mathbf{v}), p^N] \; &= \; [\mathbf{F}, \mathbf{v}] \quad \forall\, \mathbf{v} \in X \,, \\ [\mathbf{B}(\mathbf{u}^N), q] \; &= \; 0 \quad \forall\, q \in M \,, \end{aligned} \tag{2.3.4}$$

where $\mathbf{A}^N : X \to X'$ is the nonlinear operator obtained from (2.3.2) after replacing Q by Q^N, that is

$$[\mathbf{A}^N(\mathbf{w}), \mathbf{v}] := \int_{\Omega_1} \mathcal{A}(\mathbf{e}(\mathbf{w})) : \mathbf{e}(\mathbf{v}) \, dx \, + \, 2\mu \int_{\Omega_2} \mathbf{e}(\mathbf{w}) : \mathbf{e}(\mathbf{v}) \, dx \, - \, \langle Q^N(\mathbf{w}), \mathbf{v} \rangle \,, \tag{2.3.5}$$

where

$$\langle Q^N(\mathbf{w}), \mathbf{v} \rangle \; = \; - \frac{2\mu}{\pi} \sum_{j=1}^{2} \sum_{n=1}^{N} \int_0^{2\pi} \int_0^{2\pi} \frac{\cos\, n(\varphi - \theta)}{n} \frac{\partial w_j}{\partial \varphi}(r, \varphi) \frac{\partial v_j}{\partial \theta}(r, \theta) \, d\varphi \, d\theta$$

for all $\mathbf{w} := (w_1, w_2)^T$, $\mathbf{v} := (v_1, v_2)^T \in X$.

Our next goal is to show the unique solvability of the variational formulations (2.3.1) and (2.3.4). To this end, we first recall from [67] the following abstract theorem.

Theorem 2.3.1 *Let* X, M *be Hilbert spaces and let* $\mathbf{A} : X \to X'$ *and* $\mathbf{B} : X \to M'$ *be nonlinear and linear operators, respectively. Let* $V := \text{kernel}\,(\mathbf{B}) = \{\, \mathbf{v} \in X : [\mathbf{B}(\mathbf{v}), q] = 0 \quad \forall\, q \in M \,\}$. *Assume that* $\mathbf{A}$ *is Lipschitz-continuous on* X *and that for all* $\tilde{\mathbf{u}} \in X$, $\mathbf{A}(\tilde{\mathbf{u}} + \cdot)$ *is uniformly strongly monotone on* V, *that is, there exist constants* γ, $\alpha > 0$ *such that*

$$||\mathbf{A}(\mathbf{w}) \, - \, \mathbf{A}(\mathbf{v})||_{X'} \; \leq \; \gamma \, ||\mathbf{w} - \mathbf{v}||_X \quad \forall\, \mathbf{w}, \, \mathbf{v} \in X \,,$$

and

$$[\mathbf{A}(\tilde{\mathbf{u}}+\mathbf{w}) - \mathbf{A}(\tilde{\mathbf{u}}+\mathbf{v}), \mathbf{w}-\mathbf{v}] \;\geq\; \alpha\, ||\mathbf{w}-\mathbf{v}||_X^2$$

for all $\tilde{\mathbf{u}} \in X$ *and for all* $\mathbf{w}$, $\mathbf{v} \in V$. *In addition, assume that there exists* $\beta > 0$ *such that for all* $q \in M$

$$\sup_{\substack{\mathbf{v}\in X\\ \mathbf{v}\neq 0}} \frac{[\mathbf{B}(\mathbf{v}), q]}{||\mathbf{v}||_X} \;\geq\; \beta\, ||q||_M\,.$$

Then, given $(\mathbf{F}, \mathbf{G}) \in X' \times M'$, *there exists a unique* $(\mathbf{u}, p) \in X \times M$ *such that*

$$\begin{aligned} [\mathbf{A}(\mathbf{u}), \mathbf{v}] + [\mathbf{B}(\mathbf{v}), p] \;&=\; [\mathbf{F}, \mathbf{v}] \quad \forall\, \mathbf{v} \in X\,, \\ [\mathbf{B}(\mathbf{u}), q] \;&=\; [\mathbf{G}, q] \quad \forall\, q \in M\,. \end{aligned}$$

Furthermore, the following estimates hold

$$||\mathbf{u}||_X \;\leq\; \frac{1}{\alpha}\, ||\mathbf{F}|| + \frac{1}{\beta}\left(1+\frac{\gamma}{\alpha}\right)\, ||\mathbf{G}||\,,$$

$$||p||_M \;\leq\; \frac{1}{\beta}\left(1+\frac{\gamma}{\alpha}\right)\left(||\mathbf{F}|| + \frac{\gamma}{\beta}||\mathbf{G}||\right)\,.$$

Proof. It is a particular case of Proposition 2.3 in [67]. □

In order to apply the above theorem to our variational formulations (2.3.1) and (2.3.4), we need several previous results.

Lemma 2.3.1 *Let* $\tilde{\mathbf{A}} : X \to X'$ *be the nonlinear operator defined by*

$$[\tilde{\mathbf{A}}(\mathbf{w}), \mathbf{v}] := \int_{\Omega_1} \mathcal{A}(\mathbf{e}(\mathbf{w})) : \mathbf{e}(\mathbf{v})\, dx + 2\mu \int_{\Omega_2} \mathbf{e}(\mathbf{w}) : \mathbf{e}(\mathbf{v})\, dx$$

for all $\mathbf{w}$, $\mathbf{v} \in X$. *Then* $\tilde{\mathbf{A}}$ *is strongly monotone and Lipschitz-continuous on* X.

Proof. Let $a_{ij} : \mathbb{R}^{2\times 2} \to \mathbb{R}$ be the nonlinear mapping defined by

$$a_{ij}(\boldsymbol{\tau}) := \tilde{\lambda}(\mathrm{dev}(\boldsymbol{\tau}))\, \mathrm{tr}(\boldsymbol{\tau})\, \delta_{ij} + \tilde{\mu}(\mathrm{dev}(\boldsymbol{\tau}))\, \tau_{ij} \tag{2.3.6}$$

for all $\boldsymbol{\tau} := (\tau_{ij}) \in \mathbb{R}^{2\times 2}$. Then, it is easy to see, according to (2.2.7), that

$$\mathcal{A}(\mathbf{e}(\mathbf{w})) \;:=\; (a_{ij}\,(\mathbf{e}(\mathbf{w}))) \in \mathbb{R}^{2\times 2} \quad \forall\, \mathbf{w} \in X\,, \tag{2.3.7}$$

and hence, for all $\mathbf{w}$, $\mathbf{v} \in X$ we can write

$$\int_{\Omega_1} \mathcal{A}(\mathbf{e}(\mathbf{w})) : \mathbf{e}(\mathbf{v})\, dx \;=\; \sum_{i,j=1}^{2} \int_{\Omega_1} a_{ij}(\mathbf{e}(\mathbf{w}))\, e_{ij}(\mathbf{v})\, dx\,. \tag{2.3.8}$$

In addition, in virtue of the properties of κ and $\tilde{\mu}$ (cf. Section 2) one can prove (cf. Lemmas 4.1, 4.2, and 4.3 in [7]) that $a_{ij}(\cdot)$ and its first order partial derivatives are continuous in $\mathbb{R}^{2\times 2}$, and that there exist C_1, C_2, $C_3 > 0$ such that

$$|a_{ij}(\boldsymbol{\tau})| \,\leq\, C_1\,|\boldsymbol{\tau}|\,,$$

$$\sum_{i,j=1}^{2}\sum_{k,l=1}^{2}\frac{\partial}{\partial\tau_{kl}}a_{ij}(\boldsymbol{\tau})\,\beta_{kl}\,\beta_{ij} \,\geq\, C_2\sum_{i,j=1}^{2}\beta_{ij}^2\,, \tag{2.3.9}$$

and

$$\left|\frac{\partial}{\partial\tau_{kl}}\,a_{ij}(\boldsymbol{\tau})\right| \,\leq\, C_3 \quad \forall\boldsymbol{\tau}\in\mathbb{R}^{2\times 2}\,, \tag{2.3.10}$$

for all $\boldsymbol{\tau} := (\tau_{ij})$, $\boldsymbol{\beta} := (\beta_{ij}) \in \mathbb{R}^{2\times 2}$ (see, also, [44]).

The rest of the proof proceeds similarly as for Theorems 5.1 and 5.2 in [7] (see also [44]). We omit further details. □

We now recall an important result from [55].

Lemma 2.3.2 *Let* $\mathbf{A}_0$, $\mathbf{A}_0^N : X \to X'$ *be the linear operators given by* $[\mathbf{A}_0(\mathbf{w}),\mathbf{v}] := -\langle Q(\mathbf{w}),\mathbf{v}\rangle$ *and* $[\mathbf{A}_0^N(\mathbf{w}),\mathbf{v}] := -\langle Q^N(\mathbf{w}),\mathbf{v}\rangle$ *for all* $\mathbf{w}$, $\mathbf{v}\in X$. *Then,* $\mathbf{A}_0$ *and* $\mathbf{A}_0^N$ *are bounded, independently of integer* N, *and also*

$$[\mathbf{A}_0(\mathbf{w}),\mathbf{w}]\;,\quad [\mathbf{A}_0^N(\mathbf{w}),\mathbf{w}] \,\geq\, 0 \quad \forall\,\mathbf{w}\in X\,. \tag{2.3.11}$$

Proof. See Lemma 3.1 in [55]. □

As a consequence of the previous lemmas we can state the following theorem.

Theorem 2.3.2 *The nonlinear operators* $\mathbf{A}$, $\mathbf{A}^N : X \to X'$, *defined in* (2.3.2) *and* (2.3.5), *respectively, are strongly monotone and Lipschitz-continuous. More precisely, there exist* α, $M > 0$, *independent of integer* N, *such that*

$$[\mathbf{A}(\mathbf{w}) - \mathbf{A}(\mathbf{v}),\mathbf{w}-\mathbf{v}]\,,\; [\mathbf{A}^N(\mathbf{w}) - \mathbf{A}^N(\mathbf{v}),\mathbf{w}-\mathbf{v}] \,\geq\, \alpha\,||\mathbf{w}-\mathbf{v}||_X^2$$

and

$$||\mathbf{A}(\mathbf{w}) - \mathbf{A}(\mathbf{v})||_{X'}\,,\; ||\mathbf{A}^N(\mathbf{w}) - \mathbf{A}^N(\mathbf{v})||_{X'} \,\leq\, M\,||\mathbf{w}-\mathbf{v}||_X$$

for all $\mathbf{w}$, $\mathbf{v}\in X$.

Proof. It follows straightforward from Lemmas 2.3.1 and 2.3.2. □

Now, for the linear operator $\mathbf{B}$ we have the following inf-sup condition.

Lemma 2.3.3 *There exists* $\beta > 0$ *such that*

$$\sup_{\substack{\mathbf{v}\in X\\ \mathbf{v}\neq 0}} \frac{[\mathbf{B}(\mathbf{v}), q]}{||\mathbf{v}||_X} \geq \beta\, ||q||_M \qquad \forall\, q \in M\,.$$

Proof. This is a well known result which follows from classical regularity estimates (see. e.g. Lemma 3.2 in [55], Lemma 4.4 in [62] or [50]). □

At this point we can establish the main theorem concerning the unique solvability of the variational formulations (2.3.1) and (2.3.4).

Theorem 2.3.3 *There exists a unique* $(\mathbf{u}, p) \in X \times M$ *solution of* (2.3.1) *and a unique* $(\mathbf{u}^N, p^N) \in X \times M$ *solution of* (2.3.4), *for any integer* $N \geq 0$. *In addition, if* $\mathbf{u}|_\Gamma \in [H^{m+\frac{1}{2}}(\Gamma)]^2$ *for some positive integer* m, *then there exists* $C > 0$, *independent of* N *and* m, *such that the following error estimate holds*

$$||\mathbf{u} - \mathbf{u}^N||_X \,+\, ||p - p^N||_M \,\leq\, C\, \frac{1}{(N+1)^m}\, ||\mathbf{u}|_\Gamma||_{[H^{m+\frac{1}{2}}(\Gamma)]^2}\,. \tag{2.3.12}$$

Proof. The first assertions follow directly from the abstract Theorem 2.3.1, Theorem 2.3.2 and Lemma 2.3.3.

It remains to prove (2.3.12). To this end, we now recall from Lemma 3.3 in [55], that the following estimate holds for all $\mathbf{v} \in X$

$$|[\mathbf{A}_0(\mathbf{u}), \mathbf{v}] - [\mathbf{A}_0^N(\mathbf{u}), \mathbf{v}]| \,\leq\, C\, \frac{1}{(N+1)^m}\, ||\mathbf{u}|_\Gamma||_{[H^{m+\frac{1}{2}}(\Gamma)]^2}\, ||\mathbf{v}||_X\,, \tag{2.3.13}$$

where $C > 0$ is a constant independent of N and m.

On the other hand, from (2.3.1) and (2.3.4) we can write

$$[\mathbf{A}(\mathbf{u}), \mathbf{v}] - [\mathbf{A}^N(\mathbf{u}^N), \mathbf{v}] + [\mathbf{B}(\mathbf{v}), p - p^N] = 0 \quad \forall\, \mathbf{v} \in X\,, \tag{2.3.14}$$

and

$$[\mathbf{B}(\mathbf{u} - \mathbf{u}^N), q] \,=\, 0 \quad \forall\, q \in M\,. \tag{2.3.15}$$

In particular, taking $\mathbf{v} = \mathbf{u} - \mathbf{u}^N$ in (2.3.14) and $q = p - p^N$ in (2.3.15), we obtain

$$[\mathbf{A}(\mathbf{u}), \mathbf{u} - \mathbf{u}^N] \,-\, [\mathbf{A}^N(\mathbf{u}^N), \mathbf{u} - \mathbf{u}^N] \,=\, 0\,. \tag{2.3.16}$$

Then, using the strong monotonicity of $\mathbf{A}^N$, (2.3.16) and the definitions of $\mathbf{A}$ and $\mathbf{A}^N$, we deduce

$$\alpha\, ||\mathbf{u} - \mathbf{u}^N||_X^2 \,\leq\, [\mathbf{A}^N(\mathbf{u}) - \mathbf{A}^N(\mathbf{u}^N), \mathbf{u} - \mathbf{u}^N]$$

$$= [\mathbf{A}^N(\mathbf{u}), \mathbf{u} - \mathbf{u}^N] \; - \; [\mathbf{A}(\mathbf{u}), \mathbf{u} - \mathbf{u}^N] \; = \; [\mathbf{A}_0^N(\mathbf{u}), \mathbf{u} - \mathbf{u}^N] \; - \; [\mathbf{A}_0(\mathbf{u}), \mathbf{u} - \mathbf{u}^N] \,,$$

which, due to (2.3.13), yields

$$\alpha \, ||\mathbf{u} - \mathbf{u}^N||_X^2 \; \leq \; C \, \frac{1}{(N+1)^m} \, ||\mathbf{u}|_\Gamma||_{[H^{m+\frac{1}{2}}(\Gamma)]^2} \, ||\mathbf{u} - \mathbf{u}^N||_X \,,$$

and hence

$$||\mathbf{u} - \mathbf{u}^N||_X \; \leq \; \frac{C}{\alpha} \, \frac{1}{(N+1)^m} \, ||\mathbf{u}|_\Gamma||_{[H^{m+\frac{1}{2}}(\Gamma)]^2} \,. \tag{2.3.17}$$

Now, from (2.3.14) we get for all $\mathbf{v} \in X$

$$[\mathbf{B}(\mathbf{v}), p - p^N] = [\mathbf{A}^N(\mathbf{u}^N), \mathbf{v}] - [\mathbf{A}(\mathbf{u}), \mathbf{v}]$$

$$= [\mathbf{A}^N(\mathbf{u}^N), \mathbf{v}] - [\mathbf{A}^N(\mathbf{u}), \mathbf{v}] + [\mathbf{A}^N(\mathbf{u}), \mathbf{v}] - [\mathbf{A}(\mathbf{u}), \mathbf{v}]$$

$$= [\mathbf{A}^N(\mathbf{u}^N), \mathbf{v}] - [\mathbf{A}^N(\mathbf{u}), \mathbf{v}] + [\mathbf{A}_0^N(\mathbf{u}), \mathbf{v}] - [\mathbf{A}_0(\mathbf{u}), \mathbf{v}] \,.$$

Then, using the Lipschitz-continuity of $\mathbf{A}^N$ and (2.3.13), we deduce that

$$\frac{[\mathbf{B}(\mathbf{v}), p - p^N]}{||\mathbf{v}||_X} \leq \left\{ M ||\mathbf{u}^N - \mathbf{u}||_X + \frac{C}{(N+1)^m} ||\mathbf{u}|_\Gamma||_{[H^{m+\frac{1}{2}}(\Gamma)]^2} \right\}$$

for all $\mathbf{v} \in X$. Finally, the above estimate, (2.3.17) and the inf-sup condition satisfied by $\mathbf{B}$ (cf. Lemma 2.3.3) complete the proof of the theorem. □

2.4 The Galerkin approximations

The purpose of this section is to study the Galerkin approximations of the variational formulation (2.3.4). To this end, we now let X_h and M_h be finite-dimensional subspaces of X and M, respectively. Then, the Galerkin scheme of (2.3.4) reads: *Find* $(\mathbf{u}_h^N, p_h^N) \in X_h \times M_h$ *such that*

$$\begin{aligned} [\mathbf{A}^N(\mathbf{u}_h^N), \mathbf{v}_h] + [\mathbf{B}(\mathbf{v}_h), p_h^N] &= [\mathbf{F}, \mathbf{v}_h] \quad \forall\, \mathbf{v}_h \in X_h \,, \\ [\mathbf{B}(\mathbf{u}_h^N), q_h] &= 0 \quad \forall\, q_h \in M_h \,. \end{aligned} \tag{2.4.1}$$

2.4.1 The error analysis

In what follows, we assume that the operator $\mathbf{B}$ also satisfies the discrete inf-sup condition. This means that there exists $\beta^* > 0$, independent of h, such that

$$\sup_{\substack{\mathbf{v}_h \in X_h \\ \mathbf{v}_h \neq 0}} \frac{[\mathbf{B}(\mathbf{v}_h), q_h]}{||\mathbf{v}_h||_X} \geq \beta^* \, ||q_h||_M \quad \forall \, q_h \in M_h \, . \tag{2.4.2}$$

Then, we can prove the following theorem.

Theorem 2.4.1 *For all $N \in \mathbb{N}$ there exists a unique $(\mathbf{u}_h^N, p_h^N) \in X_h \times M_h$ solution of problem* (2.4.1)*. Moreover, there exists $C_0 > 0$, independent of h and N, such that*

$$\begin{aligned} &||\mathbf{u}^N - \mathbf{u}_h^N||_X + ||p^N - p_h^N||_M \\ &\leq C_0 \left\{ \inf_{\mathbf{v}_h \in X_h} ||\mathbf{u}^N - \mathbf{v}_h||_X + \inf_{q_h \in M_h} ||p^N - q_h||_M \right\} . \end{aligned} \tag{2.4.3}$$

Proof. The unique solvability of (2.4.1) is again a consequence of Theorem 2.3.1 (see also the discrete analogue given by Proposition 2.6 in [67]). In addition, the error estimate (2.4.3) constitutes a particular case of the more general result given by Theorem 2.1 in [67]. □

We are now in a position to provide the error estimate between the solutions of (2.3.1) and (2.4.1).

Theorem 2.4.2 *Let $(\mathbf{u}, p) \in X \times M$ and $(\mathbf{u}_h^N, p_h^N) \in X_h \times M_h$ be the unique solutions of* (2.3.1) *and* (2.4.1)*, respectively. Assume that $u|_\Gamma \in [H^{m+\frac{1}{2}}(\Gamma)]^2$ for some positive integer m. Then, there exists $C > 0$, independent of h, N and m, such that*

$$\begin{aligned} ||\mathbf{u} - \mathbf{u}_h^N||_X + ||p - p_h^N||_M \leq C \Big\{ &\inf_{(\mathbf{v}_h, q_h) \in X_h \times M_h} ||(\mathbf{u}, p) - (\mathbf{v}_h, q_h)||_{X \times M} \\ &+ \frac{1}{(N+1)^m} \, ||u|_\Gamma||_{[H^{m+\frac{1}{2}}(\Gamma)]^2} \Big\} . \end{aligned} \tag{2.4.4}$$

Proof. By triangle inequality we can write

$$\begin{aligned} ||\mathbf{u} - \mathbf{u}_h^N||_X + ||p - p_h^N||_M \leq \; &||\mathbf{u} - \mathbf{u}^N||_X + ||p - p^N||_M \\ &+ ||\mathbf{u}^N - \mathbf{u}_h^N||_X + ||p^N - p_h^N||_M \, , \end{aligned}$$

which, according to (2.4.3) (cf. Theorem 2.4.1), yields

$$||\mathbf{u} - \mathbf{u}_h^N||_X + ||p - p_h^N||_M \leq ||\mathbf{u} - \mathbf{u}^N||_X + ||p - p^N||_M$$

$$+ C \left\{ \inf_{\mathbf{v}_h \in X_h} ||\mathbf{u}^N - \mathbf{v}_h||_X + \inf_{q_h \in M_h} ||p^N - q_h||_M \right\}. \qquad (2.4.5)$$

Next, we observe that

$$\inf_{\mathbf{v}_h \in X_h} ||\mathbf{u}^N - \mathbf{v}_h||_X \leq ||\mathbf{u} - \mathbf{u}^N||_X + \inf_{\mathbf{v}_h \in X_h} ||\mathbf{u} - \mathbf{v}_h||_X \qquad (2.4.6)$$

and, similarly,

$$\inf_{q_h \in M_h} ||p^N - q_h||_M \leq ||p - p^N||_M + \inf_{q_h \in M_h} ||p - q_h||_M . \qquad (2.4.7)$$

Therefore, replacing (2.4.6)-(2.4.7) back into (2.4.5), and using the error estimate from Theorem 2.3.3, we obtain (2.4.4). □

2.4.2 An example of finite element subspaces

We now introduce an example of specific finite element subspaces X_h and M_h satisfying the discrete inf-sup condition (2.4.2). Furthermore, the corresponding rate of convergence is also provided. For simplicity, in what follows we assume that Γ_0 and Γ_1 are polygonal curves.

Given $n \in \mathbb{N}$, we let $0 = t_0 < t_1 < \cdots < t_n = 2\pi$ be a uniform partition of $[0, 2\pi]$ and let $\mathbf{z} : [0, 2\pi] \to \Gamma$ be the parameterization of the circle defined by $\mathbf{z}(t) = r(\cos(t), \sin(t))^T$ for all $t \in [0, 2\pi]$. Then, we denote by $\tilde{\Omega}_h$ the annular domain bounded by Γ_0 and the closed polygonal line Γ_h with vertices $\{\mathbf{z}(t_1), \mathbf{z}(t_2), \ldots, \mathbf{z}(t_n)\}$, and let $\tilde{\mathcal{T}}_h$ be a regular triangulation of $\tilde{\Omega}_h$ by triangles T of diameter h_T such that $h = \sup_{T \in \tilde{\mathcal{T}}_h} h_T$. Here, we assume that for each $T \in \mathcal{T}_h$, either $T \subseteq \bar{\Omega}_1$ or $T \subseteq \bar{\Omega}_2$. Next, each triangle with one side along Γ_h is replaced by the corresponding curved triangle with one side along Γ. In this way, we obtain a regular triangulation $\mathcal{T}_h$ of $\bar{\Omega}$ made up of straight and curved triangles so that $\bar{\Omega} = \cup \{ T : \quad T \in \mathcal{T}_h \}$.

Now, we consider the reference triangle $\hat{T}$ with vertices $\hat{P}_1 = (0,0)^T$, $\hat{P}_2 = (1,0)^T$ and $\hat{P}_3 = (0,1)^T$ and introduce a family of bijective mappings $\{F_T\}_{T \in \mathcal{T}_h}$ such that $F_T(\hat{T}) = T$. In particular, if T is a straight triangle of $\mathcal{T}_h$, then F_T is the well known invertible affine mapping defined by $F_T(\hat{x}) = B_T\, \hat{x} + b_T \ \forall\, \hat{x} \in \hat{T}$, where B_T, a square matrix of order 2, and $b_T \in \mathbb{R}^2$, depend on the vertices of T.

Similarly, if T is a curved triangle with vertices P_1, P_2 and P_3, such that $P_2 = \mathbf{z}(t_{j-1}) \in \Gamma$ and $P_3 = \mathbf{z}(t_j) \in \Gamma$, then $F_T(\hat{x}) = B_T\,\hat{x} + b_T + G_T(\hat{x})$ for all $\hat{x} := (\hat{x}_1, \hat{x}_2) \in \hat{T}$, where

$$G_T(\hat{x}) = \frac{\hat{x}_1}{1-\hat{x}_2}\Big\{\mathbf{z}(t_{j-1} + \hat{x}_2(t_j - t_{j-1})) - \left[\mathbf{z}(t_{j-1}) + \hat{x}_2\left(\mathbf{z}(t_j) - \mathbf{z}(t_{j-1})\right)\right]\Big\}.$$

It can be proved (see, e.g. Theorem 22.4 in [73]) that F_T is a diffeomorphism of class C^∞ that maps one-to-one $\hat{T}$ onto the curved triangle T in such a way that $F_T(\hat{P}_i) = P_i$ for all $i \in \{1,2,3\}$. Also, the image of edge $\hat{P}_2\hat{P}_3$ is the curved side of T and, since $G_T(\hat{x}) = (0,0)^T$ for $\hat{x}_1 = 0$ and for $\hat{x}_2 = 0$, the two other edges of $\hat{T}$ are transformed linearly under F_T to the straight sides of T.

In order to specify X_h and M_h, we first put

$$\hat{\mathbf{P}}_1(\hat{T}) := \text{span}\,\{1, \hat{x}_1, \hat{x}_2\}\,,$$

$$\hat{\mathbf{P}}_2(\hat{T}) := \text{span}\,\{1, \hat{x}_1, \hat{x}_2, \hat{x}_1^2, \hat{x}_2^2, \hat{x}_1\hat{x}_2\}\,,$$

and for each triangle $T \in \mathcal{T}_h$ we define

$$\mathbf{P}_2(T) := \{v: \quad v = \hat{v} \circ F_T^{-1}\,,\;\; \hat{v} \in \hat{\mathbf{P}}_2(\hat{T})\}$$

$$\mathbf{P}_1(T) := \{q: \quad q = \hat{q} \circ F_T^{-1}\,,\;\; \hat{q} \in \hat{\mathbf{P}}_1(\hat{T})\}\,.$$

Then, we set

$$X_h := \left\{\, \mathbf{v}_h \in [H^1_{\Gamma_0}(\Omega)]^2: \quad \mathbf{v}_h|_T \in [\mathbf{P}_2(T)]^2 \quad \forall\, T \in \mathcal{T}_h \,\right\} \tag{2.4.8}$$

and

$$M_h := \left\{\, q_h \in H^1(\Omega): \quad q_h|_T \in \mathbf{P}_1(T) \quad \forall\, T \in \mathcal{T}_h \,\right\}\,. \tag{2.4.9}$$

It is well known that X_h and M_h constitute the simplest Hood and Taylor finite element subspaces satisfying the discrete inf-sup condition (2.4.2) (see, e.g. Section VI.6 in [20] in the case of straight triangles). The analysis for curved triangles should be handled by combining the results from [20] and [73]. In addition, the following approximation properties hold

$$\inf_{\mathbf{v}_h \in X_h} ||\mathbf{u} - \mathbf{v}_h||_X \le Ch^2 |\mathbf{u}|_{[H^3(\Omega)]^2}\ \forall\, \mathbf{u} \in [H^1_{\Gamma_0}(\Omega) \cap H^3(\Omega)]^2, \tag{2.4.10}$$

$$\inf_{q_h \in M_h} ||p - q_h||_M \leq C\, h^2\, |p|_{H^2(\Omega)} \quad \forall\, p \in H^2(\Omega)\,. \tag{2.4.11}$$

Consequently, we can state the following theorem providing a rate of convergence of $O(h^2)$.

Theorem 2.4.3 *Let* $(\mathbf{u}, p) \in X \times M$ *and* $(\mathbf{u}_h^N, p_h^N) \in X_h \times M_h$ *be the unique solutions of* (2.3.1) *and* (2.4.1) *with* X_h *and* M_h *given by* (2.4.8) *and* (2.4.9). *Assume that* $\mathbf{u} \in [H^1_{\Gamma_0}(\Omega) \cap H^3(\Omega)]^2$ *and* $p \in H^2(\Omega)$. *Then, there exists* $C > 0$, *independent of* h *and* N, *such that*

$$||\mathbf{u} - \mathbf{u}_h^N||_X \,+\, ||p - p_h^N||_M$$
$$\leq\, C \left\{ h^2 \left(|\mathbf{u}|_{[H^3(\Omega)]^2} + |p|_{H^2(\Omega)} \right) \,+\, \frac{1}{(N+1)^2}\, ||\mathbf{u}|_\Gamma||_{[H^{5/2}(\Gamma)]^2} \right\}.$$

In particular, if we take $N = O(h^{-1})$, *then we obtain*

$$||\mathbf{u} - \mathbf{u}_h^N||_X \,+\, ||p - p_h^N||_M$$
$$\leq\, C\, h^2 \left\{ |\mathbf{u}|_{[H^3(\Omega)]^2} + |p|_{H^2(\Omega)} \,+\, ||\mathbf{u}|_\Gamma||_{[H^{5/2}(\Gamma)]^2} \right\}.$$

Proof. It follows from (2.4.4) (cf. Theorem 2.4.2) with $m = 2$, and the approximation properties (2.4.10)-(2.4.11). □

We end this section by remarking, as noticed from Theorems 2.4.2 and 2.4.3, that the error of our discrete scheme (2.4.1) consists of two parts, one coming from the finite element approximations (illustrated by (2.4.10)-(2.4.11)), and the other one arising from the number of terms N for the finite Fourier series on the auxiliary boundary Γ. Although the theoretical estimate given by Theorem 2.4.3 implies that a large number N is required, the numerical experiments in [55] (for the corresponding linear problem) indicates that, with a sufficiently large circle Γ, a small N may provide very good accuracy in the computations. This property has also been observed before with analogue DtN mappings for other exterior boundary value problems (see, e.g. [51], [57]).

2.5 The a-posteriori error estimator

2.5.1 Preliminaries

Let $\{\mathcal{T}_h\}_{h \in \mathtt{I}}$ be a regular family of triangulations of Ω, as defined in Subsection 2.4.2 from Chapter 2, where $\mathtt{I}$ is a at most numerable set of indexes, say $\mathtt{I} :=$

$\{h_j\}_{j\in\mathbb{N}}$, with $h_j \geq h_{j+1}\ \forall\, j \in \mathbb{N}$. Here, *regular* means that the interior angles of all the triangles of all the triangulations $\mathcal{T}_h$ are uniformly bounded from below. Thus, in what follows, C will denote a positive constant that may depend on the above uniform bound but not on the meshsize h.

Now, for any $T \in \mathcal{T}_h$ we denote by $\boldsymbol{\nu}_T$ the unit outward normal and by $\mathcal{E}(T)$ the set of its edges. Also, we put $\mathcal{E}_h := \cup\{\mathcal{E}(T) : \quad T \in \mathcal{T}_h\}$ and split this set in the following form $\mathcal{E}_h := \mathcal{E}_h(\Omega) \cup \mathcal{E}_h(\Gamma_0) \cup \mathcal{E}_h(\Gamma)$, where $\mathcal{E}_h(\Omega) := \{\, S \in \mathcal{E}_h : \quad S \subseteq \Omega\,\}$, $\mathcal{E}_h(\Gamma_0) := \{\, S \in \mathcal{E}_h : \quad S \subseteq \Gamma_0\,\}$, and $\mathcal{E}_h(\Gamma) := \{\, S \in \mathcal{E}_h : \quad S \subseteq \Gamma\,\}$.

Hereafter, we consider the finite element subspaces X_h and M_h defined in Subsection 2.4.2 from Chapter 2 (cf. (2.4.8)-(2.4.9)), which satisfy the global approximation properties (2.4.10)-(2.4.11). In addition, by applying the analysis from [29] and [17], one can show that X_h also satisfies the local properties stated in the following lemma.

Lemma 2.5.1 *Let $\mathcal{I}_h : [H^1_{\Gamma_0}(\Omega)]^2 \to X_h$ be the Clément interpolation operator as defined in* [29] *and* [17]*. Then there exist positive constants C_1 and C_2, independent of h, such that for all $\mathbf{v} \in [H^1_{\Gamma_0}(\Omega)]^2$ and for all $T \in \mathcal{T}_h$ and $S \in \mathcal{E}_h$, it holds that*

$$||\mathbf{v} - \mathcal{I}_h(\mathbf{v})||_{[L^2(T)]^2} \;\leq\; C_1\, h_T\, ||\mathbf{v}||_{[H^1(\Delta(T))]^2}$$

and

$$||\mathbf{v} - \mathcal{I}_h(\mathbf{v})||_{[L^2(S)]^2} \;\leq\; C_2\, h_S^{1/2}\, ||\mathbf{v}||_{[H^1(\Delta(S))]^2}\,,$$

where h_T (resp. h_S) denotes the diameter of T (resp. S), $\Delta(T) := \cup\{\tilde{T} \in \mathcal{T}_h : \tilde{T} \cap T \neq \phi\}$ and $\Delta(S) := \cup\{\tilde{T} \in \mathcal{T}_h : \quad \tilde{T} \cap S \neq \phi\}$.

Proof. See details in [17] and [29]. □

Since $\{\mathcal{T}_h\}_{h\in\mathrm{I}}$ is a regular family of triangulations, it is easy to see that the number of triangles in $\Delta(T)$ and in $\Delta(S)$ are bounded, independently of h. This fact will be used later for the derivation of the a-posteriori error estimate.

2.5.2 Main estimates

Our main goal is to prove the following theorem providing a reliable a-posteriori error estimate.

Theorem 2.5.1 *Let $(\mathbf{u}^N, p^N) \in X \times M$ and $(\mathbf{u}_h^N, p_h^N) \in X_h \times M_h$ be the unique solutions of the continuous and discrete formulations* (2.3.4) *and* (2.4.1)*, respectively.*

Then, there exists $C > 0$, independent of h, such that

$$||\mathbf{u}^N - \mathbf{u}_h^N||_X + ||p^N - p_h^N||_M \leq C \left\{ \sum_{T\in\mathcal{T}_h} \eta_T^2 \right\}^{1/2}, \qquad (2.5.1)$$

where for any triangle $T \in \mathcal{T}_h$ we define

$$\begin{aligned} \eta_T^2 \;:=\; & h_T^2\, ||\mathbf{f} + \mathbf{div}\,\boldsymbol{\sigma}(\mathbf{u}_h^N, p_h^N)||^2_{[L^2(T)]^2} \;+\; ||\mathbf{div}\,(\mathbf{u}_h^N)||^2_{L^2(T)} \\ & + \sum_{S\in\mathcal{E}(T)\cap\mathcal{E}_h(\Omega)} h_S ||[\![\boldsymbol{\sigma}(\mathbf{u}_h^N, p_h^N)\,\boldsymbol{\nu}_T]\!]||^2_{[L^2(S)]^2} \\ & + \sum_{S\in\mathcal{E}(T)\cap\mathcal{E}_h(\Gamma)} h_S\, ||Q^N(\mathbf{u}_h^N) \;-\; \boldsymbol{\sigma}(\mathbf{u}_h^N, p_h^N)\,\boldsymbol{\nu}||^2_{[L^2(S)]^2}\,, \end{aligned}$$

with $\mathbf{f} := \begin{cases} \mathbf{f}_1 & \text{in } \Omega_1 \\ 0 & \text{in } \Omega_2 \end{cases}$, *and* $[\![\boldsymbol{\sigma}(\mathbf{u}_h^N, p_h^N)\,\boldsymbol{\nu}_T]\!]$ *denoting the jump of the tractions* $\boldsymbol{\sigma}(\mathbf{u}_h^N, p_h^N)\;\boldsymbol{\nu}_T$ *across the edges of T.*

We remark that the question of whether $\eta := \left\{ \sum_{T\in\mathcal{T}_h} \eta_T^2 \right\}^{1/2}$ is also an efficient estimate is still an open problem. Indeed, the nonlinearity involved and the nonlocality of the approximate Dirichlet-to-Neumann mapping Q^N complicates the corresponding analysis.

Also, it is important to emphasize that an eventual a-posteriori analysis of the error $||(\mathbf{u}, p) - (\mathbf{u}_h^N, p_h^N)||_{X\times M}$ would have to deal with the additional dificulty given by the **non-conformity** of the Galerkin scheme (2.4.1).

Now, in order to prove Theorem 2.5.1, we first need some properties of a functional depending on the stored energy function Ψ (cf. (2.2.2)). More precisely, let $\mathbf{J} : [H^1(\Omega)]^2 \to \mathbb{R}$ be defined by

$$\mathbf{J}(\mathbf{v}) := \int_{\Omega_1} \Psi(\mathbf{e}(\mathbf{v}))\, dx \;+\; \mu \int_{\Omega_2} \mathbf{e}(\mathbf{v}) : \mathbf{e}(\mathbf{v})\, dx$$

for all $\mathbf{v} \in [H^1(\Omega)]^2$. Then we have the following result.

Lemma 2.5.2 *The functional $\mathbf{J}$ has continuous second order Gâteaux derivatives and there exist positive constants C_1 and C_2 such that*

$$C_1\, ||\mathbf{v}||^2_{[H^1(\Omega)]^2} \;\leq\; (D^2\mathbf{J})(\mathbf{z})(\mathbf{v}, \mathbf{v}) \;\leq\; C_2\, ||\mathbf{v}||^2_{[H^1(\Omega)]^2} \qquad (2.5.2)$$

for all $\mathbf{z}$, $\mathbf{v} \in [H^1_{\Gamma_0}(\Omega)]^2$.

Proof. The first-order Gâteaux derivative $D\mathbf{J}$ applies $[H^1(\Omega)]^2$ into its dual. Thus, given $\mathbf{z}$, $\mathbf{v} \in [H^1(\Omega)]^2$, we have $D\mathbf{J}(\mathbf{z})(\mathbf{v}) := \lim_{t\to 0} \dfrac{\mathbf{J}(\mathbf{z}+t\mathbf{v}) - \mathbf{J}(\mathbf{z})}{t}$, which yields

$$D\mathbf{J}(\mathbf{z})(\mathbf{v}) = \sum_{i,j=1}^{2} \int_{\Omega_1} a_{ij}(\mathbf{e}(\mathbf{z}))e_{ij}(\mathbf{v})dx + 2\mu \int_{\Omega_2} \mathbf{e}(\mathbf{z}) : \mathbf{e}(\mathbf{v})dx, \tag{2.5.3}$$

where $a_{ij} : \mathbb{R}^{2\times 2} \to \mathbb{R}$ are the nonlinear mappings defined in (2.3.6).

Now, the second-order Gâteaux derivative $D^2\mathbf{J}$ applies $[H^1(\Omega)]^2$ into the dual of $[H^1(\Omega)]^2 \times [H^1(\Omega)]^2$, and it is given by

$$D^2\mathbf{J}(\mathbf{z})(\mathbf{v},\mathbf{w}) := \lim_{t\to 0} \frac{D\mathbf{J}(\mathbf{z}+t\,\mathbf{v})(\mathbf{w}) - D\mathbf{J}(\mathbf{z})(\mathbf{w})}{t} \tag{2.5.4}$$

for all $\mathbf{z}$, $\mathbf{v}$, $\mathbf{w} \in [H^1(\Omega)]^2$. By using (2.5.3) in (3.5.6) we find, after some algebraic manipulations, that

$$D^2\mathbf{J}(\mathbf{z})(\mathbf{v},\mathbf{w}) = \int_{\Omega_1} \sum_{i,j=1}^{2} \sum_{k,l=1}^{2} \frac{\partial}{\partial \tau_{kl}} a_{ij}(\mathbf{e}(\mathbf{z}))\, e_{kl}(\mathbf{v})\, e_{ij}(\mathbf{w})\, dx + 2\,\mu \int_{\Omega_2} \mathbf{e}(\mathbf{v}) : \mathbf{e}(\mathbf{w})\, dx\,.$$

Hence, applying the inequalities on the nonlinear coefficients a_{ij}, given by (2.3.9) and (2.3.10), and using Korn's inequality and the continuity of the strain tensor $\mathbf{e}$, we conclude (2.5.2).

Finally, the continuity of $D^2\mathbf{J}$ follows from the fact that Φ and hence the stored energy function Ψ (cf. (2.2.2)) are of class C^2. □

We are now in a position to prove the following theorem.

Theorem 2.5.2 *There exists $C_0 > 0$, independent of h, such that*

$$\begin{aligned} ||\mathbf{u}^N - \mathbf{u}_h^N||_X + ||p^N - p_h^N||_M \le C_0 \sup_{\substack{(\mathbf{v},q)\in X\times M \\ ||(\mathbf{v},q)||\le 1}} \Big\{ &[\mathbf{A}^N(\mathbf{u}^N) - \mathbf{A}^N(\mathbf{u}_h^N), \mathbf{v}] \\ &+ [\mathbf{B}(\mathbf{v}), p^N - p_h^N] \;+\; [\mathbf{B}(\mathbf{u}^N - \mathbf{u}_h^N), q] \Big\}. \end{aligned} \tag{2.5.5}$$

Proof. Since $D^2\mathbf{J}$ is continuous, there exists $\mathbf{z}^N \in X := [H^1_{\Gamma_0}(\Omega)]^2$, $\mathbf{z}^N$ being a convex linear combination of $\mathbf{u}^N$ and $\mathbf{u}_h^N$, such that

$$(D^2\mathbf{J})(\mathbf{z}^N)(\mathbf{u}^N - \mathbf{u}_h^N, \mathbf{v}) \;=\; D\mathbf{J}(\mathbf{u}^N)(\mathbf{v}) \;-\; D\mathbf{J}(\mathbf{u}_h^N)(\mathbf{v}) \quad \forall\, \mathbf{v} \in X\,. \tag{2.5.6}$$

Next, we observe from (2.5.2) and (2.3.11) (cf. Lemma 2.3.2) that

$$(D^2\mathbf{J})(\mathbf{z}^N)(\mathbf{v},\mathbf{v}) \;-\; \langle Q^N(\mathbf{v}), \mathbf{v}\rangle \;\;\ge\;\; C_1\, ||\mathbf{v}||^2_{[H^1(\Omega)]^2}$$

for all $\mathbf{v} \in X$. Hence, since the linear operator $\mathbf{B}$ satisfies the continuous inf-sup condition (cf. Lemma 2.3.3), Brezzi's theory ([20]) implies that there exists $C_0 > 0$ such that

$$||(\tilde{\mathbf{u}}, \tilde{p})||_{X\times M} \leq C_0 \sup_{\substack{(\mathbf{v},q)\in X\times M\\ ||(\mathbf{v},q)||\leq 1}} \Big\{ (D^2\mathbf{J})(\mathbf{z}^N)(\tilde{\mathbf{u}}, \mathbf{v}) - \langle Q^N(\tilde{\mathbf{u}}), \mathbf{v}\rangle + [\mathbf{B}(\mathbf{v}), \tilde{p}] + [\mathbf{B}(\tilde{\mathbf{u}}), q] \Big\}$$

for all $(\tilde{\mathbf{u}}, \tilde{p}) \in X \times M$. In particular, for $(\tilde{\mathbf{u}}, \tilde{p}) = (\mathbf{u}^N - \mathbf{u}_h^N, p^N - p_h^N)$, we obtain

$$||\mathbf{u}^N - \mathbf{u}_h^N||_X + ||p^N - p_h^N||_M \leq C_0 \sup_{\substack{(\mathbf{v},q)\in X\times M\\ ||(\mathbf{v},q)||\leq 1}} \Big\{ (D^2\mathbf{J})(\mathbf{z}^N)(\mathbf{u}^N - \mathbf{u}_h^N, \mathbf{v}) - \langle Q^N(\mathbf{u}^N - \mathbf{u}_h^N), \mathbf{v}\rangle + [\mathbf{B}(\mathbf{v}), p^N - p_h^N] + [\mathbf{B}(\mathbf{u}^N - \mathbf{u}_h^N), q] \Big\}. \tag{2.5.7}$$

We now recall, according to (2.3.5) and (2.5.3), that

$$[\mathbf{A}^N(\mathbf{u}^N), \mathbf{v}] = D\mathbf{J}(\mathbf{u}^N)(\mathbf{v}) - \langle Q^N(\mathbf{u}^N), \mathbf{v}\rangle \tag{2.5.8}$$

and similarly,

$$[\mathbf{A}^N(\mathbf{u}_h^N), \mathbf{v}] = D\mathbf{J}(\mathbf{u}_h^N)(\mathbf{v}) - \langle Q^N(\mathbf{u}_h^N), \mathbf{v}\rangle . \tag{2.5.9}$$

Therefore, replacing (2.5.6) into (3.5.9), and using (2.5.8) and (2.5.9), we conclude (2.5.5). □

2.5.3 Proof of the main Theorem

We prove Theorem 2.5.1 by estimating the terms on the right hand side of (2.5.5). First, from (2.3.4) and (2.4.1) we get the orthogonality relation

$$0 = [\mathbf{B}(\mathbf{v}_h), p_h^N - p^N] + [\mathbf{A}^N(\mathbf{u}_h^N), \mathbf{v}_h] - [\mathbf{A}^N(\mathbf{u}^N), \mathbf{v}_h] ,$$

which yields

$$\begin{aligned} &[\mathbf{A}^N(\mathbf{u}^N), \mathbf{v}] - [\mathbf{A}^N(\mathbf{u}_h^N), \mathbf{v}] + [\mathbf{B}(\mathbf{v}), p^N - p_h^N] \\ &= [\mathbf{A}^N(\mathbf{u}^N), \mathbf{v} - \mathbf{v}_h] - [\mathbf{A}^N(\mathbf{u}_h^N), \mathbf{v} - \mathbf{v}_h] + [\mathbf{B}(\mathbf{v} - \mathbf{v}_h), p^N - p_h^N] \end{aligned} \tag{2.5.10}$$

for all $\mathbf{v}_h \in X_h$. Next, we observe from (2.3.5), (2.3.3), (2.2.5) and (2.2.8), that

$$[\mathbf{A}^N(\mathbf{z}), \mathbf{w}] + [\mathbf{B}(\mathbf{w}), q] = \int_\Omega \boldsymbol{\sigma}(\mathbf{z}, q) : \mathbf{e}(\mathbf{w})\, dx - \langle Q^N(\mathbf{z}), \mathbf{w}\rangle \tag{2.5.11}$$

for all $\mathbf{z}$, $\mathbf{w} \in X$ and for all $q \in M$.

It follows from (2.5.10) and (2.5.11) that

$$\begin{aligned}
&[\mathbf{A}^N(\mathbf{u}^N), \mathbf{v}] - [\mathbf{A}^N(\mathbf{u}_h^N), \mathbf{v}] + [\mathbf{B}(\mathbf{v}), p^N - p_h^N] \\
&= \int_\Omega [\boldsymbol{\sigma}(\mathbf{u}^N, p^N) - \boldsymbol{\sigma}(\mathbf{u}_h^N, p_h^N)] : \mathbf{e}(\mathbf{v} - \mathbf{v}_h)\, dx \\
&\quad - \langle Q^N(\mathbf{u}^N), \mathbf{v} - \mathbf{v}_h \rangle \,+\, \langle Q^N(\mathbf{u}_h^N), \mathbf{v} - \mathbf{v}_h \rangle\,. \qquad (2.5.12)
\end{aligned}$$

Now, integrating by parts on each triangle $T \in \mathcal{T}_h$, and using that $\mathbf{v} = \mathbf{v}_h = 0$ on Γ_0, $\mathbf{div}\, \boldsymbol{\sigma}(\mathbf{u}^N, p^N) = -\mathbf{f}$ in Ω, and $\boldsymbol{\sigma}(\mathbf{u}^N, p^N)\, \boldsymbol{\nu} = Q^N(\mathbf{u}^N)$ on Γ, we get from (2.5.12)

$$\begin{aligned}
&[\mathbf{A}^N(\mathbf{u}^N), \mathbf{v}] - [\mathbf{A}^N(\mathbf{u}_h^N), \mathbf{v}] + [\mathbf{B}(\mathbf{v}), p^N - p_h^N] \\
&= \sum_{T \in \mathcal{T}_h} \int_T [\mathbf{f} + \mathbf{div}\, \boldsymbol{\sigma}(\mathbf{u}_h^N, p_h^N)] \cdot (\mathbf{v} - \mathbf{v}_h)\, dx \\
&\quad + \sum_{S \in \mathcal{E}_h(\Omega)} \int_S \boldsymbol{\sigma}(\mathbf{u}^N, p^N)\, \boldsymbol{\nu}_T \cdot (\mathbf{v} - \mathbf{v}_h)\, ds \\
&\quad - \sum_{S \in \mathcal{E}_h(\Omega)} \int_S \boldsymbol{\sigma}(\mathbf{u}_h^N, p_h^N)\, \boldsymbol{\nu}_T \cdot (\mathbf{v} - \mathbf{v}_h)\, ds \\
&+ \sum_{S \in \mathcal{E}_h(\Gamma)} \int_S [Q^N(\mathbf{u}_h^N) - \boldsymbol{\sigma}(\mathbf{u}_h^N, p_h^N)\, \boldsymbol{\nu}] \cdot (\mathbf{v} - \mathbf{v}_h)\, ds\,.
\end{aligned}$$

But, according to (2.2.10), $\boldsymbol{\sigma}(\mathbf{u}^N, p^N) \in H(\mathbf{div}; \Omega)$ and hence $\boldsymbol{\sigma}(\mathbf{u}^N, p^N)\, \boldsymbol{\nu}_T$ are continuous through the edges $S \in \mathcal{E}_h(\Omega)$, which yields

$$\sum_{S \in \mathcal{E}_h(\Omega)} \int_S \boldsymbol{\sigma}(\mathbf{u}^N, p^N)\, \boldsymbol{\nu}_T \cdot (\mathbf{v} - \mathbf{v}_h)\, ds \,=\, 0\,,$$

whence

$$\begin{aligned}
&[\mathbf{A}^N(\mathbf{u}^N), \mathbf{v}] - [\mathbf{A}^N(\mathbf{u}_h^N), \mathbf{v}] + [\mathbf{B}(\mathbf{v}), p^N - p_h^N] \\
&= \sum_{T \in \mathcal{T}_h} \Bigg\{ \int_T [\mathbf{f} + \mathbf{div}\, \boldsymbol{\sigma}(\mathbf{u}_h^N, p_h^N)] \cdot (\mathbf{v} - \mathbf{v}_h)\, dx \\
&+ \sum_{S \in \mathcal{E}(T) \cap \mathcal{E}_h(\Omega)} \int_S [\![\boldsymbol{\sigma}(\mathbf{u}_h^N, p_h^N)\, \boldsymbol{\nu}_T]\!] \cdot (\mathbf{v} - \mathbf{v}_h)\, ds \\
&+ \sum_{S \in \mathcal{E}(T) \cap \mathcal{E}_h(\Gamma)} \int_S [Q^N(\mathbf{u}_h^N) - \boldsymbol{\sigma}(\mathbf{u}_h^N, p_h^N)\, \boldsymbol{\nu}] \cdot (\mathbf{v} - \mathbf{v}_h)\, ds \Bigg\}. \qquad (2.5.13)
\end{aligned}$$

Then, we take $\mathbf{v}_h = \mathcal{I}_h(\mathbf{v}) \in X_h$ in (2.5.13) and apply Cauchy-Schwarz's inequality and the local properties of the Clément interpolant (cf. Lemma 2.5.1), to deduce that

$$
\begin{aligned}
&| \, [\mathbf{A}^N(\mathbf{u}^N), \mathbf{v}] - [\mathbf{A}^N(\mathbf{u}_h^N), \mathbf{v}] + [\mathbf{B}(\mathbf{v}), p^N - p_h^N] \, | \\
&\le C \sum_{T \in \mathcal{T}_h} \Bigg\{ h_T \, ||\mathbf{f} + \mathbf{div}\, \boldsymbol{\sigma}(\mathbf{u}_h^N, p_h^N)||_{[L^2(T)]^2} \, ||\mathbf{v}||_{[H^1(\Delta(T))]^2} \\
&+ \sum_{S \in \mathcal{E}(T) \cap \mathcal{E}_h(\Omega)} h_S^{1/2} \, || \, [\![\boldsymbol{\sigma}(\mathbf{u}_h^N, p_h^N) \, \boldsymbol{\nu}_T]\!] \, ||_{[L^2(S)]^2} \, ||\mathbf{v}||_{[H^1(\Delta(S))]^2} \\
&+ \sum_{S \in \mathcal{E}(T) \cap \mathcal{E}_h(\Gamma)} h_S^{1/2} \, ||Q^N(\mathbf{u}_h^N) - \boldsymbol{\sigma}(\mathbf{u}_h^N, p_h^N) \, \boldsymbol{\nu}||_{[L^2(S)]^2} \, ||\mathbf{v}||_{[H^1(\Delta(S))]^2} \Bigg\} \\
&\le C \, \tilde{\eta} \left[\sum_{T \in \mathcal{T}_h} \left\{ ||\mathbf{v}||^2_{[H^1(\Delta(T))]^2} + ||\mathbf{v}||^2_{[H^1(\Delta(S))]^2} \right\} \right]^{1/2},
\end{aligned}
$$

where $C > 0$ is independent of h, and

$$
\begin{aligned}
\tilde{\eta}^2 \, = \, &\sum_{T \in \mathcal{T}_h} \Bigg\{ h_T^2 \, ||\mathbf{f} + \mathbf{div}\, \boldsymbol{\sigma}(\mathbf{u}_h^N, p_h^N)||^2_{[L^2(T)]^2} \\
&+ \sum_{S \in \mathcal{E}(T) \cap \mathcal{E}_h(\Omega)} h_S \, || \, [\![\boldsymbol{\sigma}(\mathbf{u}_h^N, p_h^N) \, \boldsymbol{\nu}_T]\!] \, ||^2_{[L^2(S)]^2} \\
&+ \sum_{S \in \mathcal{E}(T) \cap \mathcal{E}_h(\Gamma)} h_S \, ||Q^N(\mathbf{u}_h^N) - \boldsymbol{\sigma}(\mathbf{u}_h^N, p_h^N) \, \boldsymbol{\nu}||^2_{[L^2(S)]^2} \Bigg\} .
\end{aligned}
$$

In this way, since the number of triangles in $\Delta(T)$ and $\Delta(S)$ are bounded, independently of $\mathcal{T}_h$, we obtain

$$
| \, [\mathbf{A}^N(\mathbf{u}^N), \mathbf{v}] - [\mathbf{A}^N(\mathbf{u}_h^N), \mathbf{v}] + [\mathbf{B}(\mathbf{v}), p^N - p_h^N] \, | \; \le \; C \, \tilde{\eta} \tag{2.5.14}
$$

for all $\mathbf{v} \in X$ with $||\mathbf{v}|| \le 1$.

On the other hand, since $[\mathbf{B}(\mathbf{u}^N), q] = 0$ for all $q \in M$ (cf. (2.3.4), we have

$$
[\mathbf{B}(\mathbf{u}^N - \mathbf{u}_h^N), q] = - \, [\mathbf{B}(\mathbf{u}_h^N), q] = \int_\Omega q \, \mathbf{div}\, (\mathbf{u}_h^N) \, dx
$$

and hence

$$
| \, [\mathbf{B}(\mathbf{u}^N - \mathbf{u}_h^N), q] \, | \; \le \; || \, \mathbf{div}\, (\mathbf{u}_h^N) \, ||_{L^2(\Omega)} \, ||q||_{L^2(\Omega)} \; \le \; \left\{ \sum_{T \in \mathcal{T}_h} || \, \mathbf{div}\, (\mathbf{u}_h^N) \, ||^2_{L^2(T)} \right\}^{1/2} \tag{2.5.15}
$$

for all $q \in M$ with $||q|| \leq 1$.

Finally, replacing (2.5.14) and (2.5.15) back into (2.5.5) (cf. Theorem 2.5.2), we obtain (2.5.1) and conclude the proof of Theorem 2.5.1.

Chapter 3

A dual-mixed finite element method for nonlinear incompressible elasticity with mixed boundary conditions

In this Chapter we consider the Hu-Washizu principle and propose a new dual-mixed finite element method for nonlinear incompressible plane elasticity with mixed boundary conditions. The approach extends a related previous work on the Dirichlet problem [48] and imposes the Neumann (essential) boundary condition in a weak sense by means of an additional Lagrange multiplier. The resulting variational formulation becomes a two-fold saddle point operator equation which, for convenience of the subsequent analysis, is shown to be equivalent to a nonlinear three-fold saddle point problem. In this way, a slight generalization of the classical Babuška-Brezzi theory is applied to show the well-posedness of the continuous and discrete formulations, and to derive the corresponding a-priori error estimates. In particular, the classical PEERS space is suitably enriched to define the associated Galerkin scheme. Next, we develop a local problems-based a-posteriori error analysis and derive an implicit reliable and quasi-efficient estimate, and a fully explicit reliable one.

3.1 Introduction

The purpose of the present Chapter is to extend the approach and results from [48] to the case of mixed boundary conditions. We emphasize in advance that no further requirements are needed here for uniqueness. However, though the resulting variational formulation will present again a two-fold saddle point structure (as in [48]), the corresponding analysis will be performed through an equivalent three-fold saddle point operator equation, in which the condition $\int_\Omega \mathrm{tr}(\boldsymbol{\sigma}) = 0$ arises naturally. In order to define the model problem we let Ω be a bounded and simply connected domain in $\mathbb{R}^2$ with polygonal boundary Γ, and such that all its interior angles lie in $(0, 2\pi)$. Also, let Γ_D and Γ_N be disjoint open subsets of Γ, with $|\Gamma_D|, |\Gamma_N| \neq 0$, such that $\Gamma = \bar{\Gamma}_D \cup \bar{\Gamma}_N$. See Figure 3.1 below.

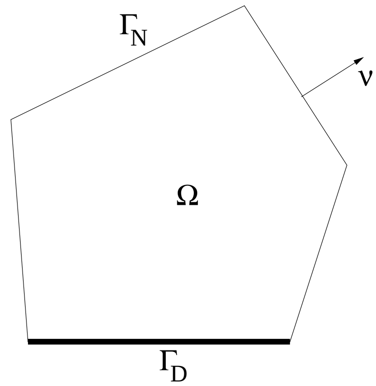

Figure 3.1. Set up of the domain.

Our goal is to determine the displacement $\mathbf{u} := (u_1, u_2)^{\mathtt{t}}$ and the pressure-like unknown p of a incompressible material occupying the region Ω, under the action of some external forces. More precisely, if $\boldsymbol{\sigma}(\mathbf{u}, p)$, $\mathbf{e}(\mathbf{u})$, and $\mathbf{I} \in \mathbb{R}^{2\times 2}$ denote the Cauchy tensor, the strain tensor of small deformations, and the identity matrix, respectively, the constitutive equation is given by:

$$\boldsymbol{\sigma}(\mathbf{u}, p) = \mathcal{N}(\mathbf{e}(\mathbf{u})) + p\,\mathbf{I} \quad \text{in} \quad \Omega\,,$$

where $\mathcal{N} : [L^2(\Omega)]^{2\times 2} \to [L^2(\Omega)]^{2\times 2}$ is a nonlinear operator such that $\mathcal{N}(\mathbf{s}) = \mathcal{N}(\mathbf{s})^{\mathtt{t}}$ for each symmetric tensor $\mathbf{s} \in [L^2(\Omega)]^{2\times 2}$. In addition, we assume that $\mathcal{N}$ induces a strongly monotone and Lipschitz continuous operator from $[L^2(\Omega)]^{2\times 2}$ into its dual, which means that there exist $\alpha_1,\ \alpha_2 > 0$ such that for all $\mathbf{r},\ \mathbf{s} \in [L^2(\Omega)]^{2\times 2}$ there hold

$$\int_\Omega [\mathcal{N}(\mathbf{r}) - \mathcal{N}(\mathbf{s})] : [\mathbf{r} - \mathbf{s}] \ \geq\ \alpha_1\, \|\mathbf{r} - \mathbf{s}\|^2_{[L^2(\Omega)]^{2\times 2}} \tag{3.1.1}$$

and

$$\|\mathcal{N}(\mathbf{r}) - \mathcal{N}(\mathbf{s})\|_{[L^2(\Omega)]^{2\times 2}} \ \leq\ \alpha_2\, \|\mathbf{r} - \mathbf{s}\|_{[L^2(\Omega)]^{2\times 2}}. \tag{3.1.2}$$

Then, given $\mathbf{f} \in [L^2(\Omega)]^2$ and $\mathbf{g} \in [H^{-1/2}(\Gamma_N)]^2$, our nonlinear boundary value problem reads as follows: Find $(\boldsymbol{\sigma}, \mathbf{u}, p)$ in appropriate spaces such that

$$\begin{gathered} \boldsymbol{\sigma} \,=\, \mathcal{N}(\mathbf{e}(\mathbf{u})) \,+\, p\,\mathbf{I} \quad \text{in} \quad \Omega, \quad \mathbf{div}\,\boldsymbol{\sigma} \,=\, -\mathbf{f} \quad \text{in} \quad \Omega, \quad \operatorname{div}\mathbf{u} \,=\, 0 \quad \text{in} \quad \Omega, \\ \mathbf{u} \,=\, \mathbf{0} \quad \text{on} \quad \Gamma_D, \quad \text{and} \quad \boldsymbol{\sigma\nu} \,=\, \mathbf{g} \quad \text{on} \quad \Gamma_N\,, \end{gathered} \tag{3.1.3}$$

where $\boldsymbol{\nu}$ is the unit outward normal to Γ_N and $\mathbf{div}$ denotes the usual divergence operator div acting along each row of the corresponding tensor. We recall here that the Sobolev space $[H^{-1/2}(\Gamma_N)]^2$ is the dual of $[H^{1/2}_{00}(\Gamma_N)]^2 := \{\mathbf{v}|_{\Gamma_N} \ :\ \mathbf{v} \in [H^1(\Omega)]^2, \quad \mathbf{v} = \mathbf{0} \quad \text{on } \Gamma_D\}$, and denote by $\langle \cdot, \cdot \rangle_{\Gamma_N}$ the corresponding duality pairing with respect to the $[L^2(\Gamma_N)]^2$-inner product.

The rest of the Chapter is organized as follows. In Section 3.2 we introduce the dual-mixed variational formulation and establish its equivalence with a three-fold saddle point operator equation. A slight extension of the usual Babuška-Brezzi theory to this kind of problem is provided in Section 3.3. This extended theory is then applied in Section 3.4 to prove the unique solvability of the continuous formulation. In Section 3.5 we define the associated Galerkin scheme and apply again the abstract theory from Section 3.3 to show that, under suitable assumptions on the meshsizes, it becomes well posed. In addition, the corresponding a-priori error estimates are also derived here. Next, Section 3.6 deals with a local problems-based a-posteriori error analysis of our formulation.

We end this section by specifying further notations, some of them already employed in the present introduction. Given any Hilbert space H, we denote by H^2 and $H^{2\times 2}$ the spaces of vectors and tensors of order two, respectively, with entries in H, provided with the product norms induced by the norm of H. In addition, for any $\boldsymbol{\tau} := (\tau_{ij}),\ \boldsymbol{\zeta} := (\zeta_{ij}) \in \mathbb{R}^{2\times 2}$, we denote $\operatorname{tr}(\boldsymbol{\tau}) := \tau_{11} + \tau_{22}$ and $\boldsymbol{\tau} : \boldsymbol{\zeta} := \sum_{i,j=1}^2 \tau_{ij}\zeta_{ij}$. The deviator of tensor $\boldsymbol{\tau}$ is denoted by $\operatorname{dev}(\boldsymbol{\tau}) := \boldsymbol{\tau} - \frac{1}{2}\operatorname{tr}(\boldsymbol{\tau})\mathbf{I}$, and satisfies $\operatorname{tr}(\operatorname{dev}(\boldsymbol{\tau})) =$

0. We also use $^{\mathtt{t}}$ to refer to the transpose of vectors and tensors. On the other hand, we recall that $H(\mathbf{div};\Omega) := \{\boldsymbol{\tau} \in [L^2(\Omega)]^{2\times 2} : \ \mathbf{div}\,\boldsymbol{\tau} \in [L^2(\Omega)]^2\}$ is a Hilbert space with the inner product $\langle \boldsymbol{\zeta}, \boldsymbol{\tau}\rangle_{H(\mathbf{div};\Omega)} := \langle \boldsymbol{\zeta}, \boldsymbol{\tau}\rangle_{[L^2(\Omega)]^{2\times 2}} + \langle \mathbf{div}\,\boldsymbol{\zeta}, \mathbf{div}\,\boldsymbol{\tau}\rangle_{[L^2(\Omega)]^2}$, where $\langle \boldsymbol{\zeta}, \boldsymbol{\tau}\rangle_{[L^2(\Omega)]^{2\times 2}} := \int_\Omega \boldsymbol{\zeta} : \boldsymbol{\tau}$ for all $\boldsymbol{\zeta}, \boldsymbol{\tau} \in [L^2(\Omega)]^{2\times 2}$, and $\langle \mathbf{v}, \mathbf{w}\rangle_{[L^2(\Omega)]^2} := \int_\Omega \mathbf{v}\cdot\mathbf{w}$ for all $\mathbf{v}, \mathbf{w} \in [L^2(\Omega)]^2$. The corresponding induced norms are denoted by $\|\cdot\|_{H(\mathbf{div};\Omega)}$, $\|\cdot\|_{[L^2(\Omega)]^{2\times 2}}$, and $\|\cdot\|_{[L^2(\Omega)]^2}$, respectively.

Finally, throughout this Chapter C, with or without subscripts, bars, tildes or hats, denote positive constants, independent of the parameters and functions involved, which may take different values at different ocurrences.

3.2 The dual-mixed variational formulation

In this section we follow the analysis from [48] and derive a dual-mixed variational formulation for problem (3.1.3). To this end, we introduce the further unknown $\mathbf{t} := \mathbf{e}(\mathbf{u})$ in Ω, whence the nonlinear constitutive equation and the incompressibility condition become, respectively,

$$\boldsymbol{\sigma} \,=\, \mathcal{N}(\mathbf{t}) \,+\, p\,\mathbf{I} \qquad \text{and} \qquad \operatorname{tr}(\mathbf{t}) \,=\, 0 \quad \text{in} \quad \Omega\,. \tag{3.2.1}$$

Then, according to the definition of $\mathbf{e}(\mathbf{u})$ we can write

$$\mathbf{t} \,=\, \nabla\mathbf{u} \,-\, \boldsymbol{\gamma}\,, \tag{3.2.2}$$

where $\boldsymbol{\gamma} := \dfrac{1}{2}\,(\nabla\mathbf{u} - (\nabla\mathbf{u})^{\mathtt{t}})$ represents rotations and lives in the space $\mathcal{R} := \{\boldsymbol{\delta} \in [L^2(\Omega)]^{2\times 2} : \ \boldsymbol{\delta} + \boldsymbol{\delta}^{\mathtt{t}} = \mathbf{0}\}$, which is equipped with the norm $\|\cdot\|_{\mathcal{R}} := \|\cdot\|_{[L^2(\Omega)]^{2\times 2}}$.

Next, we multiply (3.2.2) by a test function $\boldsymbol{\tau} \in H(\mathbf{div};\Omega)$, integrate by parts, and obtain

$$-\int_\Omega \mathbf{t} : \boldsymbol{\tau} - \int_\Omega \mathbf{u}\cdot\mathbf{div}\,\boldsymbol{\tau} - \int_\Omega \boldsymbol{\gamma} : \boldsymbol{\tau} \,-\, \langle \boldsymbol{\tau}\boldsymbol{\nu}, \boldsymbol{\xi}\rangle_{\Gamma_N} \,=0 \qquad \forall\,\boldsymbol{\tau} \in H(\mathbf{div};\Omega)\,, \tag{3.2.3}$$

where $\boldsymbol{\xi} := -\mathbf{u}|_{\Gamma_N} \in [H_{00}^{1/2}(\Gamma_N)]^2$ is introduced here as another auxiliary unknown. Actually, the trace $\boldsymbol{\xi}$ plays the role of the Lagrange multiplier associated to the weak formulation of the Neumann condition, which is given by

$$\langle \boldsymbol{\sigma}\boldsymbol{\nu}, \boldsymbol{\lambda}\rangle_{\Gamma_N} = \langle \mathbf{g}, \boldsymbol{\lambda}\rangle_{\Gamma_N} \qquad \forall\,\boldsymbol{\lambda} \in [H_{00}^{1/2}(\Gamma_N)]^2\,. \tag{3.2.4}$$

Now, the equations in (3.2.1) yield

$$\int_\Omega \mathcal{N}(\mathbf{t}) : \mathbf{s} - \int_\Omega \boldsymbol{\sigma} : \mathbf{s} + \int_\Omega p\,\mathrm{tr}(\mathbf{s}) \;=\; 0 \qquad \forall\, \mathbf{s} \in [L^2(\Omega)]^{2\times 2}\,, \tag{3.2.5}$$

and

$$\int_\Omega q\,\mathrm{tr}(\mathbf{t}) \;=\; 0 \qquad \forall\, q \,\in\, L^2(\Omega)\,, \tag{3.2.6}$$

whereas the equilibrium equation becomes

$$-\int_\Omega \mathbf{v}\cdot\mathbf{div}\,\boldsymbol{\sigma} \;=\; \int_\Omega \mathbf{f}\cdot\mathbf{v} \qquad \forall\, \mathbf{v} \,\in\, [L^2(\Omega)]^2\,. \tag{3.2.7}$$

Finally, the symmetry of $\boldsymbol{\sigma}$ is weakly required through the relation

$$\int_\Omega \boldsymbol{\delta} : \boldsymbol{\sigma} \;=\; 0 \qquad \forall\, \boldsymbol{\delta} \,\in\, \mathcal{R}\,. \tag{3.2.8}$$

Consequently, collecting (3.2.3), (3.2.4), (3.2.5), (3.2.6), (3.2.7) and (3.2.8), we arrive to the following variational formulation of the boundary value problem (3.1.3): Find $(\mathbf{t}, (\boldsymbol{\sigma}, p), (\mathbf{u}, \boldsymbol{\gamma}, \boldsymbol{\xi})) \in X_1 \times \mathcal{M}_1 \times M$ such that

$$\begin{aligned} \int_\Omega \mathcal{N}(\mathbf{t}) : \mathbf{s} - \int_\Omega \boldsymbol{\sigma} : \mathbf{s} + \int_\Omega p\,\mathrm{tr}(\mathbf{s}) \;&=\; 0\,,\\ -\int_\Omega \mathbf{t} : \boldsymbol{\tau} + \int_\Omega q\,\mathrm{tr}(\mathbf{t}) - \int_\Omega \mathbf{u}\cdot\mathbf{div}\,\boldsymbol{\tau} - \int_\Omega \boldsymbol{\gamma} : \boldsymbol{\tau} - \langle \boldsymbol{\tau}\,\boldsymbol{\nu}, \boldsymbol{\xi}\rangle_{\Gamma_N} \;&=\; 0\,,\\ -\int_\Omega \mathbf{v}\cdot\mathbf{div}\,\boldsymbol{\sigma} - \int_\Omega \boldsymbol{\delta} : \boldsymbol{\sigma} - \langle \boldsymbol{\sigma}\,\boldsymbol{\nu}, \boldsymbol{\lambda}\rangle_{\Gamma_N} \;&=\; \mathcal{G}(\mathbf{v}, \boldsymbol{\delta}, \boldsymbol{\lambda})\,, \end{aligned} \tag{3.2.9}$$

for all $(\mathbf{s}, (\boldsymbol{\tau}, q), (\mathbf{v}, \boldsymbol{\delta}, \boldsymbol{\lambda})) \in X_1 \times \mathcal{M}_1 \times M$, where $X_1 := [L^2(\Omega)]^{2\times 2}$, $\mathcal{M}_1 := H(\mathbf{div};\Omega) \times L^2(\Omega)$, $M := [L^2(\Omega)]^2 \times \mathcal{R} \times [H^{1/2}_{00}(\Gamma_N)]^2$, and $\mathcal{G} \in M'$ is defined by

$$\mathcal{G}(\mathbf{v}, \boldsymbol{\delta}, \boldsymbol{\lambda}) \;:=\; \int_\Omega \mathbf{f}\cdot\mathbf{v} \,-\, \langle \mathbf{g}, \boldsymbol{\lambda}\rangle_{\Gamma_N} \qquad \forall\, (\mathbf{v}, \boldsymbol{\delta}, \boldsymbol{\lambda}) \,\in\, M\,.$$

It is not difficult to realize that, similarly to the variational formulation obtained in [48], (3.2.9) has also a two-fold saddle point structure. However, differently from [48], (3.2.9) does not include the condition $\int_\Omega \mathrm{tr}(\boldsymbol{\sigma}) \,=\, 0$, which, besides guaranteeing uniqueness of the Dirichlet problem, is fundamental for the analysis in [48]. Therefore, in order to prove that our present formulation (3.2.9) is well posed, we consider instead of it an equivalent three-fold saddle point operator equation, in which that condition arises naturally. The corresponding equivalence result is established below in Theorem 3.2.1.

Let us first observe that there holds $H(\mathbf{div};\Omega) = \tilde{H}(\mathbf{div};\Omega) + \mathbb{R}\mathbf{I}$, where

$$\tilde{H}(\mathbf{div};\Omega) := \left\{ \boldsymbol{\tau} \in H(\mathbf{div};\Omega) : \quad \int_\Omega \mathrm{tr}(\boldsymbol{\tau}) \ = 0 \right\}.$$

In fact, given $\boldsymbol{\tau} \in H(\mathbf{div};\Omega)$ we have the unique decomposition $\boldsymbol{\tau} = \tilde{\boldsymbol{\tau}} + \tau_0 \mathbf{I}$ with $\tilde{\boldsymbol{\tau}} \in \tilde{H}(\mathbf{div};\Omega)$ and $\tau_0 = \frac{1}{2|\Omega|} \int_\Omega \mathrm{tr}(\boldsymbol{\tau}) \in \mathbb{R}$. Further, it is easy to see that $\|\boldsymbol{\tau}\|^2_{H(\mathbf{div};\Omega)} = \|\tilde{\boldsymbol{\tau}}\|^2_{H(\mathbf{div};\Omega)} + 2\tau_0^2|\Omega|$, and hence $\|\boldsymbol{\tau}\|_{H(\mathbf{div};\Omega)}$ and $\|(\tilde{\boldsymbol{\tau}}, \tau_0)\|_{H(\mathbf{div};\Omega)\times\mathbb{R}} := \|\tilde{\boldsymbol{\tau}}\|_{H(\mathbf{div};\Omega)} + |\tau_0|$ are equivalent.

Analogously, we have $L^2(\Omega) = L^2_0(\Omega) + \mathbb{R}$, where

$$L^2_0(\Omega) := \left\{ q \in L^2(\Omega) : \quad \int_\Omega q \ = 0 \right\} .$$

Indeed, given $q \in L^2(\Omega)$ there holds the unique decomposition $q = \tilde{q} + q_0$ with $\tilde{q} \in L^2_0(\Omega)$ and $q_0 := \frac{1}{|\Omega|} \int_\Omega q \in \mathbb{R}$. Also, it is easy to see that $\|q\|^2_{L^2(\Omega)} = \|\tilde{q}\|^2_{L^2(\Omega)} + q_0^2|\Omega|$, and hence $\|q\|_{L^2(\Omega)}$ and $\|(\tilde{q}, q_0)\|_{L^2(\Omega)\times\mathbb{R}} := \|\tilde{q}\|_{L^2(\Omega)} + |q_0|$ are equivalent.

On the other hand, we notice that taking $\boldsymbol{\tau} = \mathbf{I}$ and $q = 1$ in the second equation of (3.2.9) we obtain

$$\langle \boldsymbol{\nu}, \boldsymbol{\xi} \rangle_{\Gamma_N} \ = \ 0 \,. \tag{3.2.10}$$

Then, writing $\boldsymbol{\sigma} = \tilde{\boldsymbol{\sigma}} + \sigma_0 \mathbf{I}$, with $\tilde{\boldsymbol{\sigma}} \in \tilde{H}(\mathbf{div};\Omega)$ and $\sigma_0 \in \mathbb{R}$, and $p = \tilde{p} + p_0$, with $\tilde{p} \in L^2_0(\Omega)$ and $p_0 \in \mathbb{R}$, and considering the equation (3.2.10), we can introduce the following alternative formulation: Find $(\mathbf{t}, (\tilde{\boldsymbol{\sigma}}, \tilde{p}, c), (\mathbf{u}, \boldsymbol{\gamma}, \boldsymbol{\xi}), \sigma_0) \in X_1 \times M_1 \times M \times \mathbb{R}$ such that

$$\begin{aligned}
\int_\Omega \mathcal{N}(\mathbf{t}) : \mathbf{s} - \int_\Omega \tilde{\boldsymbol{\sigma}} : \mathbf{s} + \int_\Omega \tilde{p}\, \mathrm{tr}(\mathbf{s}) + c \int_\Omega \mathrm{tr}(\mathbf{s}) = 0 \,, \\
- \int_\Omega \mathbf{t} : \tilde{\boldsymbol{\tau}} + \int_\Omega \tilde{q}\, \mathrm{tr}(\mathbf{t}) + d \int_\Omega \mathrm{tr}(\mathbf{t}) - \int_\Omega \mathbf{u} \cdot \mathbf{div}\, \tilde{\boldsymbol{\tau}} - \int_\Omega \boldsymbol{\gamma} : \tilde{\boldsymbol{\tau}} - \langle \tilde{\boldsymbol{\tau}} \boldsymbol{\nu}, \boldsymbol{\xi} \rangle_{\Gamma_N} = 0 \,, \\
- \int_\Omega \mathbf{v} \cdot \mathbf{div}\, \tilde{\boldsymbol{\sigma}} - \int_\Omega \boldsymbol{\delta} : \tilde{\boldsymbol{\sigma}} - \langle \tilde{\boldsymbol{\sigma}}\, \boldsymbol{\nu}, \boldsymbol{\lambda} \rangle_{\Gamma_N} - \sigma_0 \, \langle \boldsymbol{\nu}, \boldsymbol{\lambda} \rangle_{\Gamma_N} = \mathcal{G}(\mathbf{v}, \boldsymbol{\delta}, \boldsymbol{\lambda}) \,, \\
- \tau_0 \, \langle \boldsymbol{\nu}, \boldsymbol{\xi} \rangle_{\Gamma_N} = 0 \,,
\end{aligned} \tag{3.2.11}$$

for all $(\mathbf{s}, (\tilde{\boldsymbol{\tau}}, \tilde{q}, d), (\mathbf{v}, \boldsymbol{\delta}, \boldsymbol{\lambda}), \tau_0) \in X_1 \times M_1 \times M \times \mathbb{R}$, where $M_1 := \tilde{H}(\mathbf{div};\Omega) \times L^2_0(\Omega) \times \mathbb{R}$.

Theorem 3.2.1 *Problems* (3.2.9) *and* (3.2.11) *are equivalent. More precisely:*

i) *If* $(\mathbf{t}, (\boldsymbol{\sigma}, p), (\mathbf{u}, \boldsymbol{\gamma}, \boldsymbol{\xi})) \in X_1 \times \mathcal{M}_1 \times M$ *is a solution of* (3.2.9), *where* $\boldsymbol{\sigma} := \tilde{\boldsymbol{\sigma}} + \sigma_0 \mathbf{I}$, *with* $\tilde{\boldsymbol{\sigma}} \in \tilde{H}(\mathbf{div}; \Omega)$ *and* $\sigma_0 \in \mathrm{I\!R}$, *and* $p = \tilde{p} + p_0$, *with* $\tilde{p} \in L^2_0(\Omega)$ *and* $p_0 \in \mathrm{I\!R}$, *then* $(\mathbf{t}, (\tilde{\boldsymbol{\sigma}}, \tilde{p}, c), (\mathbf{u}, \boldsymbol{\gamma}, \boldsymbol{\xi}), \sigma_0) \in X_1 \times M_1 \times M \times \mathrm{I\!R}$, *where* $c := p_0 - \sigma_0$, *is a solution of* (3.2.11).

ii) *If* $(\mathbf{t}, (\tilde{\boldsymbol{\sigma}}, \tilde{p}, c), (\mathbf{u}, \boldsymbol{\gamma}, \boldsymbol{\xi}), \sigma_0) \in X_1 \times M_1 \times M \times \mathrm{I\!R}$ *is a solution of* (3.2.11), *then* $(\mathbf{t}, (\boldsymbol{\sigma}, p), (\mathbf{u}, \boldsymbol{\gamma}, \boldsymbol{\xi})) \in X_1 \times \mathcal{M}_1 \times M$, *with* $\boldsymbol{\sigma} := \tilde{\boldsymbol{\sigma}} + \sigma_0 \mathbf{I}$ *and* $p := \tilde{p} + (\sigma_0 + c)$, *is a solution of* (3.2.9).

Proof. It follows from a direct application of the decompositions $H(\mathbf{div}; \Omega) = \tilde{H}(\mathbf{div}; \Omega) \oplus \mathrm{I\!R}\mathbf{I}$ and $L^2(\Omega) = L^2_0(\Omega) \oplus \mathrm{I\!R}$. □

We now show that (3.2.11) can be rewritten as a three-fold saddle point operator equation. To this purpose, we let $\mathcal{O}$ denote a generic null operator/functional, and define the operators $\mathcal{A}_1 : X_1 \to X_1'$, $\mathcal{B}_1 : X_1 \to M_1'$, $\mathcal{B} : M_1 \to M'$, and $\mathcal{C} : M \to \mathcal{M}'$, where $\mathcal{M} := \mathrm{I\!R}$, as follows:

$$[\mathcal{A}_1(\mathbf{r}), \mathbf{s}] := \int_\Omega \mathcal{N}(\mathbf{r}) : \mathbf{s} \qquad \forall\, \mathbf{r}, \mathbf{s} \in X_1\,, \tag{3.2.12}$$

$$[\mathcal{B}_1(\mathbf{r}), (\tilde{\boldsymbol{\tau}}, \tilde{q}, d)] := -\int_\Omega \mathbf{r} : \tilde{\boldsymbol{\tau}} + \int_\Omega \tilde{q}\, \mathrm{tr}(\mathbf{r}) + d \int_\Omega \mathrm{tr}(\mathbf{r}) \quad \forall\, \mathbf{r} \in X_1,\ \forall\, (\tilde{\boldsymbol{\tau}}, \tilde{q}, d) \in M_1\,,$$

$$[\mathcal{B}(\tilde{\boldsymbol{\tau}}, \tilde{q}, d), (\mathbf{v}, \boldsymbol{\delta}, \boldsymbol{\lambda})] := -\int_\Omega \mathbf{v} \cdot \mathbf{div}\, \tilde{\boldsymbol{\tau}} - \int_\Omega \boldsymbol{\delta} : \tilde{\boldsymbol{\tau}} - \langle \tilde{\boldsymbol{\tau}} \boldsymbol{\nu}, \boldsymbol{\lambda} \rangle_{\Gamma_N} \quad \forall\, (\tilde{\boldsymbol{\tau}}, \tilde{q}, d) \in M_1,$$

$\forall\, (\mathbf{v}, \boldsymbol{\delta}, \boldsymbol{\lambda}) \in M$, and

$$[\mathcal{C}(\mathbf{v}, \boldsymbol{\delta}, \boldsymbol{\lambda}), \tau_0] := -\tau_0 \langle \boldsymbol{\nu}, \boldsymbol{\lambda} \rangle_{\Gamma_N} \qquad \forall\, (\mathbf{v}, \boldsymbol{\delta}, \boldsymbol{\lambda}) \in M,\ \forall\, \tau_0 \in \mathcal{M}\,.$$

Hereafter, $[\cdot, \cdot]$ stands for the duality pairing induced by the operators involved. It is worth remarking that $\mathcal{A}_1$ is nonlinear and that $\mathcal{B}_1$, $\mathcal{B}$, $\mathcal{C}$, and the corresponding transposes $\mathcal{B}_1' : M_1 \to X_1'$, $\mathcal{B}' : M \to M_1'$ and $\mathcal{C}' : \mathcal{M} \to M'$, are all linear and bounded operators.

According to the above notations, the mixed variational formulation (3.2.11) can be stated as the matrix operator equation: Find $(\mathbf{t}, (\tilde{\boldsymbol{\sigma}}, \tilde{p}, c), (\mathbf{u}, \boldsymbol{\gamma}, \boldsymbol{\xi}), \sigma_0) \in X_1 \times M_1 \times M \times \mathcal{M}$ such that

$$\begin{bmatrix} \mathcal{A}_1 & \mathcal{B}_1' & \mathcal{O} & \mathcal{O} \\ \mathcal{B}_1 & \mathcal{O} & \mathcal{B}' & \mathcal{O} \\ \mathcal{O} & \mathcal{B} & \mathcal{O} & \mathcal{C}' \\ \mathcal{O} & \mathcal{O} & \mathcal{C} & \mathcal{O} \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ (\tilde{\boldsymbol{\sigma}}, \tilde{p}, c) \\ (\mathbf{u}, \boldsymbol{\gamma}, \boldsymbol{\xi}) \\ \sigma_0 \end{bmatrix} = \begin{bmatrix} \mathcal{O} \\ \mathcal{O} \\ \mathcal{G} \\ \mathcal{O} \end{bmatrix}, \tag{3.2.13}$$

which clearly shows a three-fold saddle point structure.

3.3 An extension of the Babuška-Brezzi theory

In order to prove that (3.2.13) is well posed we need a further extension of the classical Babuška-Brezzi theory. To this end, we let X_1, M_1, M, and $\mathcal{M}$ be arbitrary Hilbert spaces, and consider a nonlinear operator $\mathcal{A}_1 : X_1 \to X_1'$, and linear bounded operators $\mathcal{B}_1 : X_1 \to M_1'$, $\mathcal{B} : M_1 \to M'$, and $\mathcal{C} : M \to \mathcal{M}'$, with transposes $\mathcal{B}_1' : M_1 \to X_1'$, $\mathcal{B}' : M \to M_1'$ and $\mathcal{C}' : \mathcal{M} \to M'$, respectively. Then, given $(\mathcal{F}_1, \mathcal{G}_1, \mathcal{G}, \mathbf{G}) \in X_1' \times M_1' \times M' \times \mathcal{M}'$, we are interested in the following nonlinear variational problem: Find $(\vec{\mathbf{t}}, \vec{\boldsymbol{\sigma}}, \vec{\mathbf{u}}, \vec{\eta}) \in X_1 \times M_1 \times M \times \mathcal{M}$ such that

$$\begin{bmatrix} \mathcal{A}_1 & \mathcal{B}_1' & \mathcal{O} & \mathcal{O} \\ \mathcal{B}_1 & \mathcal{O} & \mathcal{B}' & \mathcal{O} \\ \mathcal{O} & \mathcal{B} & \mathcal{O} & \mathcal{C}' \\ \mathcal{O} & \mathcal{O} & \mathcal{C} & \mathcal{O} \end{bmatrix} \begin{bmatrix} \vec{\mathbf{t}} \\ \vec{\boldsymbol{\sigma}} \\ \vec{\mathbf{u}} \\ \vec{\eta} \end{bmatrix} = \begin{bmatrix} \mathcal{F}_1 \\ \mathcal{G}_1 \\ \mathcal{G} \\ \mathbf{G} \end{bmatrix}. \tag{3.3.1}$$

Sufficient conditions for the unique solvability of (3.3.1) are provided in the following theorem.

Theorem 3.3.1 *Let* $\tilde{M} := \{\mathbf{v} \in M : \quad [\mathcal{C}(\mathbf{v}), \mu] = 0 \quad \forall\, \mu \in \mathcal{M}\}$, $\tilde{M}_1 := \{\boldsymbol{\tau} \in M_1 : [\mathcal{B}(\boldsymbol{\tau}), \mathbf{v}] = 0 \quad \forall\, \mathbf{v} \in \tilde{M}\}$, *and* $\tilde{X}_1 := \{\mathbf{s} \in X_1 : \quad [\mathcal{B}_1(\mathbf{s}), \boldsymbol{\tau}] = 0 \quad \forall\, \boldsymbol{\tau} \in \tilde{M}_1\}$. *Also, let* $\tilde{\mathbf{p}} : \tilde{X}_1 \to X_1$ *be the canonical injection with adjoint* $\tilde{\mathbf{p}}' : X_1' \to \tilde{X}_1'$. *Assume that:*

i) $\mathcal{C}$ *satisfies an inf-sup condition on* $M \times \mathcal{M}$, *that is, there exists* $\beta_0 > 0$ *such that*

$$\sup_{\substack{\mathbf{v} \in M \\ \mathbf{v} \neq 0}} \frac{[\mathcal{C}(\mathbf{v}), \mu]}{\|\mathbf{v}\|_M} \geq \beta_0 \|\mu\|_{\mathcal{M}} \quad \forall\, \mu \in \mathcal{M}.$$

ii) $\mathcal{B}$ *satisfies an inf-sup condition on* $M_1 \times \tilde{M}$, *that is, there exists* $\beta > 0$ *such that*

$$\sup_{\substack{\boldsymbol{\tau} \in M_1 \\ \boldsymbol{\tau} \neq 0}} \frac{[\mathcal{B}(\boldsymbol{\tau}), \mathbf{v}]}{\|\boldsymbol{\tau}\|_{M_1}} \geq \beta \|\mathbf{v}\|_M \quad \forall\, \mathbf{v} \in \tilde{M}.$$

iii) $\mathcal{B}_1$ *satisfies an inf-sup condition on* $X_1 \times \tilde{M}_1$, *that is, there exists* $\beta_1 > 0$ *such that*

$$\sup_{\substack{\mathbf{s} \in X_1 \\ \mathbf{s} \neq 0}} \frac{[\mathcal{B}_1(\mathbf{s}), \boldsymbol{\tau}]}{\|\mathbf{s}\|_{X_1}} \geq \beta_1 \|\boldsymbol{\tau}\|_{M_1} \quad \forall\, \boldsymbol{\tau} \in \tilde{M}_1.$$

iv) $\mathcal{A}_1 : X_1 \to X_1'$ *is Lipschitz-continuous with a Lipschitz constant* $\gamma > 0$, *and for any* $\mathbf{s} \in X_1$, *the nonlinear operator* $\tilde{\mathbf{p}}' \mathcal{A}_1(\cdot + \mathbf{s})\, \tilde{\mathbf{p}} : \tilde{X}_1 \to \tilde{X}_1'$ *is strongly monotone with a monotonicity constant* $\alpha > 0$.

Then, there exists a unique $(\vec{\mathbf{t}}, \vec{\boldsymbol{\sigma}}, \vec{\mathbf{u}}, \vec{\eta}) \in X_1 \times M_1 \times M \times \mathcal{M}$ *solution of* (3.3.1). *In addition, there exists* $\tilde{C} > 0$, *depending only on* γ, α, $\|\mathcal{B}_1\|$, $\|\mathcal{B}\|$, β_0, β, *and* β_1, *such that*

$$\|(\vec{\mathbf{t}}, \vec{\boldsymbol{\sigma}}, \vec{\mathbf{u}}, \vec{\eta})\| \;\le\; \tilde{C}\, \left\{ \|\mathcal{F}_1\|_{X_1'} + \|\mathcal{G}_1\|_{M_1'} + \|\mathcal{G}\|_{M'} + \|\mathbf{G}\|_{\mathcal{M}'} + \|\mathcal{A}_1(0)\|_{X_1'} \right\}. \quad (3.3.2)$$

Proof. We proceed similarly to the proof of Theorem 2.4 in [36] and adapt the analysis from [50] to the present situation. First, according to the inf-sup condition for $\mathcal{C}$ (cf. i)) and Lemma 4.1 in Chapter I of [50], we know that $\mathcal{C} : \tilde{M}^{\perp} \to \mathcal{M}'$ and $\mathcal{C}' : \mathcal{M} \to \tilde{M}^{o}$ are isomorphisms with

$$\|\mathcal{C}^{-1}\|, \quad \|(\mathcal{C}')^{-1}\| \le \frac{1}{\beta_0}. \quad (3.3.3)$$

Now, let $\mathbf{u}_0 := \mathcal{C}^{-1}(\mathbf{G}) \in \tilde{M}^{\perp}$. We remark, due to (3.3.3), that

$$\|\mathbf{u}_0\|_M \le \frac{1}{\beta_0} \|\mathbf{G}\|_{\mathcal{M}'}. \quad (3.3.4)$$

With this, we consider the functionals $(\mathcal{F}_1, \tilde{\mathcal{G}}_1, \tilde{\mathcal{G}}) \in X_1' \times M_1' \times \tilde{M}'$, where $\tilde{\mathcal{G}}_1 = \mathcal{G}_1 - \mathcal{B}'(\mathbf{u}_0)$ and $\tilde{\mathcal{G}} := \mathcal{G}|_{\tilde{M}'}$. Then, according to the hypotheses ii), iii), and iv), we can apply Theorem 2.4 in [36] (see also Theorem 1 in [35]), to conclude that there exists a unique $(\tilde{\mathbf{t}}, \tilde{\boldsymbol{\sigma}}, \tilde{\mathbf{u}}) \in X_1 \times M_1 \times \tilde{M}$ such that

$$\begin{bmatrix} \mathcal{A}_1 & \mathcal{B}_1' & \mathcal{O} \\ \mathcal{B}_1 & \mathcal{O} & \mathcal{B}' \\ \mathcal{O} & \mathcal{B} & \mathcal{O} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{t}} \\ \tilde{\boldsymbol{\sigma}} \\ \tilde{\mathbf{u}} \end{bmatrix} = \begin{bmatrix} \mathcal{F}_1 \\ \tilde{\mathcal{G}}_1 \\ \tilde{\mathcal{G}} \end{bmatrix}. \quad (3.3.5)$$

In addition, there exists $C_1 > 0$, depending only on γ, α, β, β_1, and $\|\mathcal{B}_1\|$, such that

$$\|\tilde{\mathbf{t}}\|_{X_1} + \|\tilde{\boldsymbol{\sigma}}\|_{M_1} + \|\tilde{\mathbf{u}}\|_M \le C_1 \left\{ \|\mathcal{F}_1\|_{X_1'} + \|\mathcal{G}_1 - \mathcal{B}'(\mathbf{u}_0)\|_{M_1'} + \|\mathcal{G}\|_{M'} + \|\mathcal{A}_1(0)\|_{X_1'} \right\}. \quad (3.3.6)$$

Since $(\mathcal{G} - \mathcal{B}(\tilde{\boldsymbol{\sigma}})) \in \tilde{M}^{o}$, we can take $\tilde{\eta} := (\mathcal{C}')^{-1}(\mathcal{G} - \mathcal{B}(\tilde{\boldsymbol{\sigma}})) \in \mathcal{M}$. Thus, it is clear that

$$[\mathcal{C}'(\tilde{\eta}), \mathbf{v}] = [\mathcal{G} - \mathcal{B}(\tilde{\boldsymbol{\sigma}}), \mathbf{v}], \quad \forall\, \mathbf{v} \in M \quad (3.3.7)$$

and, due to (3.3.3),

$$\|\tilde{\eta}\|_{\mathcal{M}} \leq \frac{1}{\beta_0}\|\mathcal{G} - \mathcal{B}(\tilde{\boldsymbol{\sigma}})\|_{M'}. \tag{3.3.8}$$

In this way, noting also that $\mathcal{C}(\tilde{\mathbf{u}} + \mathbf{u}_0) = \mathcal{C}(\mathbf{u}_0) = \mathbf{G}$, we conclude that $(\tilde{\mathbf{t}}, \tilde{\boldsymbol{\sigma}}, \tilde{\mathbf{u}} + \mathbf{u}_0, \tilde{\eta}) \in X_1 \times M_1 \times M \times \mathcal{M}$ is a solution of (3.3.1).

For the uniqueness, we now let $(\mathbf{t}, \boldsymbol{\sigma}, \mathbf{u}, \eta) \in X_1 \times M_1 \times M \times \mathcal{M}$ be another solution of (3.3.1). Since $[\mathcal{C}'(\eta), \mathbf{v}] = [\mathcal{C}(\mathbf{v}), \eta] = 0$ for all $\mathbf{v} \in \tilde{M}$, it follows that $(\mathbf{t}, \boldsymbol{\sigma}, \mathbf{u} - \mathbf{u}_0) \in X_1 \times M_1 \times \tilde{M}$ is also a solution of (3.3.5), and hence $(\mathbf{t}, \boldsymbol{\sigma}, \mathbf{u} - \mathbf{u}_0) = (\tilde{\mathbf{t}}, \tilde{\boldsymbol{\sigma}}, \tilde{\mathbf{u}})$. This relation and the fact that $\mathcal{C}'(\eta) = \mathcal{G} - \mathcal{B}(\boldsymbol{\sigma}) = \mathcal{G} - \mathcal{B}(\tilde{\boldsymbol{\sigma}}) = \mathcal{C}'(\tilde{\eta})$ show that $(\tilde{\mathbf{t}}, \tilde{\boldsymbol{\sigma}}, \tilde{\mathbf{u}} + \mathbf{u}_0, \tilde{\eta})$ is the unique solution of (3.3.1).

Finally, the estimate (3.3.2) follows from (3.3.4), (3.3.6) and (3.3.8). □

At this point we remark that hypothesis iv) in Theorem 3.3.1 simply means

$$|\,[\mathcal{A}_1(\mathbf{t}) - \mathcal{A}_1(\mathbf{r}), \mathbf{s}]\,| \;\leq\; \gamma\,\|\mathbf{t} - \mathbf{r}\|\,\|\mathbf{s}\| \quad \forall\,\mathbf{t},\,\mathbf{r},\,\mathbf{s} \in X_1\,, \tag{3.3.9}$$

and

$$[\mathcal{A}_1(\mathbf{t} + \mathbf{s}) - \mathcal{A}_1(\mathbf{r} + \mathbf{s}), \mathbf{t} - \mathbf{r}] \;\geq\; \alpha\,\|\mathbf{t} - \mathbf{r}\|^2 \qquad \forall\,\mathbf{t},\,\mathbf{r} \in \tilde{X}_1\,, \quad \forall\,\mathbf{s} \in X_1\,. \tag{3.3.10}$$

Now, for the finite element approximation of (3.3.1), we let $X_{1,h}$, $M_{1,h}$, M_h, and $\mathcal{M}_h$ be finite dimensional subspaces of X_1, M_1, M, and $\mathcal{M}$, respectively. We assume that the index h is taken in a numerable family $\mathcal{I} := \{h_j\}_{j\in\mathbb{N}}$ such that $h_j \geq h_{j+1}$ for all $j \in \mathbb{N}$. Then, the Galerkin scheme associated with (3.3.1) reads as follows: Find $(\vec{\mathbf{t}}_h, \vec{\boldsymbol{\sigma}}_h, \vec{\mathbf{u}}_h, \vec{\eta}_h) \in X_{1,h} \times M_{1,h} \times M_h \times \mathcal{M}_h$ such that

$$\begin{array}{llllll}
[\mathcal{A}_1(\vec{\mathbf{t}}_h), \mathbf{s}_h] & +\,[\mathcal{B}_1(\mathbf{s}_h), \vec{\boldsymbol{\sigma}}_h] & & & = [\mathcal{F}_1, \mathbf{s}_h] & \forall\,\mathbf{s}_h \in X_{1,h}\,, \\
[\mathcal{B}_1(\vec{\mathbf{t}}_h), \boldsymbol{\tau}_h] & + & [\mathcal{B}(\boldsymbol{\tau}_h), \vec{\mathbf{u}}_h] & & = [\mathcal{G}_1, \boldsymbol{\tau}_h] & \forall\,\boldsymbol{\tau}_h \in M_{1,h}\,, \\
 & [\mathcal{B}(\vec{\boldsymbol{\sigma}}_h), \mathbf{v}_h] & + & [\mathcal{C}(\mathbf{v}_h), \vec{\eta}_h] & = [\mathcal{G}, \mathbf{v}_h] & \forall\,\mathbf{v}_h \in M_h\,, \\
 & & [\mathcal{C}(\vec{\mathbf{u}}_h), \mu_h] & & = [\mathbf{G}, \mu_h] & \forall\,\mu_h \in \mathcal{M}_h\,.
\end{array} \tag{3.3.11}$$

The discrete analogue of Theorem 3.3.1 is established next.

Theorem 3.3.2 *Let* $\tilde{M}_h := \{\mathbf{v}_h \in M_h : \quad [\mathcal{C}(\mathbf{v}_h), \mu_h] = 0 \quad \forall\,\mu_h \in \mathcal{M}_h\}$, $\tilde{M}_{1,h} := \{\boldsymbol{\tau}_h \in M_{1,h} : \; [\mathcal{B}(\boldsymbol{\tau}_h), \mathbf{v}_h] = 0 \quad \forall\,\mathbf{v}_h \in \tilde{M}_h\}$, *and* $\tilde{X}_{1,h} := \{\mathbf{s}_h \in X_{1,h} : \; [\mathcal{B}_1(\mathbf{s}_h), \boldsymbol{\tau}_h] = 0 \quad \forall\,\boldsymbol{\tau}_h \in \tilde{M}_{1,h}\}$. *Also, let* $\mathbf{p}_h : X_{1,h} \to X_1$ *and* $\tilde{\mathbf{p}}_h : \tilde{X}_{1,h} \to X_1$ *be the canonical injections with adjoints* $\mathbf{p}'_h : X'_1 \to X'_{1,h}$ *and* $\tilde{\mathbf{p}}'_h : X'_1 \to \tilde{X}'_{1,h}$, *respectively. Assume that:*

i) *$\mathcal{C}$ satisfies an inf-sup condition on $M_h \times \mathcal{M}_h$, that is, there exists $\bar{\beta}_0 > 0$ such that*

$$\sup_{\substack{\mathbf{v}_h \in M_h \\ \mathbf{v} \neq 0}} \frac{[\mathcal{C}(\mathbf{v}_h), \mu_h]}{\|\mathbf{v}_h\|_M} \geq \bar{\beta}_0 \|\mu_h\|_{\mathcal{M}} \quad \forall\, \mu_h \in \mathcal{M}_h\,.$$

ii) *$\mathcal{B}$ satisfies an inf-sup condition on $M_{1,h} \times \tilde{M}_h$, that is, there exists $\bar{\beta} > 0$ such that*

$$\sup_{\substack{\boldsymbol{\tau}_h \in M_{1,h} \\ \boldsymbol{\tau}_h \neq 0}} \frac{[\mathcal{B}(\boldsymbol{\tau}_h), \mathbf{v}_h]}{\|\boldsymbol{\tau}_h\|_{M_1}} \geq \bar{\beta} \|\mathbf{v}_h\|_M \quad \forall\, \mathbf{v}_h \in \tilde{M}_h\,.$$

iii) *$\mathcal{B}_1$ satisfies an inf-sup condition on $X_{1,h} \times \tilde{M}_{1,h}$, that is, there exists $\bar{\beta}_1 > 0$ such that*

$$\sup_{\substack{\mathbf{s}_h \in X_{1,h} \\ \mathbf{s}_h \neq 0}} \frac{[\mathcal{B}_1(\mathbf{s}_h), \boldsymbol{\tau}_h]}{\|\mathbf{s}_h\|_{X_1}} \geq \bar{\beta}_1 \|\boldsymbol{\tau}_h\|_{M_1} \quad \forall\, \boldsymbol{\tau}_h \in \tilde{M}_{1,h}\,.$$

iv) *$\mathcal{A}_{1,h} := \mathbf{p}_h' \,\mathcal{A}_1\, \mathbf{p}_h : X_{1,h} \to X_{1,h}'$ is Lipschitz-continuous with a Lipschitz constant $\bar{\gamma} > 0$, and for any $\mathbf{s}_h \in X_{1,h}$, the nonlinear operator $\tilde{\mathbf{p}}_h' \,\mathcal{A}_1(\cdot + \mathbf{s}_h)\, \tilde{\mathbf{p}}_h : \tilde{X}_{1,h} \to \tilde{X}_{1,h}'$ is strongly monotone with a monotonicity constant $\bar{\alpha} > 0$.*

Then, there exists a unique $(\vec{\mathbf{t}}_h, \vec{\boldsymbol{\sigma}}_h, \vec{\mathbf{u}}_h, \vec{\eta}_h) \in X_{1,h} \times M_{1,h} \times M_h \times \mathcal{M}_h$ solution of (3.3.11). *In addition, there exists $\bar{C} > 0$, depending only on $\bar{\gamma}$, $\bar{\alpha}$, $\|\mathcal{B}_1\|$, $\|\mathcal{B}\|$, $\bar{\beta}_0$, $\bar{\beta}$, and $\bar{\beta}_1$, such that*

$$\|(\vec{\mathbf{t}}_h, \vec{\boldsymbol{\sigma}}_h, \vec{\mathbf{u}}_h, \vec{\eta}_h)\| \;\leq\; \bar{C} \left\{ \|\mathcal{F}_{1,h}\|_{X_{1,h}'} + \|\mathcal{G}_{1,h}\|_{M_{1,h}'} + \|\mathcal{G}_h\|_{M_h'} + \|\mathbf{G_h}\|_{\mathcal{M}_h'} + \|\mathcal{A}_{1,h}(0)\|_{X_{1,h}'} \right\}, \quad (3.3.12)$$

where $\mathcal{F}_{1,h} := \mathcal{F}_1|_{X_{1,h}}$, $\mathcal{G}_{1,h} := \mathcal{G}_1|_{M_{1,h}}$, $\mathcal{G}_h := \mathcal{G}|_{M_h}$, and $\mathbf{G}_h := \mathbf{G}|_{\mathcal{M}_h}$.

Proof. We now proceed similarly to the proof of Theorem 3.2 in [36] and adapt again the analysis from [50] to the present situation. Indeed, we first let $k_h : M_h \to M$ and $\mathbf{j}_h : \mathcal{M}_h \to \mathcal{M}$ be the canonical injections with adjoints $k_h' : M' \to M_h'$ and $\mathbf{j}_h' : \mathcal{M}' \to \mathcal{M}_h'$, respectively. Then, according to the discrete inf-sup condition for $\mathcal{C}$ (cf. i)) and Lemma 4.1 in Chapter I of [50], we know that $\mathcal{C}_h := \mathbf{j}_h' \,\mathcal{C} k_h : \tilde{M}_h^{\perp} \cap M_h \to \mathcal{M}_h'$ and $\mathcal{C}_h' := k_h' \,\mathcal{C}' \mathbf{j}_h : \mathcal{M}_h \to \tilde{M}_h^{o} \cap M_h'$ are isomorphisms with

$$\|\mathcal{C}_h^{-1}\|, \quad \|(\mathcal{C}_h')^{-1}\| \leq \frac{1}{\bar{\beta}_0}\,.$$

Next, we take $\mathbf{u}_{0,h} := \mathcal{C}_h^{-1}(\mathbf{j}_h'\mathbf{G}) \in \tilde{M}_h^{\perp} \cap M_h$, which satisfies

$$\|\mathbf{u}_{0,h}\|_M \leq \frac{1}{\bar{\beta}_0}\|\mathbf{j}_h'\mathbf{G}\|_{\mathcal{M}_h'},$$

and define the functionals $\tilde{\mathcal{G}}_1 := \mathcal{G}_1 - \mathcal{B}'(\mathbf{u}_{0,h})$ and $\tilde{\mathcal{G}} := \mathcal{G}|_{\tilde{M}_h}$. Then, according to the hypotheses ii), iii), and iv), we can apply Theorem 3.2 in [36] (see also Theorem 3 in [35]), to conclude that there exists a unique $(\tilde{\mathbf{t}}_h, \tilde{\boldsymbol{\sigma}}_h, \tilde{\mathbf{u}}_h) \in X_{1,h} \times M_{1,h} \times \tilde{M}_h$ such that

$$\begin{array}{llcll}
[\mathcal{A}_1(\tilde{\mathbf{t}}_h), \mathbf{s}_h] & +[\mathcal{B}_1(\mathbf{s}_h), \tilde{\boldsymbol{\sigma}}_h] & & = [\mathcal{F}_1, \mathbf{s}_h] & \forall\, \mathbf{s}_h \in X_{1,h}\,, \\
[\mathcal{B}_1(\tilde{\mathbf{t}}_h), \boldsymbol{\tau}_h] & \qquad + & [\mathcal{B}(\boldsymbol{\tau}_h), \tilde{\mathbf{u}}_h] & = [\tilde{\mathcal{G}}_1, \boldsymbol{\tau}_h] & \forall\, \boldsymbol{\tau}_h \in M_{1,h}\,, \\
 & [\mathcal{B}(\tilde{\boldsymbol{\sigma}}_h), \mathbf{v}_h] & & = [\tilde{\mathcal{G}}, \mathbf{v}_h] & \forall\, \mathbf{v}_h \in \tilde{M}_h\,.
\end{array}$$

The rest is similar to the proof of Theorem 3.3.1. □

Similarly as for the continuous case, we notice now that hypothesis iv) in Theorem 3.3.2 means

$$|\,[\mathcal{A}_1(\mathbf{t}_h) - \mathcal{A}_1(\mathbf{r}_h), \mathbf{s}_h]\,| \;\leq\; \bar{\gamma}\,\|\mathbf{t}_h - \mathbf{r}_h\|\,\|\mathbf{s}_h\| \quad \forall\, \mathbf{t}_h,\, \mathbf{r}_h,\, \mathbf{s}_h \,\in X_{1,h}\,, \tag{3.3.13}$$

and

$$[\mathcal{A}_1(\mathbf{t}_h+\mathbf{s}_h)-\mathcal{A}_1(\mathbf{r}_h+\mathbf{s}_h), \mathbf{t}_h-\mathbf{r}_h] \;\geq\; \bar{\alpha}\,\|\mathbf{t}_h-\mathbf{r}_h\|^2 \qquad \forall\, \mathbf{t}_h,\, \mathbf{r}_h \,\in \tilde{X}_{1,h}\,, \quad \forall\, \mathbf{s}_h \in X_{1,h}\,. \tag{3.3.14}$$

The following theorem establishes the corresponding Cea estimate.

Theorem 3.3.3 *Let* $(\vec{\mathbf{t}}, \vec{\boldsymbol{\sigma}}, \vec{\mathbf{u}}, \vec{\eta})$ *and* $(\vec{\mathbf{t}}_h, \vec{\boldsymbol{\sigma}}_h, \vec{\mathbf{u}}_h, \vec{\eta}_h)$ *be the unique solutions of* (3.3.1) *and* (3.3.11)*, respectively. Then, there exists* $\hat{C} > 0$*, independent of* $h \in \mathcal{I}$*, such that*

$$\|(\vec{\mathbf{t}}, \vec{\boldsymbol{\sigma}}, \vec{\mathbf{u}}, \vec{\eta}) - (\vec{\mathbf{t}}_h, \vec{\boldsymbol{\sigma}}_h, \vec{\mathbf{u}}_h, \vec{\eta}_h)\| \;\leq\; \hat{C} \inf_{\substack{(\mathbf{s}_h, \boldsymbol{\tau}_h, \mathbf{v}_h, \mu_h) \in \\ X_{1,h}\times M_{1,h}\times M_h\times \mathcal{M}_h}} \|(\vec{\mathbf{t}}, \vec{\boldsymbol{\sigma}}, \vec{\mathbf{u}}, \vec{\eta}) - (\mathbf{s}_h, \boldsymbol{\tau}_h, \mathbf{v}_h, \mu_h)\|\,. \tag{3.3.15}$$

Proof. It follows from a suitable extension of the analysis in Section 4 of [36]. Alternatively, assuming Gâteaux differentiability of the nonlinear operator $\mathcal{A}_1$, one could apply the approach from [42] (see Theorem 3.3 there). □

3.4 Unique solvability of the continuous formulation

In this section we show that (3.2.13) (equivalently (3.2.11)) satisfies the hypotheses of Theorem 3.3.1, and thus, according to the equivalence result given by Theorem 3.2.1, we conclude the unique solvability of (3.2.9). The continuous inf-sup condition for $\mathcal{C}$ is established first.

Lemma 3.4.1 *There exists $\beta_0 > 0$ such that for all $\tau_0 \in \mathbb{R}$ there holds*

$$\sup_{\substack{\vec{\mathbf{v}}\in M\\ \vec{\mathbf{v}}\neq 0}} \frac{[\mathcal{C}(\vec{\mathbf{v}}), \tau_0]}{\|\vec{\mathbf{v}}\|_M} \geq \beta_0\, |\tau_0|.$$

Proof. Let $\tau_0 \in \mathbb{R}$, $\tau_0 \neq 0$. Then $-\tau_0\,\boldsymbol{\lambda} \in [H^{1/2}_{00}(\Gamma_N)]^2$ for all $\boldsymbol{\lambda} \in [H^{1/2}_{00}(\Gamma_N)]^2$ and we have

$$\sup_{\substack{\vec{\mathbf{v}}\in M\\ \vec{\mathbf{v}}\neq 0}} \frac{[\mathcal{C}(\vec{\mathbf{v}}), \tau_0]}{\|\vec{\mathbf{v}}\|_M} \geq \sup_{\substack{\boldsymbol{\lambda}\in[H^{1/2}_{00}(\Gamma_N)]^2\\ \boldsymbol{\lambda}\neq 0}} \frac{[\mathcal{C}(0,0,-\tau_0\boldsymbol{\lambda}), \tau_0]}{\|-\tau_0\boldsymbol{\lambda}\|_{[H^{1/2}_{00}(\Gamma_N)]^2}}$$

$$= |\tau_0| \sup_{\substack{\boldsymbol{\lambda}\in[H^{1/2}_{00}(\Gamma_N)]^2\\ \boldsymbol{\lambda}\neq 0}} \frac{\langle \boldsymbol{\nu}, \boldsymbol{\lambda}\rangle_{\Gamma_N}}{\|\boldsymbol{\lambda}\|_{[H^{1/2}_{00}(\Gamma_N)]^2}} = \|\boldsymbol{\nu}\|_{[H^{-1/2}(\Gamma_N)]^2}\, |\tau_0|\,,$$

which proves the required estimate with $\beta_0 = \|\boldsymbol{\nu}\|_{[H^{-1/2}(\Gamma_N)]^2}$. □

We now characterize the null space of the operator $\mathcal{C}$, which is needed to prove next the continuous inf-sup condition for $\mathcal{B}$. Indeed, recalling that $M = [L^2(\Omega)]^2 \times \mathcal{R} \times [H^{1/2}_{00}(\Gamma_N)]^2$ and $\mathcal{M} = \mathbb{R}$, we easily deduce that

$$\tilde{M} := \{\vec{\mathbf{v}} := (\mathbf{v}, \boldsymbol{\delta}, \boldsymbol{\lambda}) \in M : [\mathcal{C}(\vec{\mathbf{v}}), \tau_0] = 0\ \forall\, \tau_0 \in \mathcal{M}\} = [L^2(\Omega)]^2 \times \mathcal{R} \times [\tilde{H}^{1/2}_{00}(\Gamma_N)]^2\,,$$

where

$$[\tilde{H}^{1/2}_{00}(\Gamma_N)]^2 := \left\{ \boldsymbol{\lambda} \in [H^{1/2}_{00}(\Gamma_N)]^2 : \quad \langle \boldsymbol{\nu}, \boldsymbol{\lambda}\rangle_{\Gamma_N} = 0 \right\}.$$

Lemma 3.4.2 *There exists $\beta > 0$ such that for all $\vec{\mathbf{v}} \in \tilde{M}$ there holds*

$$\sup_{\substack{\vec{\boldsymbol{\tau}}\in M_1\\ \vec{\boldsymbol{\tau}}\neq 0}} \frac{[\mathcal{B}(\vec{\boldsymbol{\tau}}), \vec{\mathbf{v}}]}{\|\vec{\boldsymbol{\tau}}\|_{M_1}} = \sup_{\substack{\tilde{\boldsymbol{\tau}}\in \tilde{H}(\mathbf{div};\Omega)\\ \tilde{\boldsymbol{\tau}}\neq 0}} \frac{[\mathcal{B}(\tilde{\boldsymbol{\tau}}, 0, 0), \vec{\mathbf{v}}]}{\|\tilde{\boldsymbol{\tau}}\|_{H(\mathbf{div};\Omega)}} \geq \beta\, \|\vec{\mathbf{v}}\|_M\,.$$

Proof. Let $\vec{\mathbf{v}} := (\mathbf{v}, \boldsymbol{\delta}, \boldsymbol{\lambda}) \in \tilde{M}$ and $\boldsymbol{\tau} := \tilde{\boldsymbol{\tau}} + \boldsymbol{\tau}_0\,\mathbf{I} \in H(\mathbf{div};\Omega)$, with $\tilde{\boldsymbol{\tau}} \in \tilde{H}(\mathbf{div};\Omega)$ and $\boldsymbol{\tau}_0 \in \mathrm{I\!R}$. It follows that $[\mathcal{B}(\tilde{\boldsymbol{\tau}},0,0),\vec{\mathbf{v}}] = [\mathcal{B}(\boldsymbol{\tau},0,0),\vec{\mathbf{v}}]$, and since $\|\boldsymbol{\tau}\|_{H(\mathbf{div};\Omega)} \geq \|\tilde{\boldsymbol{\tau}}\|_{H(\mathbf{div};\Omega)}$ we can prove that

$$\sup_{\substack{\tilde{\boldsymbol{\tau}} \in \tilde{H}(\mathbf{div};\Omega)\\ \tilde{\boldsymbol{\tau}} \neq 0}} \frac{[\mathcal{B}(\tilde{\boldsymbol{\tau}},0,0),\vec{\mathbf{v}}]}{\|\tilde{\boldsymbol{\tau}}\|_{H(\mathbf{div};\Omega)}} = \sup_{\substack{\boldsymbol{\tau} \in H(\mathbf{div};\Omega)\\ \boldsymbol{\tau} \neq 0}} \frac{[\mathcal{B}(\boldsymbol{\tau},0,0),\vec{\mathbf{v}}]}{\|\boldsymbol{\tau}\|_{H(\mathbf{div};\Omega)}},$$

which yields

$$\sup_{\substack{\vec{\boldsymbol{\tau}} \in M_1\\ \vec{\boldsymbol{\tau}} \neq 0}} \frac{[\mathcal{B}(\vec{\boldsymbol{\tau}}),\vec{\mathbf{v}}]}{\|\vec{\boldsymbol{\tau}}\|_{M_1}} = \sup_{\substack{\boldsymbol{\tau} \in H(\mathbf{div};\Omega)\\ \boldsymbol{\tau} \neq 0}} \frac{[\mathcal{B}(\boldsymbol{\tau},0,0),\vec{\mathbf{v}}]}{\|\boldsymbol{\tau}\|_{H(\mathbf{div};\Omega)}}$$

$$= \sup_{\substack{\boldsymbol{\tau} \in H(\mathbf{div};\Omega)\\ \boldsymbol{\tau} \neq 0}} \frac{-\displaystyle\int_\Omega \mathbf{v}\cdot\mathbf{div}\,\boldsymbol{\tau} - \int_\Omega \boldsymbol{\delta} : \boldsymbol{\tau} - \langle \boldsymbol{\tau}\boldsymbol{\nu}, \boldsymbol{\lambda}\rangle_{\Gamma_N}}{\|\boldsymbol{\tau}\|_{H(\mathbf{div};\Omega)}}. \tag{3.4.1}$$

Now, let $\boldsymbol{\tau}(\mathbf{v}) := \mathbf{e}(\mathbf{z})$ where $\mathbf{z} \in [H^1(\Omega)]^2$ is the unique weak solution of the boundary value problem: $-\mathbf{div}\,\mathbf{e}(\mathbf{z}) = \mathbf{v}$ in Ω, $\quad \mathbf{z} = \mathbf{0}$ on Γ_D, $\quad \mathbf{e}(\mathbf{z})\boldsymbol{\nu} = \mathbf{0}$ on Γ_N. Then there hold $\mathbf{div}\,\boldsymbol{\tau}(\mathbf{v}) = -\mathbf{v}$ in Ω, $\boldsymbol{\tau}(\mathbf{v}) = \boldsymbol{\tau}(\mathbf{v})^{\mathtt{t}}$ in Ω, and $\boldsymbol{\tau}(\mathbf{v})\boldsymbol{\nu} = \mathbf{0}$ on Γ_N. In addition, Korn's inequality and the corresponding continuous dependence result imply the existence of $\bar{C} > 0$ such that $\|\boldsymbol{\tau}(\mathbf{v})\|_{H(\mathbf{div};\Omega)} \leq \bar{C}\,\|\mathbf{v}\|_{[L^2(\Omega)]^2}$, whence we deduce from (3.4.1):

$$\sup_{\substack{\vec{\boldsymbol{\tau}} \in M_1\\ \vec{\boldsymbol{\tau}} \neq 0}} \frac{[\mathcal{B}(\vec{\boldsymbol{\tau}}),\vec{\mathbf{v}}]}{\|\vec{\boldsymbol{\tau}}\|_{M_1}} \geq \frac{[\mathcal{B}(\boldsymbol{\tau}(\mathbf{v}),0,0),\vec{\mathbf{v}}]}{\|\boldsymbol{\tau}(\mathbf{v})\|_{H(\mathbf{div};\Omega)}} \geq \frac{1}{\bar{C}}\,\|\mathbf{v}\|_{[L^2(\Omega)]^2}. \tag{3.4.2}$$

Next, according to Lemma 4.4 in [49] (see also Lemma 4.2 in [15]), there exists $\boldsymbol{\tau}(\boldsymbol{\delta}) \in H(\mathbf{div};\Omega)$ such that $\mathbf{div}\,\boldsymbol{\tau}(\boldsymbol{\delta}) = \mathbf{0}$ in Ω, $\frac{1}{2}(\boldsymbol{\tau}(\boldsymbol{\delta})^{\mathtt{t}} - \boldsymbol{\tau}(\boldsymbol{\delta})) = \boldsymbol{\delta}$ in Ω, and $\boldsymbol{\tau}(\boldsymbol{\delta})\boldsymbol{\nu} = \mathbf{0}$ on Γ_N. In addition, there exists $\tilde{C} > 0$, independent of $\boldsymbol{\delta}$, such that $\|\boldsymbol{\tau}(\boldsymbol{\delta})\|_{H(\mathbf{div};\Omega)} \leq \tilde{C}\,\|\boldsymbol{\delta}\|_{[L^2(\Omega)]^{2\times 2}}$. Then, noting that $-\boldsymbol{\delta} : \boldsymbol{\tau}(\boldsymbol{\delta}) = \boldsymbol{\delta} : \boldsymbol{\delta}$, we get from (3.4.1):

$$\sup_{\substack{\vec{\boldsymbol{\tau}} \in M_1\\ \vec{\boldsymbol{\tau}} \neq 0}} \frac{[\mathcal{B}(\vec{\boldsymbol{\tau}}),\vec{\mathbf{v}}]}{\|\vec{\boldsymbol{\tau}}\|_{M_1}} \geq \frac{[\mathcal{B}(\boldsymbol{\tau}(\boldsymbol{\delta}),0,0),\vec{\mathbf{v}}]}{\|\boldsymbol{\tau}(\boldsymbol{\delta})\|_{H(\mathbf{div};\Omega)}} \geq \frac{1}{\tilde{C}}\,\|\boldsymbol{\delta}\|_{[L^2(\Omega)]^{2\times 2}}. \tag{3.4.3}$$

On the other hand, given $\boldsymbol{\psi} \in [H^{-1/2}(\Gamma_N)]^2$, we let $\boldsymbol{\tau}(\boldsymbol{\psi}) := \mathbf{e}(z)$ where $\mathbf{z} \in [H^1(\Omega)]^2$ is the unique weak solution of the boundary value problem: $\mathbf{div}\,\mathbf{e}(\mathbf{z}) = \mathbf{0}$ in Ω, $\quad \mathbf{z} = \mathbf{0}$ on Γ_D, $\quad \mathbf{e}(\mathbf{z})\boldsymbol{\nu} = \boldsymbol{\psi}$ on Γ_N. Then there hold $\mathbf{div}\,\boldsymbol{\tau}(\boldsymbol{\psi}) = \mathbf{0}$ in Ω, $\boldsymbol{\tau}(\boldsymbol{\psi}) = \boldsymbol{\tau}(\boldsymbol{\psi})^{\mathtt{t}}$ in Ω, and $\boldsymbol{\tau}(\boldsymbol{\psi})\boldsymbol{\nu} = \boldsymbol{\psi}$ on Γ_N. In addition, similarly as before, there exists $\hat{C} > 0$ such that $\|\boldsymbol{\tau}(\boldsymbol{\psi})\|_{H(\mathbf{div};\Omega)} \leq \hat{C}\,\|\boldsymbol{\psi}\|_{[H^{-1/2}(\Gamma_N)]^2}$, and hence we obtain from (3.4.1)

$$\sup_{\substack{\vec{\boldsymbol{\tau}}\in M_1\\ \vec{\boldsymbol{\tau}}\neq 0}} \frac{[\mathcal{B}(\vec{\boldsymbol{\tau}}),\vec{\mathbf{v}}]}{\|\vec{\boldsymbol{\tau}}\|_{M_1}} \geq \frac{|\,[\mathcal{B}(\boldsymbol{\tau}(\boldsymbol{\psi}),0,0),\vec{\mathbf{v}}]\,|}{\|\boldsymbol{\tau}(\boldsymbol{\psi})\|_{H(\mathbf{div};\Omega)}} \geq \frac{1}{\hat{C}}\,\frac{|\,\langle\boldsymbol{\psi},\boldsymbol{\lambda}\rangle_{\Gamma_N}\,|}{\|\boldsymbol{\psi}\|_{[H^{-1/2}(\Gamma_N)]^2}} \quad \forall\,\boldsymbol{\psi}\in[H^{-1/2}(\Gamma_N)]^2\,,$$

which gives

$$\sup_{\substack{\vec{\boldsymbol{\tau}}\in M_1\\ \vec{\boldsymbol{\tau}}\neq 0}} \frac{[\mathcal{B}(\vec{\boldsymbol{\tau}}),\vec{\mathbf{v}}]}{\|\vec{\boldsymbol{\tau}}\|_{M_1}} \geq \frac{1}{\hat{C}} \sup_{\substack{\boldsymbol{\psi}\in[H^{-1/2}(\Gamma_N)]^2\\ \boldsymbol{\psi}\neq 0}} \frac{|\,\langle\boldsymbol{\psi},\boldsymbol{\lambda}\rangle_{\Gamma_N}\,|}{\|\boldsymbol{\psi}\|_{[H^{-1/2}(\Gamma_N)]^2}} = \frac{1}{\hat{C}}\,\|\boldsymbol{\lambda}\|_{[H_{00}^{1/2}(\Gamma_N)]^2}\,. \tag{3.4.4}$$

Consequently, (3.4.2), (3.4.3), and (3.4.4) provide the required inequality, thus completing the proof. □

In order to prove the continuous inf-sup condition for $\mathcal{B}_1$, we now require to identify the space

$$\tilde{M}_1 := \left\{\vec{\boldsymbol{\tau}} := (\tilde{\boldsymbol{\tau}},\tilde{q},d)\in M_1 : \quad [\mathcal{B}(\vec{\boldsymbol{\tau}}),\vec{\mathbf{v}}] = 0 \quad \forall\,\vec{\mathbf{v}} := (\mathbf{v},\boldsymbol{\delta},\boldsymbol{\lambda})\in\tilde{M}\right\}.$$

In fact, recalling that $M_1 = \tilde{H}(\mathbf{div};\Omega)\times L_0^2(\Omega)\times\mathrm{I\!R}$ and $\tilde{M} = [L^2(\Omega)]^2\times\mathcal{R}\times[\tilde{H}_{00}^{1/2}(\Gamma_N)]^2$, we find that $\tilde{M}_1 = \tilde{M}_1^{\boldsymbol{\sigma}}\times L_0^2(\Omega)\times\mathrm{I\!R}$, where

$$\tilde{M}_1^{\boldsymbol{\sigma}} := \Big\{\tilde{\boldsymbol{\tau}}\in\tilde{H}(\mathbf{div};\Omega):\ \mathbf{div}\,\tilde{\boldsymbol{\tau}} = \mathbf{0} \text{ in } \Omega\,,\ \ \tilde{\boldsymbol{\tau}} = \tilde{\boldsymbol{\tau}}^{\mathrm{t}} \text{ in } \Omega,\ \ \langle\tilde{\boldsymbol{\tau}}\boldsymbol{\nu},\boldsymbol{\lambda}\rangle_{\Gamma_N} = 0 \quad \forall\,\boldsymbol{\lambda}\in[\tilde{H}_{00}^{1/2}(\Gamma_N)]^2\Big\}.$$

At this point we remark that the space $\tilde{M}_1^{\boldsymbol{\sigma}}$ is nontrivial. In fact, let $\mathbf{z}\in[H^1(\Omega)]^2$ be the unique weak solution of the boundary value problem :

$$\mathbf{div}\,\mathbf{e}(\mathbf{z}) = \mathbf{0} \quad \text{in} \quad \Omega\,, \quad \mathbf{z} = \mathbf{1} \quad \text{on} \quad \Gamma_D\,, \quad \mathbf{e}(\mathbf{z})\boldsymbol{\nu} = \mathbf{0} \quad \text{on} \quad \Gamma_N\,.$$

Then, defining $\boldsymbol{\tau} := \mathbf{e}(\mathbf{z})$ and writing $\boldsymbol{\tau} = \tilde{\boldsymbol{\tau}} + c\,\mathbf{I}$ with $\tilde{\boldsymbol{\tau}}\in\tilde{H}(\mathbf{div};\Omega)$ and $c\in\mathrm{I\!R}$, we easily see that $\tilde{\boldsymbol{\tau}}\in\tilde{M}_1^{\boldsymbol{\sigma}}$.

Lemma 3.4.3 *There exists $\beta_1 > 0$ such that for all $\vec{\boldsymbol{\tau}}\in\tilde{M}_1$ there holds*

$$\sup_{\substack{\mathbf{s}\in X_1\\ \mathbf{s}\neq 0}} \frac{[\mathcal{B}_1(\mathbf{s}),\vec{\boldsymbol{\tau}}]}{\|\mathbf{s}\|_{X_1}} \geq \beta_1\,\|\vec{\boldsymbol{\tau}}\|_{M_1}\,.$$

Proof. Let $\vec{\boldsymbol{\tau}} := (\tilde{\boldsymbol{\tau}}, \tilde{q}, d) \in \tilde{M}_1$ such that $\|\tilde{\boldsymbol{\tau}}\|_{H(\mathbf{div};\Omega)} \geq \|\tilde{q}\|_{L^2(\Omega)}$. Since $\mathrm{tr}(\mathrm{dev}(\tilde{\boldsymbol{\tau}})) = 0$, we easily obtain that $[\mathcal{B}_1(-\mathrm{dev}(\tilde{\boldsymbol{\tau}})), \vec{\boldsymbol{\tau}}] = \|\mathrm{dev}(\tilde{\boldsymbol{\tau}})\|^2_{[L^2(\Omega)]^{2\times 2}}$. In addition, recalling that $\int_\Omega \mathrm{tr}(\tilde{\boldsymbol{\tau}}) = 0$, and noting from the definition of $\tilde{M}_1^{\boldsymbol{\sigma}}$ that $\mathbf{div}\, \tilde{\boldsymbol{\tau}} = \mathbf{0}$ in Ω, we deduce from Lemma 3.1 in [10] (see also Proposition 3.1 in Chapter IV of [20]) that there exists $C_1 > 0$, depending only on Ω, such that $\|\mathrm{dev}(\tilde{\boldsymbol{\tau}})\|_{[L^2(\Omega)]^{2\times 2}} \geq C_1 \|\tilde{\boldsymbol{\tau}}\|_{H(\mathbf{div};\Omega)}$. Hence, assuming that $\mathrm{dev}(\tilde{\boldsymbol{\tau}}) \neq \mathbf{0}$, it follows that

$$\begin{aligned} \sup_{\substack{\mathbf{s}\in X_1\\ \mathbf{s}\neq 0}} \frac{[\mathcal{B}_1(\mathbf{s}), \vec{\boldsymbol{\tau}}]}{\|\mathbf{s}\|_{X_1}} &\geq \frac{[\mathcal{B}_1(-\mathrm{dev}(\tilde{\boldsymbol{\tau}})), \vec{\boldsymbol{\tau}}]}{\|\mathrm{dev}(\tilde{\boldsymbol{\tau}})\|_{X_1}} = \|\mathrm{dev}(\tilde{\boldsymbol{\tau}})\|_{[L^2(\Omega)]^{2\times 2}} \geq C_1 \|\tilde{\boldsymbol{\tau}}\|_{H(\mathbf{div};\Omega)} \\ &\geq \frac{C_1}{2} \|(\tilde{\boldsymbol{\tau}}, \tilde{q})\|_{H(\mathbf{div};\Omega)\times L^2(\Omega)}. \end{aligned} \tag{3.4.5}$$

If $\mathrm{dev}(\tilde{\boldsymbol{\tau}}) := (\tilde{\boldsymbol{\tau}} - \frac{1}{2}\mathrm{tr}(\tilde{\boldsymbol{\tau}})\,\mathbf{I}) = \mathbf{0}$ then necessarily $\tilde{\boldsymbol{\tau}} = \mathbf{0}$ and the estimate (3.4.5) is trivially satisfied.

We now take $\vec{\boldsymbol{\tau}} := (\tilde{\boldsymbol{\tau}}, \tilde{q}, d) \in \tilde{M}_1$ such that $\|\tilde{\boldsymbol{\tau}}\|_{H(\mathbf{div};\Omega)} \leq \|\tilde{q}\|_{L^2(\Omega)}$. Since $\int_\Omega \tilde{q} = \int_\Omega \mathrm{tr}(\tilde{\boldsymbol{\tau}}) = 0$, we obtain $[\mathcal{B}_1(\tilde{q}\,\mathbf{I} + \tilde{\boldsymbol{\tau}}), \vec{\boldsymbol{\tau}}] = 2\|\tilde{q}\|^2_{L^2(\Omega)} - \|\tilde{\boldsymbol{\tau}}\|^2_{H(\mathbf{div};\Omega)} \geq \|\tilde{q}\|^2_{L^2(\Omega)}$, and hence, assuming $\tilde{q}\,\mathbf{I} + \tilde{\boldsymbol{\tau}} \neq \mathbf{0}$,

$$\sup_{\substack{\mathbf{s}\in X_1\\ \mathbf{s}\neq 0}} \frac{[\mathcal{B}_1(\mathbf{s}), \vec{\boldsymbol{\tau}}]}{\|\mathbf{s}\|_{X_1}} \geq \frac{[\mathcal{B}_1(\tilde{q}\,\mathbf{I} + \tilde{\boldsymbol{\tau}}), \vec{\boldsymbol{\tau}}]}{\|\tilde{q}\,\mathbf{I} + \tilde{\boldsymbol{\tau}}\|_{X_1}} \geq C_2 \|\tilde{q}\|_{L^2(\Omega)} \geq \frac{C_2}{2} \|(\tilde{\boldsymbol{\tau}}, \tilde{q})\|_{H(\mathbf{div};\Omega)\times L^2(\Omega)}. \tag{3.4.6}$$

If $\tilde{q}\,\mathbf{I} + \tilde{\boldsymbol{\tau}} = \mathbf{0}$ then necessarily $\tilde{q} = 0$ and the estimate (3.4.6) is obviously satisfied.

Finally, using again that $\int_\Omega \tilde{q} = \int_\Omega \mathrm{tr}(\tilde{\boldsymbol{\tau}}) = 0$, we find that $[\mathcal{B}_1(d\,\mathbf{I}), \vec{\boldsymbol{\tau}}] = 2\,|\Omega|\,d^2$ $\forall\, d \in \mathbb{R}$, and hence for each $d \neq 0$ there holds

$$\sup_{\substack{\mathbf{s}\in X_1\\ \mathbf{s}\neq 0}} \frac{[\mathcal{B}_1(\mathbf{s}), \vec{\boldsymbol{\tau}}]}{\|\mathbf{s}\|_{X_1}} \geq \frac{[\mathcal{B}_1(d\,\mathbf{I}), \vec{\boldsymbol{\tau}}]}{\|d\,\mathbf{I}\|_{X_1}} = (2\,|\Omega|)^{1/2}\,|d|\,. \tag{3.4.7}$$

In this way, (3.4.5), (3.4.6), and (3.4.7) imply the inf-sup condition for $\mathcal{B}_1$. □

The hypotheses of Theorem 3.3.1 are completed with the following result.

Lemma 3.4.4 *The nonlinear operator $\mathcal{A}_1 : X_1 \to X_1'$ is strongly monotone and Lipschitz continuous, that is, there exist α, $\gamma > 0$ such that*

$$[\mathcal{A}_1(\mathbf{t}) - \mathcal{A}_1(\mathbf{r}), \mathbf{t} - \mathbf{r}] \geq \alpha \|\mathbf{t} - \mathbf{r}\|^2_{X_1},$$

and

$$|\,[\mathcal{A}_1(\mathbf{t}) - \mathcal{A}_1(\mathbf{r}), \mathbf{s}]\,| \leq \gamma \|\mathbf{t} - \mathbf{r}\|_{X_1} \|\mathbf{s}\|_{X_1}$$

for all $\mathbf{t}$, $\mathbf{r}$, $\mathbf{s}$ $\in X_1 := [L^2(\Omega)]^{2\times 2}$.

Proof. It is a straigthforward consequence of assumptions (3.1.1) and (3.1.2). □

The unique solvability of (3.2.9) (equivalently (3.2.11) or (3.2.13)) is established now.

Theorem 3.4.1 *There exists a unique* $(\mathbf{t}, (\boldsymbol{\sigma}, p), (\mathbf{u}, \boldsymbol{\gamma}, \boldsymbol{\xi})) \in X_1 \times \mathcal{M}_1 \times M$ *solution of* (3.2.9)*. Moreover, there exists* $C > 0$ *such that*

$$\|(\mathbf{t}, (\boldsymbol{\sigma}, p), (\mathbf{u}, \boldsymbol{\gamma}, \boldsymbol{\xi}))\| \,\leq\, C\, \left\{ \|\mathcal{G}\|_{M'} + \|\mathcal{N}(0)\|_{[L^2(\Omega)]^{2\times 2}} \right\}, \tag{3.4.8}$$

where $\|\mathcal{G}\|_{M'} \,\leq\, \|\mathbf{f}\|_{[L^2(\Omega)]^2} + \|\mathbf{g}\|_{[H^{-1/2}(\Gamma_N)]^2}$.

Proof. It follows from Lemmata 3.4.1, 3.4.2, 3.4.3, and 3.4.4, a direct application of the abstract result given by Theorem 3.3.1, and the equivalence provided by Theorem 3.2.1. □

3.5 The dual-mixed finite element scheme

We now let $\{\mathcal{T}_h\}_{h>0}$ be a regular family of triangulations of $\bar{\Omega}$, made up of triangles T of diameter h_T, such that $h := \max\{h_T : \quad T \in \mathcal{T}_h\}$, and assume that all the points in $\bar{\Gamma}_D \cap \bar{\Gamma}_N$ become vertices of $\mathcal{T}_h$ for all $h > 0$. Also, for reasons that will become clear below (cf. Lemmata 3.5.1 and 3.5.3), we introduce an independent partition $\left\{\tilde{\Gamma}_1, \tilde{\Gamma}_2, \cdots, \tilde{\Gamma}_m\right\}$ of Γ_N and denote $\tilde{h} := \max\{|\tilde{\Gamma}_j| : \quad j \in \{1, \cdots, m\}\}$. Next, we let $X_{1,h}$, $\mathcal{M}^{\boldsymbol{\sigma}}_{1,h}$, $\mathcal{M}^{p}_{1,h}$, $M^{\mathbf{u}}_h$, $M^{\boldsymbol{\gamma}}_h$, and $M^{\boldsymbol{\xi}}_{\tilde{h}}$ be finite element subspaces for the unknowns $\mathbf{t}$, $\boldsymbol{\sigma}$, p, $\mathbf{u}$, $\boldsymbol{\gamma}$, and $\boldsymbol{\xi}$, respectively, and define the product spaces $\mathcal{M}_{1,h} := \mathcal{M}^{\boldsymbol{\sigma}}_{1,h} \times \mathcal{M}^{p}_{1,h}$ and $M_{h,\tilde{h}} := M^{\mathbf{u}}_h \times M^{\boldsymbol{\gamma}}_h \times M^{\boldsymbol{\xi}}_{\tilde{h}}$. Then, the Galerkin scheme associated with (3.2.9) reads as follows : Find $(\mathbf{t}_h, (\boldsymbol{\sigma}_h, p_h), (\mathbf{u}_h, \boldsymbol{\gamma}_h, \boldsymbol{\xi}_{\tilde{h}})) \in X_{1,h} \times \mathcal{M}_{1,h} \times M_{h,\tilde{h}}$ such that

$$\begin{aligned}
\int_\Omega \mathcal{N}(\mathbf{t}_h) : \mathbf{s} - \int_\Omega \boldsymbol{\sigma}_h : \mathbf{s} + \int_\Omega p_h \operatorname{tr}(\mathbf{s}) \;&=\; 0\,, \\
-\int_\Omega \mathbf{t}_h : \boldsymbol{\tau} + \int_\Omega q_h \operatorname{tr}(\mathbf{t}_h) - \int_\Omega \mathbf{u}_h \cdot \mathbf{div}\, \boldsymbol{\tau} - \int_\Omega \boldsymbol{\gamma}_h : \boldsymbol{\tau} - \langle \boldsymbol{\tau}\, \boldsymbol{\nu}, \boldsymbol{\xi}_{\tilde{h}} \rangle_{\Gamma_N} \;&=\; 0\,, \\
-\int_\Omega \mathbf{v} \cdot \mathbf{div}\, \boldsymbol{\sigma}_h - \int_\Omega \boldsymbol{\delta} : \boldsymbol{\sigma}_h - \langle \boldsymbol{\sigma}_h\, \boldsymbol{\nu}, \boldsymbol{\lambda} \rangle_{\Gamma_N} \;&=\; \mathcal{G}(\mathbf{v}, \boldsymbol{\delta}, \boldsymbol{\lambda})
\end{aligned} \tag{3.5.1}$$

for all $(\mathbf{s}, (\boldsymbol{\tau}, q), (\mathbf{v}, \boldsymbol{\delta}, \boldsymbol{\lambda})) \in X_{1,h} \times \mathcal{M}_{1,h} \times M_{h,\tilde{h}}$.

Similarly to the continuous case, we now introduce an alternative formulation. To this end, we first let $M_{1,h}^{\tilde{\boldsymbol{\sigma}}} := \left\{ \boldsymbol{\tau} \in \mathcal{M}_{1,h}^{\boldsymbol{\sigma}} : \quad \int_\Omega \operatorname{tr}(\boldsymbol{\tau}) \,=\, 0 \right\}$, $M_{1,h}^{\tilde{p}} := \left\{ q \in \mathcal{M}_{1,h}^{p} : \int_\Omega q \,=\, 0 \right\}$, and define the product space $M_{1,h} := M_{1,h}^{\tilde{\boldsymbol{\sigma}}} \times M_{1,h}^{\tilde{p}} \times \mathrm{I\!R}$. Then, the discrete analogue of (3.2.11) becomes: Find $(\mathbf{t}_h, (\tilde{\boldsymbol{\sigma}}_h, \tilde{p}_h, c_h), (\mathbf{u}_h, \boldsymbol{\gamma}_h, \boldsymbol{\xi}_{\tilde{h}}), \sigma_{0,h}) \in X_{1,h} \times M_{1,h} \times M_{h,\tilde{h}} \times \mathrm{I\!R}$ such that

$$
\begin{aligned}
\int_\Omega \mathcal{N}(\mathbf{t}_h) : \mathbf{s} - \int_\Omega \tilde{\boldsymbol{\sigma}}_h : \mathbf{s} + \int_\Omega \tilde{p}_h \operatorname{tr}(\mathbf{s}) + c_h \int_\Omega \operatorname{tr}(\mathbf{s}) \quad &= \quad 0\,, \\
- \int_\Omega \mathbf{t}_h : \tilde{\boldsymbol{\tau}} + \int_\Omega \tilde{q} \operatorname{tr}(\mathbf{t}_h) + d \int_\Omega \operatorname{tr}(\mathbf{t}_h) - \int_\Omega \mathbf{u}_h \cdot \mathbf{div}\, \tilde{\boldsymbol{\tau}} - \int_\Omega \boldsymbol{\gamma}_h : \tilde{\boldsymbol{\tau}} & \\
- \langle \tilde{\boldsymbol{\tau}} \boldsymbol{\nu}, \boldsymbol{\xi}_{\tilde{h}} \rangle_{\Gamma_N} \quad &= \quad 0\,, \\
- \int_\Omega \mathbf{v} \cdot \mathbf{div}\, \tilde{\boldsymbol{\sigma}}_h - \int_\Omega \boldsymbol{\delta} : \tilde{\boldsymbol{\sigma}}_h - \langle \tilde{\boldsymbol{\sigma}}_h \boldsymbol{\nu}, \boldsymbol{\lambda} \rangle_{\Gamma_N} - \sigma_{0,h} \langle \boldsymbol{\nu}, \boldsymbol{\lambda} \rangle_{\Gamma_N} \quad &= \quad \mathcal{G}(\mathbf{v}, \boldsymbol{\delta}, \boldsymbol{\lambda})\,, \\
- \tau_0 \langle \boldsymbol{\nu}, \boldsymbol{\xi}_{\tilde{h}} \rangle_{\Gamma_N} \quad &= \quad 0\,,
\end{aligned}
\tag{3.5.2}
$$

for all $(\mathbf{s}, (\tilde{\boldsymbol{\tau}}, \tilde{q}, d), (\mathbf{v}, \boldsymbol{\delta}, \boldsymbol{\lambda}), \tau_0) \in X_{1,h} \times M_{1,h} \times M_{h,\tilde{h}} \times \mathrm{I\!R}$.

Theorem 3.5.1 *Assume that* $\mathrm{I\!R}\,\mathbf{I} \subseteq \mathcal{M}_{1,h}^{\boldsymbol{\sigma}}$ *and* $\mathrm{I\!R} \subseteq \mathcal{M}_{1,h}^{p}$. *Then problems* (3.5.1) *and* (3.5.2) *are equivalent. More precisely :*

i) *If* $(\mathbf{t}_h, (\boldsymbol{\sigma}_h, p_h), (\mathbf{u}_h, \boldsymbol{\gamma}_h, \boldsymbol{\xi}_{\tilde{h}})) \in X_{1,h} \times \mathcal{M}_{1,h} \times M_{h,\tilde{h}}$ *is a solution of* (3.5.1), *where* $\boldsymbol{\sigma}_h := \tilde{\boldsymbol{\sigma}}_h + \sigma_{0,h}\mathbf{I}$, *with* $\tilde{\boldsymbol{\sigma}}_h \in M_{1,h}^{\tilde{\boldsymbol{\sigma}}}$ *and* $\sigma_{0,h} \in \mathrm{I\!R}$, *and* $p_h = \tilde{p}_h + p_{0,h}$, *with* $\tilde{p}_h \in M_{1,h}^{\tilde{p}}$ *and* $p_{0,h} \in \mathrm{I\!R}$, *then* $(\mathbf{t}_h, (\tilde{\boldsymbol{\sigma}}_h, \tilde{p}_h, c_h), (\mathbf{u}_h, \boldsymbol{\gamma}_h, \boldsymbol{\xi}_{\tilde{h}}), \sigma_{0,h}) \in X_{1,h} \times M_{1,h} \times M_{h,\tilde{h}} \times \mathrm{I\!R}$, *where* $c_h := p_{0,h} - \sigma_{0,h}$, *is a solution of* (3.5.2).

ii) *If* $(\mathbf{t}_h, (\tilde{\boldsymbol{\sigma}}_h, \tilde{p}_h, c_h), (\mathbf{u}_h, \boldsymbol{\gamma}_h, \boldsymbol{\xi}_{\tilde{h}}), \sigma_{0,h}) \in X_{1,h} \times M_{1,h} \times M_{h,\tilde{h}} \times \mathrm{I\!R}$ *is a solution of* (3.5.2), *then* $(\mathbf{t}_h, (\boldsymbol{\sigma}_h, p_h), (\mathbf{u}_h, \boldsymbol{\gamma}_h, \boldsymbol{\xi}_{\tilde{h}})) \in X_{1,h} \times \mathcal{M}_{1,h} \times M_{h,\tilde{h}}$, *with* $\boldsymbol{\sigma}_h := \tilde{\boldsymbol{\sigma}}_h + \sigma_{0,h}\mathbf{I}$ *and* $p_h := \tilde{p}_h + (\sigma_{0,h} + c_h)$, *is a solution of* (3.5.1).

Proof. It follows similarly to the proof of Theorem 3.2.1 since it is based on the decompositions $\mathcal{M}_{1,h}^{\boldsymbol{\sigma}} := M_{1,h}^{\tilde{\boldsymbol{\sigma}}} \oplus \mathrm{I\!R}\,\mathbf{I}$ and $\mathcal{M}_{1,h}^{p} := M_{1,h}^{\tilde{p}} \oplus \mathrm{I\!R}$. □

Throughout the rest of this section we show that (3.5.2) satisfies the hypotheses of Theorem 3.3.2, which certainly depends on the specific finite element subspaces to be utilized. Actually, the explicit definition of these subspaces will be derived from the need of verifying the discrete inf-sup conditions. To this end, our analysis follows some of the arguments employed in Section 3.4 and extend them, as much as possible, to the present discrete case.

We begin with the discrete inf-sup condition for the bilinear form $\mathcal{C}$. Indeed, proceeding similarly to the proof of Lemma 3.4.1, we observe that

$$\sup_{\substack{\vec{\mathbf{v}}\in M_{h,\tilde{h}}\\ \vec{\mathbf{v}}\neq 0}} \frac{[\mathcal{C}(\vec{\mathbf{v}}),\tau_0]}{\|\vec{\mathbf{v}}\|_M} \;\geq\; \sup_{\substack{\boldsymbol{\lambda}_{\tilde{h}}\in M_{\tilde{h}}^{\boldsymbol{\xi}}\\ \boldsymbol{\lambda}_{\tilde{h}}\neq 0}} \frac{[\mathcal{C}(0,0,-\tau_0\boldsymbol{\lambda}_{\tilde{h}}),\tau_0]}{\|-\tau_0\boldsymbol{\lambda}_{\tilde{h}}\|_{[H_{00}^{1/2}(\Gamma_N)]^2}} \;=\; |\tau_0| \sup_{\substack{\boldsymbol{\lambda}_{\tilde{h}}\in M_{\tilde{h}}^{\boldsymbol{\xi}}\\ \boldsymbol{\lambda}_{\tilde{h}}\neq 0}} \frac{\langle \boldsymbol{\nu},\boldsymbol{\lambda}_{\tilde{h}}\rangle_{\Gamma_N}}{\|\boldsymbol{\lambda}_{\tilde{h}}\|_{[H_{00}^{1/2}(\Gamma_N)]^2}}, \tag{3.5.3}$$

which, however, does not become $|\tau_0|\;\|\boldsymbol{\nu}\|_{[H^{-1/2}(\Gamma_N)]^2}$ since the supremum is taken only on the subspace $M_{\tilde{h}}^{\boldsymbol{\xi}}$ of $[H_{00}^{1/2}(\Gamma_N)]^2$. Nevertheless, we prove next that for sufficiently small meshsize $\tilde{h}$, the above expression is bounded below by

$$C\,|\tau_0|\,\|\boldsymbol{\nu}\|_{[H^{-1/2}(\Gamma_N)]^2}.$$

For this purpose, we now assume that $M_{\tilde{h}}^{\boldsymbol{\xi}}$ satisfies the following approximation property:

$(AP_{\tilde{h}}^{\boldsymbol{\xi}})$ for each $t\in(-\frac{1}{2},\frac{3}{2}]$ and for each $\boldsymbol{\lambda}\in[H^t(\Gamma_N)]^2\cap[H_{00}^{1/2}(\Gamma_N)]^2$ there exists $\boldsymbol{\lambda}_{\tilde{h}}\in M_{\tilde{h}}^{\boldsymbol{\xi}}$ such that

$$\|\boldsymbol{\lambda}-\boldsymbol{\lambda}_{\tilde{h}}\|_{[H_{00}^{1/2}(\Gamma_N)]^2} \;\leq\; C\,\tilde{h}^{t-1/2}\;\|\boldsymbol{\lambda}\|_{[H^t(\Gamma_N)]^2}\,.$$

Lemma 3.5.1 *There exist $\bar{C}_0$, $h_0>0$ such that*

$$\sup_{\substack{\boldsymbol{\lambda}_{\tilde{h}}\in M_{\tilde{h}}^{\boldsymbol{\xi}}\\ \boldsymbol{\lambda}_{\tilde{h}}\neq 0}} \frac{\langle \boldsymbol{\nu},\boldsymbol{\lambda}_{\tilde{h}}\rangle_{\Gamma_N}}{\|\boldsymbol{\lambda}_{\tilde{h}}\|_{[H_{00}^{1/2}(\Gamma_N)]^2}} \;\geq\; \bar{\beta}_0 \;:=\; \bar{C}_0\,\|\boldsymbol{\nu}\|_{[H^{-1/2}(\Gamma_N)]^2} \qquad \forall\,\tilde{h}\,\leq\,h_0\,, \tag{3.5.4}$$

and hence

$$\sup_{\substack{\vec{\mathbf{v}}\in M_{h,\tilde{h}}\\ \vec{\mathbf{v}}\neq 0}} \frac{[\mathcal{C}(\vec{\mathbf{v}}),\tau_0]}{\|\vec{\mathbf{v}}\|_M} \;\geq\; \bar{\beta}_0\,|\tau_0| \qquad \forall\,\tau_0\in\mathbb{R}\,,\quad \forall\,\tilde{h}\,\leq\,h_0\,. \tag{3.5.5}$$

Proof. We follow the analysis of Lemma 3.1 in [47] (see also Lemma 3.3 in [12]). In fact, we first let $\mathbf{z}\in[H^1(\Omega)]^2$ be the unique weak solution of the boundary value problem:

$$-\,\Delta\mathbf{z}\,+\,\mathbf{z}\,=\,\mathbf{0}\quad\text{in}\quad\Omega\,,\qquad \mathbf{z}=\mathbf{0}\quad\text{on}\quad\Gamma_D,\qquad \frac{\partial\mathbf{z}}{\partial\boldsymbol{\nu}}\,=\,\boldsymbol{\nu}\quad\text{on}\quad\Gamma_N\,.$$

Since Γ is polygonal, $\boldsymbol{\nu}$ becomes piecewise constant, whence $\boldsymbol{\nu} \in [L^2(\Gamma_N)]^2$. It follows, according to the classical regularity results in [53] (see also [54]), that $\mathbf{z} \in [H^{1+\delta}(\Omega)]^2$ and $\|\mathbf{z}\|_{[H^{1+\delta}(\Omega)]^2} \le C\,\|\boldsymbol{\nu}\|_{[H^{-1/2+\delta}(\Gamma_N)]^2}$, where $\delta := \min\{\frac{1}{2}, \frac{\pi}{2\omega}\}$ and $\omega \in (0, 2\pi)$ is the largest interior angle of Ω. In addition, there also holds $\|\mathbf{z}\|_{[H^1(\Omega)]^2} \le C\,\|\boldsymbol{\nu}\|_{[H^{-1/2}(\Gamma_N)]^2}$. Thus, we let $\boldsymbol{\psi} := \mathbf{z}|_{\Gamma_N} \in [H^{1/2+\delta}(\Gamma_N)]^2 \cap [H^{1/2}_{00}(\Gamma_N)]^2$, and apply $(AP^{\boldsymbol{\xi}}_{\tilde{h}})$ with $t = 1/2 + \delta$ to deduce that there exists $\boldsymbol{\psi}_{\tilde{h}} \in M^{\boldsymbol{\xi}}_{\tilde{h}}$ such that

$$\|\boldsymbol{\psi} - \boldsymbol{\psi}_{\tilde{h}}\|_{[H^{1/2}_{00}(\Gamma_N)]^2} \le C\,\tilde{h}^{\delta}\,\|\boldsymbol{\psi}\|_{[H^{1/2+\delta}(\Gamma_N)]^2}\,. \tag{3.5.6}$$

Next, we fix a uniformly regular partition $\left\{\hat{\Gamma}_1, \hat{\Gamma}_2, \cdots, \hat{\Gamma}_{\hat{n}}\right\}$ of Γ_N such that $\boldsymbol{\nu}|_{\hat{\Gamma}_j} \in \mathbb{R}^2$ for each $j \in \{1, ..., \hat{n}\}$, and denote $\hat{h} := \max\{|\hat{\Gamma}_j| : \quad j \in \{1, \cdots, \hat{n}\}\}$. Then, due to the well known inverse inequality for piecewise constant functions on Γ_N, we obtain that

$$\|\boldsymbol{\nu}\|_{[H^{-1/2+\delta}(\Gamma_N)]^2} \le C\,\hat{h}^{-\delta}\,\|\boldsymbol{\nu}\|_{[H^{-1/2}(\Gamma_N)]^2}\,. \tag{3.5.7}$$

Hence, using (3.5.6), trace theorem, and (3.5.7), we find that

$$\begin{aligned}\|\boldsymbol{\psi} - \boldsymbol{\psi}_{\tilde{h}}\|_{[H^{1/2}_{00}(\Gamma_N)]^2} \le C\,\tilde{h}^{\delta}\,\|\mathbf{z}\|_{[H^{1+\delta}(\Omega)]^2} &\le C\,\tilde{h}^{\delta}\,\|\boldsymbol{\nu}\|_{[H^{-1/2+\delta}(\Gamma_N)]^2}\\ &\le \hat{C}\,\tilde{h}^{\delta}\,\|\boldsymbol{\nu}\|_{[H^{-1/2}(\Gamma_N)]^2}\,,\end{aligned} \tag{3.5.8}$$

with $\hat{C} = C\,\hat{h}^{-\delta}$. In addition, applying triangle inequality, estimate (3.5.8), and trace theorem, we get

$$\|\boldsymbol{\psi}_{\tilde{h}}\|_{[H^{1/2}_{00}(\Gamma_N)]^2} \le \hat{C}\,\tilde{h}^{\delta}\,\|\boldsymbol{\nu}\|_{[H^{-1/2}(\Gamma_N)]^2} + C\,\|\mathbf{z}\|_{[H^1(\Omega)]^2} \le C\,\|\boldsymbol{\nu}\|_{[H^{-1/2}(\Gamma_N)]^2}\,. \tag{3.5.9}$$

Also, since $-\Delta\mathbf{z} + \mathbf{z} = \mathbf{0}$ in Ω, it is easy to see that

$$\langle \boldsymbol{\nu}, \boldsymbol{\psi}\rangle_{\Gamma_N} = \langle \frac{\partial \mathbf{z}}{\partial \boldsymbol{\nu}}, \mathbf{z}\rangle_{\Gamma} = \|\mathbf{z}\|^2_{[H^1(\Omega)]^2} \ge C\,\left\|\frac{\partial \mathbf{z}}{\partial \boldsymbol{\nu}}\right\|^2_{[H^{-1/2}(\Gamma_N)]^2} = C\,\|\boldsymbol{\nu}\|^2_{[H^{-1/2}(\Gamma_N)]^2}\,. \tag{3.5.10}$$

Therefore, employing (3.5.8), (3.5.9), and (3.5.10), we can write

$$\begin{aligned}\sup_{\substack{\boldsymbol{\lambda}_{\tilde{h}} \in M^{\boldsymbol{\xi}}_{\tilde{h}}\\ \boldsymbol{\lambda}_{\tilde{h}} \neq 0}} \frac{\langle \boldsymbol{\nu}, \boldsymbol{\lambda}_{\tilde{h}}\rangle_{\Gamma_N}}{\|\boldsymbol{\lambda}_{\tilde{h}}\|_{[H^{1/2}_{00}(\Gamma_N)]^2}} &\ge \frac{|\,\langle \boldsymbol{\nu}, \boldsymbol{\psi}_{\tilde{h}}\rangle_{\Gamma_N}\,|}{\|\boldsymbol{\psi}_{\tilde{h}}\|_{[H^{1/2}_{00}(\Gamma_N)]^2}} \ge C\,\frac{|\,\langle \boldsymbol{\nu}, \boldsymbol{\psi}_{\tilde{h}}\rangle_{\Gamma_N}\,|}{\|\boldsymbol{\nu}\|_{[H^{-1/2}(\Gamma_N)]^2}}\\ &\ge \frac{C}{\|\boldsymbol{\nu}\|_{[H^{-1/2}(\Gamma_N)]^2}}\left\{\,|\,\langle \boldsymbol{\nu}, \boldsymbol{\psi}\rangle_{\Gamma_N}\,| \;-\; |\,\langle \boldsymbol{\nu}, \boldsymbol{\psi}_{\tilde{h}} - \boldsymbol{\psi}\rangle_{\Gamma_N}\,|\right\}\end{aligned}$$

$$\geq \left\{ \bar{C} - \tilde{C}\,\tilde{h}^{\delta} \right\} \|\boldsymbol{\nu}\|_{[H^{-1/2}(\Gamma_N)]^2},$$

which yields (3.5.4) with $\bar{C}_0$ and h_0 depending on $\bar{C}$, $\tilde{C}$, and δ. Finally, the discrete inf-sup condition (3.5.5) follows from (3.5.3) and (3.5.4). □

At this point we recall that the simplest finite element subspace of $[H_{00}^{1/2}(\Gamma_N)]^2$ satisfying the approximation property $(AP)_{\tilde{h}}^{\boldsymbol{\xi}}$, and hence the discrete inf-sup condition for $\mathcal{C}$ (cf. (3.5.5)), independently of the choices of $M_h^{\mathbf{u}}$ and $M_h^{\boldsymbol{\gamma}}$, is given by the continuous piecewise linear functions on Γ_N vanishing at the end points of Γ_N. According to it, from now on we consider

$$M_{\tilde{h}}^{\boldsymbol{\xi}} := \left\{ \boldsymbol{\lambda}_{\tilde{h}} \in [H_{00}^{1/2}(\Gamma_N)]^2 : \quad \boldsymbol{\lambda}_{\tilde{h}}|_{\tilde{\Gamma}_j} \in [\mathbb{P}_1(\tilde{\Gamma}_j)]^2 \quad \forall\, j \in \{1, \cdots, m\} \right\}. \tag{3.5.11}$$

Hereafter, given an integer $\ell \geq 0$ and a subset S of $\mathbb{R}^2$, $\mathbb{P}_\ell(S)$ stands for the space of polynomials defined in S of total degree $\leq \ell$.

On the other hand, in order to deal with the discrete inf-sup condition for $\mathcal{B}$, we need to characterize the discrete kernel of $\mathcal{C}$. Thus, recalling that $M_{h,\tilde{h}} = M_h^{\mathbf{u}} \times M_h^{\boldsymbol{\gamma}} \times M_{\tilde{h}}^{\boldsymbol{\xi}}$, we easily find that

$$\tilde{M}_{h,\tilde{h}} := \left\{ \vec{\mathbf{v}} \in M_{h,\tilde{h}} : \quad [\mathcal{C}(\vec{\mathbf{v}}), \tau_0] = 0 \quad \forall\, \tau_0 \in \mathbb{R} \right\} = M_h^{\mathbf{u}} \times M_h^{\boldsymbol{\gamma}} \times \tilde{M}_{\tilde{h}}^{\boldsymbol{\xi}},$$

where

$$\tilde{M}_{\tilde{h}}^{\boldsymbol{\xi}} := \left\{ \boldsymbol{\lambda}_{\tilde{h}} \in M_{\tilde{h}}^{\boldsymbol{\xi}} : \quad \langle \boldsymbol{\nu}, \boldsymbol{\lambda}_{\tilde{h}} \rangle_{\Gamma_N} = 0 \right\}.$$

Next, proceeding similarly to the analysis in Lemma 3.4.2, and using that $M_{1,h} := M_{1,h}^{\tilde{\boldsymbol{\sigma}}} \times M_{1,h}^{\tilde{p}} \times \mathbb{R}$, we can prove that for each $\vec{\mathbf{v}} := (\mathbf{v}_h, \boldsymbol{\delta}_h, \boldsymbol{\lambda}_{\tilde{h}}) \in \tilde{M}_{h,\tilde{h}}$ there holds the discrete analogue of (3.4.1), that is

$$\sup_{\substack{\vec{\boldsymbol{\tau}} \in M_{1,h} \\ \vec{\boldsymbol{\tau}} \neq 0}} \frac{[\mathcal{B}(\vec{\boldsymbol{\tau}}), \vec{\mathbf{v}}]}{\|\vec{\boldsymbol{\tau}}\|_{M_1}} = \sup_{\substack{\tilde{\boldsymbol{\tau}} \in M_{1,h}^{\tilde{\boldsymbol{\sigma}}} \\ \tilde{\boldsymbol{\tau}} \neq 0}} \frac{[\mathcal{B}(\tilde{\boldsymbol{\tau}}, 0, 0), \vec{\mathbf{v}}]}{\|\tilde{\boldsymbol{\tau}}\|_{H(\mathbf{div};\Omega)}}$$

$$= \sup_{\substack{\boldsymbol{\tau} \in \mathcal{M}_{1,h}^{\boldsymbol{\sigma}} \\ \boldsymbol{\tau} \neq 0}} \frac{[\mathcal{B}(\boldsymbol{\tau}, 0, 0), \vec{\mathbf{v}}]}{\|\boldsymbol{\tau}\|_{H(\mathbf{div};\Omega)}} = \sup_{\substack{\boldsymbol{\tau} \in \mathcal{M}_{1,h}^{\boldsymbol{\sigma}} \\ \boldsymbol{\tau} \neq 0}} \frac{-\displaystyle\int_\Omega \mathbf{v}_h \cdot \mathbf{div}\,\boldsymbol{\tau} - \int_\Omega \boldsymbol{\delta}_h : \boldsymbol{\tau} - \langle \boldsymbol{\tau}\boldsymbol{\nu}, \boldsymbol{\lambda}_{\tilde{h}} \rangle_{\Gamma_N}}{\|\boldsymbol{\tau}\|_{H(\mathbf{div};\Omega)}}. \tag{3.5.12}$$

Then we observe that, except for the term involving $\boldsymbol{\lambda}_{\tilde{h}}$, the expresion (3.5.12) coincides with that in [9] for which the well known PEERS finite element subspace satisfies the corresponding discrete inf-sup condition. Motivated by this fact, in what follows we let $\mathcal{M}^{\boldsymbol{\sigma}}_{1,h} \times M^{\mathbf{u}}_h \times M^{\boldsymbol{\gamma}}_h$ be exactly the PEERS space introduced in [9]. More precisely, we define

$$\mathcal{M}^{\boldsymbol{\sigma}}_{1,h} := \left\{ \boldsymbol{\tau}_h \in H(\mathbf{div};\Omega) : \ \boldsymbol{\tau}_h|_T \in [\mathcal{RT}_0(T)^{\mathrm{t}}]^2 \oplus [\mathbb{P}_0(T)\,\mathbf{curl}^{\mathrm{t}}\, b_T]^2 \quad \forall\, T \in \mathcal{T}_h \right\}, \tag{3.5.13}$$

$$M^{\mathbf{u}}_h := \left\{ \mathbf{v}_h \in [L^2(\Omega)]^2 : \quad \mathbf{v}_h|_T \in [\mathbb{P}_0(T)]^2 \quad \forall\, T \in \mathcal{T}_h \right\}, \tag{3.5.14}$$

and

$$M^{\boldsymbol{\gamma}}_h := \left\{ \begin{pmatrix} 0 & \delta_h \\ -\delta_h & 0 \end{pmatrix} \in [H^1(\Omega)]^{2\times 2} : \quad \delta_h|_T \in \mathbb{P}_1(T) \quad \forall\, T \in \mathcal{T}_h \right\}, \tag{3.5.15}$$

where $\mathcal{RT}_0(T)$ is the local Raviart-Thomas space of order 0 (cf. [20], [66]), b_T is the usual cubic bubble function on $T \in \mathcal{T}_h$, and $\mathbf{curl}^{\mathrm{t}}\, b_T := (\frac{\partial b_T}{\partial x_2}, -\frac{\partial b_T}{\partial x_1})$.

Then, we have the following preliminary estimate.

Lemma 3.5.2 *There exists $\hat{C} > 0$, independent of h, such that for all $\vec{\mathbf{v}} := (\mathbf{v}_h, \boldsymbol{\delta}_h, \boldsymbol{\lambda}_{\tilde{h}}) \in \tilde{M}_{h,\tilde{h}}$ there holds*

$$\sup_{\substack{\vec{\boldsymbol{\tau}} \in M_{1,h} \\ \vec{\boldsymbol{\tau}} \neq 0}} \frac{[\mathcal{B}(\vec{\boldsymbol{\tau}}), \vec{\mathbf{v}}]}{\|\vec{\boldsymbol{\tau}}\|_{M_1}} \ \geq \ \hat{C}\, \|(\mathbf{v}_h, \boldsymbol{\delta}_h)\|_{[L^2(\Omega)]^2 \times \mathcal{R}}\,. \tag{3.5.16}$$

Proof. We recall from Lemma 4.4 in [9] that given $(\mathbf{v}_h, \boldsymbol{\delta}_h) \in M^{\mathbf{u}}_h \times M^{\boldsymbol{\gamma}}_h$ there exists $\hat{\boldsymbol{\tau}}_h \in \mathcal{M}^{\boldsymbol{\sigma}}_{1,h}$, $\hat{\boldsymbol{\tau}}_h \neq \mathbf{0}$, such that

$$[\mathcal{B}(\hat{\boldsymbol{\tau}}_h, 0, 0), (\mathbf{v}_h, \boldsymbol{\delta}_h, 0)] \ \geq \ \hat{C}\, \|\hat{\boldsymbol{\tau}}_h\|_{H(\mathbf{div};\Omega)}\, \|(\mathbf{v}_h, \boldsymbol{\delta}_h)\|_{[L^2(\Omega)]^2 \times \mathcal{R}}\,,$$

with a constant $\hat{C} > 0$, independent of h and $(\mathbf{v}_h, \boldsymbol{\delta}_h)$. Moreover, $\hat{\boldsymbol{\tau}}_h$ can be chosen so that $\hat{\boldsymbol{\tau}}_h \boldsymbol{\nu} = \mathbf{0}$ on Γ_N, and hence

$$\sup_{\substack{\boldsymbol{\tau} \in \mathcal{M}^{\boldsymbol{\sigma}}_{1,h} \\ \boldsymbol{\tau} \neq 0}} \frac{[\mathcal{B}(\boldsymbol{\tau}, 0, 0), \vec{\mathbf{v}}]}{\|\boldsymbol{\tau}\|_{H(\mathbf{div};\Omega)}} \ \geq \ \frac{[\mathcal{B}(\hat{\boldsymbol{\tau}}_h, 0, 0), \vec{\mathbf{v}}]}{\|\hat{\boldsymbol{\tau}}_h\|_{H(\mathbf{div};\Omega)}} \ \geq \ \hat{C}\, \|(\mathbf{v}_h, \boldsymbol{\delta}_h)\|_{[L^2(\Omega)]^2 \times \mathcal{R}}\,,$$

which, according to (3.5.12), yields (3.5.16). □

In order to continue the analysis of the discrete inf-sup condition for $\mathcal{B}$, we now proceed as in [12]. To this end, we introduce the finite element subspace of

$[H^{-1/2}(\Gamma_N)]^2$ given by the piecewise constant functions. In other words, if $\{\Gamma_1, \Gamma_2, \cdots, \Gamma_n\}$ is the partition on Γ_N induced by the triangulation $\mathcal{T}_h$, we define

$$H_h^{-1/2} := \left\{ \boldsymbol{\psi}_h \in [L^2(\Gamma_N)]^2 : \qquad \boldsymbol{\psi}_h|_{\Gamma_j} \in [\mathbb{P}_0(\Gamma_j)]^2 \quad \forall j \in \{1, \cdots, n\} \right\} ,$$

which satisfies the following approximation property (see [11]):

$(AP_h^{-1/2})$ for each $s \in (-\frac{1}{2}, \frac{1}{2}]$ and for each $\boldsymbol{\psi} \in [H^s(\Gamma_N)]^2$ there exists $\boldsymbol{\psi}_h \in H_h^{-1/2}$ such that

$$\|\boldsymbol{\psi} - \boldsymbol{\psi}_h\|_{[H^{-1/2}(\Gamma_N)]^2} \le C\, h^{s+1/2}\, \|\boldsymbol{\psi}\|_{[H^s(\Gamma_N)]^2} .$$

Also, we assume that $\{\Gamma_1, \Gamma_2, \cdots, \Gamma_n\}$ and $\{\tilde\Gamma_1, \tilde\Gamma_2, \cdots, \tilde\Gamma_m\}$ are uniformly regular, which means that there exist C, $\tilde C > 0$, independent of h and $\tilde h$, such that $|\Gamma_j| \ge C\, h$ for all $j \in \{1, \cdots, n\}$ and $|\tilde\Gamma_j| \ge \tilde C\, \tilde h$ for all $j \in \{1, \cdots, m\}$, for all h, $\tilde h > 0$. These conditions yield the inverse inequalities for the spaces $H_h^{-1/2}$ and $M_{\tilde h}^{\boldsymbol{\xi}}$ (see Theorem 3.5 in [2] and [11]), respectively, that is, for any real numbers s and t with $-\frac{1}{2} \le s \le t \le 0$, there exists $C > 0$ such that

$$\|\boldsymbol{\psi}_h\|_{[H^t(\Gamma_N)]^2} \le C\, h^{s-t}\, \|\boldsymbol{\psi}_h\|_{[H^s(\Gamma_N)]^2} \qquad \forall\, \boldsymbol{\psi}_h \in H_h^{-1/2} , \tag{3.5.17}$$

and for any real numbers s and t with $0 \le s \le t \le 1$, there exists $\tilde C > 0$ such that

$$\|\boldsymbol{\lambda}_{\tilde h}\|_{[H^t(\Gamma_N)]^2} \le \tilde C\, \tilde h^{s-t}\, \|\boldsymbol{\lambda}_{\tilde h}\|_{[H^s(\Gamma_N)]^2} \qquad \forall\, \boldsymbol{\lambda}_{\tilde h} \in M_{\tilde h}^{\boldsymbol{\xi}} . \tag{3.5.18}$$

The following lemma provides a second preliminary estimate.

Lemma 3.5.3 *There exist $C_0 \in (0,1)$, $C_1 > 0$, independent of h and $\tilde h$, such that for all $h \le C_0\, \tilde h$ and for all $\vec{\mathbf{v}} := (\mathbf{v}_h, \boldsymbol{\delta}_h, \boldsymbol{\lambda}_{\tilde h}) \in \tilde M_{h,\tilde h}$ there holds*

$$\sup_{\substack{\vec{\boldsymbol{\tau}} \in M_{1,h} \\ \vec{\boldsymbol{\tau}} \neq 0}} \frac{[\mathcal{B}(\vec{\boldsymbol{\tau}}), \vec{\mathbf{v}}]}{\|\vec{\boldsymbol{\tau}}\|_{M_1}} \ge C_1\, \|\boldsymbol{\lambda}_{\tilde h}\|_{[H_{00}^{1/2}(\Gamma_N)]^2} - \|\boldsymbol{\delta}_h\|_{\mathcal{R}} . \tag{3.5.19}$$

Proof. We proceed as in the last part of the proof of Lemma 3.4.2. Thus, given $\boldsymbol{\psi}_h \in H_h^{-1/2}$ we let $\mathbf{z} \in [H^1_{\Gamma_D}(\Omega)]^2$ be the unique weak solution of the boundary value problem:

$$\mathbf{div}\, \mathbf{e}(\mathbf{z}) = \mathbf{0} \quad \text{in} \quad \Omega, \qquad \mathbf{z} = \mathbf{0} \quad \text{on} \quad \Gamma_D, \qquad \mathbf{e}(\mathbf{z})\, \boldsymbol{\nu} = \boldsymbol{\psi}_h \quad \text{on} \quad \Gamma_N .$$

Since $H_h^{-1/2} \subseteq [L^2(\Gamma_N)]^2$, the classical regularity results in [53] (see also [54]) imply that $\mathbf{z} \in [H^{1+\delta}(\Omega)]^2$ and $\|\mathbf{z}\|_{[H^{1+\delta}(\Omega)]^2} \leq C \, \|\boldsymbol{\psi}_h\|_{[H^{-1/2+\delta}(\Gamma_N)]^2}$, where $\delta := \min\{\frac{1}{2}, \frac{\pi}{2\omega}\}$ and $\omega \in (0, 2\pi)$ is the largest interior angle of Ω. In addition, the usual continuous dependence result establishes that $\|\mathbf{z}\|_{[H^1(\Omega)]^2} \leq C \, \|\boldsymbol{\psi}_h\|_{[H^{-1/2}(\Gamma_N)]^2}$. Thus, we let $\boldsymbol{\tau}(\boldsymbol{\psi}_h) := \mathbf{e}(\mathbf{z}) \in [H^\delta(\Omega)]^{2\times 2}$ and observe that $\mathbf{div}\, \boldsymbol{\tau}(\boldsymbol{\psi}_h) = \mathbf{0}$ in Ω and $\boldsymbol{\tau}(\boldsymbol{\psi}_h)\boldsymbol{\nu} = \boldsymbol{\psi}_h$ on Γ_N. Moreover, according to the previous estimates for $\mathbf{z}$, we obtain that

$$\|\boldsymbol{\tau}(\boldsymbol{\psi}_h)\|_{H(\mathbf{div};\Omega)} = \|\boldsymbol{\tau}(\boldsymbol{\psi}_h)\|_{[L^2(\Omega)]^{2\times 2}} \leq \tilde{C} \, \|\boldsymbol{\psi}_h\|_{[H^{-1/2}(\Gamma_N)]^2} \,, \tag{3.5.20}$$

and

$$\|\boldsymbol{\tau}(\boldsymbol{\psi}_h)\|_{[H^\delta(\Omega)]^{2\times 2}} \leq C \, \|\boldsymbol{\psi}_h\|_{[H^{-1/2+\delta}(\Gamma_N)]^2} \,. \tag{3.5.21}$$

Next, we let $\bar{\mathcal{M}}^{\boldsymbol{\sigma}}_{1,h} := \{\boldsymbol{\tau}_h \in H(\mathbf{div};\Omega) : \quad \boldsymbol{\tau}_h|_T \in [\mathcal{RT}_0(T)^{\mathtt{t}}]^2 \quad \forall\, T \in \mathcal{T}_h\}$ be the usual Raviart-Thomas subspace, which is certainly contained in $\mathcal{M}^{\boldsymbol{\sigma}}_{1,h}$, and consider the equilibrium interpolation operator $\mathcal{E}_h : [H^\delta(\Omega)]^{2\times 2} \cap H(\mathbf{div};\Omega) \to \bar{\mathcal{M}}^{\boldsymbol{\sigma}}_{1,h}$ (see [20], [66], and Theorem 3.1 in [2]). It is well known, as proved by Theorem 6.3 in [66] and Theorem 3.4 in [2], that $\mathcal{E}_h$ satisfies the following approximation property

$$\|\boldsymbol{\tau} - \mathcal{E}_h(\boldsymbol{\tau})\|_{[L^2(\Omega)]^{2\times 2}} \leq C\, h^\delta \, \|\boldsymbol{\tau}\|_{[H^\delta(\Omega)]^{2\times 2}} \qquad \forall\, \boldsymbol{\tau} \in [H^\delta(\Omega)]^{2\times 2} \cap H(\mathbf{div};\Omega) \,. \tag{3.5.22}$$

Also, there hold $\mathbf{div}\, \mathcal{E}_h(\boldsymbol{\tau}(\boldsymbol{\psi}_h)) = \mathbf{div}\, \boldsymbol{\tau}(\boldsymbol{\psi}_h) = \mathbf{0}$ in Ω and $\mathcal{E}_h(\boldsymbol{\tau}(\boldsymbol{\psi}_h))\boldsymbol{\nu} = \boldsymbol{\tau}(\boldsymbol{\psi}_h)\boldsymbol{\nu} = \boldsymbol{\psi}_h$ on Γ_N. Then, using (3.5.22), (3.5.20), and (3.5.21), we get

$$\begin{aligned}
\|\mathcal{E}_h(\boldsymbol{\tau}(\boldsymbol{\psi}_h))\|_{H(\mathbf{div};\Omega)} &\leq \|\boldsymbol{\tau}(\boldsymbol{\psi}_h) - \mathcal{E}_h(\boldsymbol{\tau}(\boldsymbol{\psi}_h))\|_{[L^2(\Omega)]^{2\times 2}} + \|\boldsymbol{\tau}(\boldsymbol{\psi}_h)\|_{[L^2(\Omega)]^{2\times 2}} \\
&\leq C\, h^\delta \, \|\boldsymbol{\tau}(\boldsymbol{\psi}_h)\|_{[H^\delta(\Omega)]^{2\times 2}} + \|\boldsymbol{\tau}(\boldsymbol{\psi}_h)\|_{[L^2(\Omega)]^{2\times 2}} \\
&\leq C\, h^\delta \, \|\boldsymbol{\psi}_h\|_{[H^{-1/2+\delta}(\Gamma_N)]^2} + \tilde{C} \, \|\boldsymbol{\psi}_h\|_{[H^{-1/2}(\Gamma_N)]^2} \,,
\end{aligned}$$

which, applying inverse inequality (3.5.17), yields

$$\|\mathcal{E}_h(\boldsymbol{\tau}(\boldsymbol{\psi}_h))\|_{H(\mathbf{div};\Omega)} \leq C \, \|\boldsymbol{\psi}_h\|_{[H^{-1/2}(\Gamma_N)]^2} \,.$$

Therefore, given $\vec{\mathbf{v}} := (\mathbf{v}_h, \boldsymbol{\delta}_h, \boldsymbol{\lambda}_{\tilde{h}}) \in \tilde{M}_{h,\tilde{h}}$, we find that

$$\sup_{\substack{\boldsymbol{\tau} \in \mathcal{M}^{\boldsymbol{\sigma}}_{1,h} \\ \boldsymbol{\tau} \neq 0}} \frac{[\mathcal{B}(\boldsymbol{\tau}, 0, 0), \vec{\mathbf{v}}]}{\|\boldsymbol{\tau}\|_{H(\mathbf{div};\Omega)}} \geq \frac{|\, [\mathcal{B}(\mathcal{E}_h(\boldsymbol{\tau}(\boldsymbol{\psi}_h)), 0, 0), \vec{\mathbf{v}}] \,|}{\|\mathcal{E}_h(\boldsymbol{\tau}(\boldsymbol{\psi}_h))\|_{H(\mathbf{div};\Omega)}}$$

$$= \frac{\Big| - \langle \boldsymbol{\psi}_h \,,\, \boldsymbol{\lambda}_{\tilde{h}} \rangle_{\Gamma_N} - \displaystyle\int_\Omega \boldsymbol{\delta}_h : \mathcal{E}_h(\boldsymbol{\tau}(\boldsymbol{\psi}_h)) \Big|}{\|\mathcal{E}_h(\boldsymbol{\tau}(\boldsymbol{\psi}_h))\|_{H(\mathbf{div};\Omega)}} \geq \frac{1}{C} \frac{|\langle \boldsymbol{\psi}_h, \boldsymbol{\lambda}_{\tilde{h}} \rangle_{\Gamma_N}|}{\|\boldsymbol{\psi}_h\|_{[H^{-1/2}(\Gamma_N)]^2}} - \|\boldsymbol{\delta}_h\|_{\mathcal{R}} \,,$$

for all $\boldsymbol{\psi}_h \in H_h^{-1/2}$, and hence,

$$\sup_{\substack{\boldsymbol{\tau} \in \mathcal{M}^{\boldsymbol{\sigma}}_{1,h} \\ \boldsymbol{\tau} \neq 0}} \frac{[\mathcal{B}(\boldsymbol{\tau}, 0, 0), \vec{\mathbf{v}}]}{\|\boldsymbol{\tau}\|_{H(\mathbf{div};\Omega)}} \geq \frac{1}{C} \sup_{\substack{\boldsymbol{\psi}_h \in H_h^{-1/2} \\ \boldsymbol{\psi}_h \neq 0}} \frac{|\langle \boldsymbol{\psi}_h \,,\, \boldsymbol{\lambda}_{\tilde{h}} \rangle_{\Gamma_N}|}{\|\boldsymbol{\psi}_h\|_{[H^{-1/2}(\Gamma_N)]^2}} - \|\boldsymbol{\delta}_h\|_{\mathcal{R}} \,. \tag{3.5.23}$$

On the other hand, since the normal trace on Γ_N is well defined and continuous from $[H^\delta(\Omega)]^{2\times 2} \cap H(\mathbf{div};\Omega)$ onto $[H^{-1/2+\delta}(\Gamma_N)]^2$ for $\delta \neq \frac{1}{2}$ (see Theorem 2.4 and the corresponding remark in [2]), we can apply the vector version of Lemma 3.3 in [12], making use of the approximation property $(AP_h^{-1/2})$ and the inverse inequality (3.5.18), to deduce that there exist $C_0 \in (0,1)$, $\tilde{C} > 0$, independent of h and $\tilde{h}$, such that for all $h \leq C_0 \,\tilde{h}$ there holds

$$\sup_{\substack{\boldsymbol{\psi}_h \in H_h^{-1/2} \\ \boldsymbol{\psi}_h \neq 0}} \frac{|\langle \boldsymbol{\psi}_h \,,\, \boldsymbol{\lambda}_{\tilde{h}} \rangle_{\Gamma_N}|}{\|\boldsymbol{\psi}_h\|_{[H^{-1/2}(\Gamma_N)]^2}} \geq \tilde{C} \,\|\boldsymbol{\lambda}_{\tilde{h}}\|_{[H_{00}^{1/2}(\Gamma_N)]^2} \,. \tag{3.5.24}$$

In this way, (3.5.23) and (3.5.24) yield (3.5.19) and complete the proof. □

It is important to remark here that the eventual use of local inverse inequalities, instead of (3.5.17) and (3.5.18), is limited by the fact that the norms $\|\cdot\|_{[H^t(\Gamma_N)]^2}$ are non-local.

As a consequence of the previous lemmata we can establish the discrete inf-sup condition for $\mathcal{B}$.

Lemma 3.5.4 *Let $C_0 \in (0,1)$ be the constant provided by Lemma 3.5.3. Then, there exists $\bar{\beta} > 0$, independent of h and $\tilde{h}$, such that for all $h \leq C_0 \,\tilde{h}$ and for all $\vec{\mathbf{v}} := (\mathbf{v}_h, \boldsymbol{\delta}_h, \boldsymbol{\lambda}_{\tilde{h}}) \in \tilde{M}_{h,\tilde{h}}$ there holds*

$$\sup_{\substack{\vec{\boldsymbol{\tau}} \in M_{1,h} \\ \vec{\boldsymbol{\tau}} \neq 0}} \frac{[\mathcal{B}(\vec{\boldsymbol{\tau}}), \vec{\mathbf{v}}]}{\|\vec{\boldsymbol{\tau}}\|_{M_1}} \geq \bar{\beta} \,\|\vec{\mathbf{v}}\|_M \,. \tag{3.5.25}$$

Proof. Let $C_1 > 0$ be the constant in (3.5.19) (cf. Lemma 3.5.3). The proof proceeds by assuming two possible cases. If $\|(\mathbf{v}_h, \boldsymbol{\delta}_h)\|_{[L^2(\Omega)]^2 \times \mathcal{R}} \leq \dfrac{C_1}{2} \|\boldsymbol{\lambda}_{\tilde{h}}\|_{[H_{00}^{1/2}(\Gamma_N)]^2}$, then

(3.5.25) follows directly from (3.5.19). On the other hand, if $\|(\mathbf{v}_h, \boldsymbol{\delta}_h)\|_{[L^2(\Omega)]^2\times\mathcal{R}} \geq \frac{C_1}{2}\,\|\boldsymbol{\lambda}_{\tilde{h}}\|_{[H_{00}^{1/2}(\Gamma_N)]^2}$, then (3.5.25) is a trivial consequence of (3.5.16). □

Next, we identify the discrete kernel of the operator $\mathcal{B}$, which is needed to prove the discrete inf-sup condition for $\mathcal{B}_1$. Indeed, recalling that $M_{1,h} = M_{1,h}^{\tilde{\boldsymbol{\sigma}}} \times M_{1,h}^{\tilde{p}} \times \mathrm{IR}$ and $\tilde{M}_{h,\tilde{h}} = M_h^{\mathbf{u}} \times M_h^{\boldsymbol{\gamma}} \times \tilde{M}_{\tilde{h}}^{\boldsymbol{\xi}}$, we obtain that

$$\tilde{M}_{1,h} \;:=\; \left\{\vec{\boldsymbol{\tau}} \in M_{1,h} : \quad [\mathcal{B}(\vec{\boldsymbol{\tau}}), \vec{\mathbf{v}}] = 0 \quad \forall\, \vec{\mathbf{v}} \,\in\, \tilde{M}_{h,\tilde{h}}\right\} \;=\; \tilde{M}_{1,h}^{\tilde{\boldsymbol{\sigma}}} \times M_{1,h}^{\tilde{p}} \times \mathrm{IR}\,,$$

where

$$\tilde{M}_{1,h}^{\tilde{\boldsymbol{\sigma}}} \;:=\; \left\{\boldsymbol{\tau} \in M_{1,h}^{\tilde{\boldsymbol{\sigma}}} : \int_\Omega \mathbf{v}_h \cdot \mathbf{div}\,\boldsymbol{\tau} \,+\, \int_\Omega \boldsymbol{\delta}_h : \boldsymbol{\tau} \,+\, \langle \boldsymbol{\tau\nu}, \boldsymbol{\lambda}_{\tilde{h}}\rangle_{\Gamma_N} \;=\; 0 \right.$$
$$\left. \forall\, (\mathbf{v}_h, \boldsymbol{\delta}_h, \boldsymbol{\lambda}_{\tilde{h}}) \in \tilde{M}_{h,\tilde{h}}\right\}.$$

Since $(\mathbf{div}\,\boldsymbol{\tau})|_T$ and $\mathbf{v}_h|_T$ are constant vectors $\forall\,\boldsymbol{\tau} \,\in\, M_{1,h}^{\tilde{\boldsymbol{\sigma}}}$ and $\forall\,\mathbf{v}_h \,\in\, M_h^{\mathbf{u}}$, we deduce that $\mathbf{div}\,\boldsymbol{\tau}$ vanishes in Ω for all $\boldsymbol{\tau} \,\in\, \tilde{M}_{1,h}^{\tilde{\boldsymbol{\sigma}}}$, and hence

$$\tilde{M}_{1,h}^{\tilde{\boldsymbol{\sigma}}} := \left\{\boldsymbol{\tau} \in M_{1,h}^{\tilde{\boldsymbol{\sigma}}} :\; \mathbf{div}\,\boldsymbol{\tau} = \mathbf{0} \text{ in } \Omega,\; \int_\Omega \boldsymbol{\delta}_h : \boldsymbol{\tau} \;=\; 0 \;\forall\, \boldsymbol{\delta}_h \in M_h^{\boldsymbol{\gamma}}\,, \text{ and}\right.$$
$$\left.\langle \boldsymbol{\tau\nu}, \boldsymbol{\lambda}_{\tilde{h}}\rangle_{\Gamma_N} = 0 \;\forall\, \boldsymbol{\lambda}_{\tilde{h}} \in \tilde{M}_{\tilde{h}}^{\boldsymbol{\xi}}\right\}.$$

We now observe from the proof of Lemma 3.4.3 that, in order to extend the arguments yielding the continuous inf-sup condition for $\mathcal{B}_1$ to the present discrete case, we require that

$$\mathrm{dev}(\boldsymbol{\tau})\,,\;\; (q\,\mathbf{I} + \boldsymbol{\tau})\,,\;\; d\,\mathbf{I} \;\in X_{1,h} \qquad \forall\,(\boldsymbol{\tau}, q, d) \,\in\, \tilde{M}_{1,h} \;:=\; \tilde{M}_{1,h}^{\tilde{\boldsymbol{\sigma}}} \times M_{1,h}^{\tilde{p}} \times \mathrm{IR}\,. \tag{3.5.26}$$

In addition, since the only requirement on $\mathcal{M}_{1,h}^{p}$ is given by $\mathrm{IR} \subseteq \mathcal{M}_{1,h}^{p}$ (cf. Theorem 3.5.1), we can take this space as the simplest finite element subspace of $L^2(\Omega)$, that is

$$\mathcal{M}_{1,h}^{p} \;:=\; \left\{q_h \,\in\, L^2(\Omega) : \quad q_h|_T \,\in\, \mathrm{IP}_0(T) \qquad \forall\, T \,\in\, \mathcal{T}_h \right\}\,. \tag{3.5.27}$$

Therefore, according to the definitions of $\tilde{M}_{1,h}^{\tilde{\boldsymbol{\sigma}}}$ and $\mathcal{M}_{1,h}^{p}$, we deduce that (3.5.26) is verified if the finite element subspace for the unknown $\mathbf{t} \,\in\, [L^2(\Omega)]^{2\times 2}$ is defined as follows:

$$X_{1,h} \;:=\; \left\{\,\mathbf{s} \in [L^2(\Omega)]^{2\times 2} : \quad \mathbf{s}|_T \,\in\, \mathcal{X}(T) \qquad \forall\; T \,\in\, \mathcal{T}_h \right\}\,, \tag{3.5.28}$$

where $\quad \mathcal{X}(T) \;:=\; [\mathrm{IP}_0(T)]^{2\times 2} \,\oplus\, [\mathrm{IP}_0(T)\,\mathbf{curl}^{\mathbf{t}}\, b_T]^2 \,\oplus\, \mathrm{dev}\,([\mathrm{IP}_0(T)\,\mathbf{curl}^{\mathbf{t}}\, b_T]^2)$.

Lemma 3.5.5 *There exists $\bar{\beta}_1 > 0$, independent of h and $\tilde{h}$, such that for all $\vec{\boldsymbol{\tau}} \in \tilde{M}_{1,h}$ there holds*

$$\sup_{\substack{\mathbf{s}\in X_{1,h}\\ \mathbf{s}\neq 0}} \frac{[\mathcal{B}_1(\mathbf{s}), \vec{\boldsymbol{\tau}}]}{\|\mathbf{s}\|_{X_1}} \geq \bar{\beta}_1 \, \|\vec{\boldsymbol{\tau}}\|_{M_1} \, .$$

Proof. The choice (3.5.28) of the subspace $X_{1,h}$ guarantees that (3.5.26) holds and hence it suffices to apply the same arguments of the proof of Lemma 3.4.3. We omit details and refer to that proof. □

Since the strong monotonicity and Lipschitz-continuity of $\mathcal{A}_1$ (cf. Lemma 3.4.4) are certainly valid on any subspace of X_1, we conclude that hypothesis iv) in Theorem 3.3.2 (equivalently (3.3.13) and (3.3.14)) is clearly satisfied.

The unique solvability, stability, and convergence of (3.5.1) (equivalently (3.5.2)) is provided next. For clarity reasons we recall that the definitions of the corresponding finite element subspaces are given in (3.5.11), (3.5.13), (3.5.14), (3.5.15), (3.5.27), and (3.5.28).

Theorem 3.5.2 *There exist $C_0 \in (0,1)$ and $h_0 > 0$ such that for all $\tilde{h} \leq h_0$ and for all $h \leq C_0\,\tilde{h}$, the mixed finite element scheme* (3.5.1) *has a unique solution $(\mathbf{t}_h, (\boldsymbol{\sigma}_h, p_h), (\mathbf{u}_h, \boldsymbol{\gamma}_h, \boldsymbol{\xi}_{\tilde{h}})) \in X_{1,h} \times \mathcal{M}_{1,h} \times M_{h,\tilde{h}}$. In addition, there exist C_1, $C_2 > 0$, independent of h and $\tilde{h}$, such that*

$$\|(\mathbf{t}_h, (\boldsymbol{\sigma}_h, p_h), (\mathbf{u}_h, \boldsymbol{\gamma}_h, \boldsymbol{\xi}_{\tilde{h}}))\| \leq C_1 \left\{ \|\mathcal{G}|_{M_{h,\tilde{h}}}\|_{M'_{h,\tilde{h}}} + \|\mathcal{N}(0)\|_{[L^2(\Omega)]^{2\times 2}} \right\} ,$$

with $\|\mathcal{G}|_{M_{h,\tilde{h}}}\|_{M'_{h,\tilde{h}}} \leq \|\mathbf{f}\|_{[L^2(\Omega)]^2} + \|\mathbf{g}\|_{[H^{-1/2}(\Gamma_N)]^2}$, and the following Cea estimate holds

$$\|(\mathbf{t}, (\boldsymbol{\sigma}, p), (\mathbf{u}, \boldsymbol{\gamma}, \boldsymbol{\xi})) - (\mathbf{t}_h, (\boldsymbol{\sigma}_h, p_h), (\mathbf{u}_h, \boldsymbol{\gamma}_h, \boldsymbol{\xi}_{\tilde{h}}))\|$$

$$\leq C_2 \inf_{\substack{(\mathbf{s}_h,(\boldsymbol{\tau}_h,q_h),(\mathbf{v}_h,\boldsymbol{\delta}_h,\boldsymbol{\lambda}_{\tilde{h}}))\\ \in X_{1,h}\times\mathcal{M}_{1,h}\times M_{h,\tilde{h}}}} \|(\mathbf{t}, (\boldsymbol{\sigma}, p), (\mathbf{u}, \boldsymbol{\gamma}, \boldsymbol{\xi})) - (\mathbf{s}_h, (\boldsymbol{\tau}_h, q_h), (\mathbf{v}_h, \boldsymbol{\delta}_h, \boldsymbol{\lambda}_{\tilde{h}}))\| \, .$$

Proof. It follows from Lemmata 2.5.1, 3.5.3, and 3.5.5, a direct application of the abstract results given by Theorems 3.3.2 and 3.3.3, and the equivalence provided by Theorem 3.5.1. □

We end this section with a result on the rate of convergence of the finite element scheme (3.5.1).

Theorem 3.5.3 *Assume that* $\mathbf{t}|_T \in [H^1(T)]^{2\times 2}$ $\forall T \in \mathcal{T}_h$, $\boldsymbol{\sigma} \in [H^1(\Omega)]^{2\times 2}$, $\mathbf{div}\,\boldsymbol{\sigma} \in [H^1(\Omega)]^2$, $p \in H^1(\Omega)$, $\mathbf{u} \in [H^1(\Omega)]^2$, $\boldsymbol{\gamma} \in [H^1(\Omega)]^{2\times 2}$ *and* $\boldsymbol{\xi} \in [H^{3/2}(\Gamma_N)]^2$. *Then, there exists* $C > 0$, *independent of* h *and* $\tilde{h}$, *such that*

$$\|(\mathbf{t},(\boldsymbol{\sigma},p),(\mathbf{u},\boldsymbol{\gamma},\boldsymbol{\xi})) - (\mathbf{t}_h,(\boldsymbol{\sigma}_h,p_h),(\mathbf{u}_h,\boldsymbol{\gamma}_h,\boldsymbol{\xi}_{\tilde{h}}))\| \leq C\,h \Bigg\{ \sum_{T\in\mathcal{T}} \|\mathbf{t}\|_{[H^1(T)]^{2\times 2}}$$

$$+ \|\boldsymbol{\sigma}\|_{[H^1(\Omega)]^{2\times 2}} + \|\mathbf{div}\,\boldsymbol{\sigma}\|_{[H^1(\Omega)]^2} + \|p\|_{H^1(\Omega)} + \|\mathbf{u}\|_{[H^1(\Omega)]^2} + \|\boldsymbol{\gamma}\|_{[H^1(\Omega)]^{2\times 2}} \Bigg\}$$

$$+ C\,\tilde{h}\,\|\boldsymbol{\xi}\|_{[H^{3/2}(\Gamma_N)]^2}\,.$$

Proof. It follows straightforward from the Cea estimate in Theorem 3.5.2 and the usual approximation properties of the finite element subspaces (see, e.g. [9], [11], [66]). In particular, for $X_{1,h}$ we just consider the approximation property satisfied by the subspace of piecewise constant tensors. □

Because of the condition $h \leq C_0\,\tilde{h}$, we assume from now on, without loss of generality, that each edge Γ_i is contained in an edge $\tilde{\Gamma}_j$, for some $j \in \{1,\cdots,m\}$. Certainly, this requires implicitly that the end points of $\tilde{\Gamma}_j$ be vertices of $\mathcal{T}_h$, which is also assumed in what follows.

3.6 The a-posteriori error analysis

In this section we develop a local problems-based a-posteriori error analysis for the mixed finite element scheme (3.5.1). As in [40] and [48], our approach follows the technique from [22], [23], and [24], which is a modification of the original Bank-Weiser method proposed in [14].

We first introduce some notations. Given $T \in \mathcal{T}_h$ we denote by $E(T)$ be the set of its edges, and let E_h be the set of all edges of the triangulation $\mathcal{T}_h$. Then we write $E_h = E_h(\Omega) \cup E_h(\Gamma_D) \cup E_h(\Gamma_N)$, where $E_h(\Omega) := \{e \in E_h : e \subseteq \Omega\}$, $E_h(\Gamma_D) := \{e \in E_h : e \subseteq \Gamma_D\}$, and $E_h(\Gamma_N) := \{e \in E_h : e \subseteq \Gamma_N\}$. Also, given $T \in \mathcal{T}_h$, we let $\boldsymbol{\nu}_T$ be the unit outward normal to ∂T, and denote by $\langle\cdot,\cdot\rangle_{[L^2(T)]^{2\times 2}}$, $\langle\cdot,\cdot\rangle_{H(\mathbf{div};T)}$, and $\langle\cdot,\cdot\rangle_{L^2(T)}$ the inner products of $[L^2(T)]^{2\times 2}$, $H(\mathbf{div};T)$, and $L^2(T)$, respectively.

On the other hand, given a polygonal domain $\mathcal{S} \subset \mathbb{R}^2$ and $s \in (1,\infty)$, the Sobolev space $[W^{1,s}(\mathcal{S})]^2$ is the Banach space of functions $\mathbf{v} \in [L^s(\mathcal{S})]^2$ such that the first order distributional derivatives of $\mathbf{v}$ belong to $[L^s(\mathcal{S})]^2$ (see [61]). The trace Theorem ensures that there exists a linear continuous map $\gamma : [W^{1,s}(\mathcal{S})]^2 \to [L^2(\partial\mathcal{S})]^2$ such that $\gamma\mathbf{v} = \mathbf{v}|_{\partial\mathcal{S}}$ for each $\mathbf{v} \in [W^{1,s}(\mathcal{S}) \cap C(\bar{\mathcal{S}})]^2$. Hence, it is usual to denote $[W^{1-1/s,s}(\partial\mathcal{S})]^2 := \gamma([W^{1,s}(\mathcal{S})]^2)$ which is a strict subspace of $[L^s(\partial\mathcal{S})]^2$ (see [61]). Furthermore, the Sobolev imbedding theorem establishes that $[W^{1,s}(\mathcal{S})]^2 \subset [C(\bar{\mathcal{S}})]^2$ if $s > 2$. In particular, when $\mathcal{S} := T \in \mathcal{T}_h$ and $s = 2$ we use the standard notation and write $[H^{1/2}(\partial T)]^2$ instead of $[W^{1/2,2}(\partial T)]^2$. The dual space of $[H^{1/2}(\partial T)]^2$ is $[H^{-1/2}(\partial T)]^2$, and we denote by $\langle \cdot, \cdot \rangle_{\partial T}$ the corresponding duality pairing with respect to the $[L^2(\partial T)]^2$-inner product. Further, given $e \in E(T)$, we let $H^{1/2}_{00}(e)$ be the space of functions in $H^{1/2}(e)$ whose extensions by zero to the rest of ∂T are in $H^{1/2}(\partial T)$. It is important to retain here that the restriction of an element in $[H^{-1/2}(\partial T)]^2$ over edge e does not belong in general to $[H^{-1/2}(e)]^2$, but to the dual of $[H^{1/2}_{00}(e)]^2$, denoted by $[H^{-1/2}_{00}(e)]^2$, and which is larger than $[H^{-1/2}(e)]^2$. In what follows we denote by $\langle \cdot, \cdot \rangle_e$ the duality pairing between $[H^{-1/2}_{00}(e)]^2$ and $[H^{1/2}_{00}(e)]^2$ with respect to the $[L^2(e)]^2$-inner product. Finally, h_e denotes the diameter of the edge $e \in E_h$, and for each $e \in E_h(\Gamma_N)$ we set $\tilde{h}_e := |\tilde{\Gamma}_j|$, where $\tilde{\Gamma}_j$ is the segment containing e.

We now let $\mathbf{X} := X_1 \times \mathcal{M}_1 = [L^2(\Omega)]^{2\times 2} \times H(\mathbf{div}; \Omega) \times L^2(\Omega)$ and introduce the operators $\mathbf{A} : \mathbf{X} \to \mathbf{X}'$ and $\mathbf{B} : \mathbf{X} \to M'$ given by:

$$[\mathbf{A}(\mathbf{r}, \boldsymbol{\zeta}, \rho), (\mathbf{s}, \boldsymbol{\tau}, q)] := \int_\Omega \mathcal{N}(\mathbf{r}) : \mathbf{s} - \int_\Omega \boldsymbol{\zeta} : \mathbf{s} + \int_\Omega \rho \,\mathrm{tr}(\mathbf{s}) - \int_\Omega \mathbf{r} : \boldsymbol{\tau} + \int_\Omega q \,\mathrm{tr}(\mathbf{r}), \tag{3.6.1}$$

and

$$[\mathbf{B}(\mathbf{s}, \boldsymbol{\tau}, q), (\mathbf{v}, \boldsymbol{\delta}, \boldsymbol{\lambda})] := - \int_\Omega \mathbf{v} \cdot \mathbf{div}\, \boldsymbol{\tau} - \int_\Omega \boldsymbol{\delta} : \boldsymbol{\tau} - \langle \boldsymbol{\tau\nu}, \boldsymbol{\lambda} \rangle_{\Gamma_N}, \tag{3.6.2}$$

for all $(\mathbf{r}, \boldsymbol{\zeta}, \rho)$, $(\mathbf{s}, \boldsymbol{\tau}, q) \in \mathbf{X}$ and for all $(\mathbf{v}, \boldsymbol{\delta}, \boldsymbol{\lambda}) \in M$. It follows that our continuous formulation (3.2.9) can be stated, equivalently, as: Find $((\mathbf{t}, \boldsymbol{\sigma}, p), (\mathbf{u}, \boldsymbol{\gamma}, \boldsymbol{\xi})) \in \mathbf{X} \times M$ such that

$$\begin{bmatrix} \mathbf{A} & \mathbf{B}' \\ \mathbf{B} & \mathcal{O} \end{bmatrix} \begin{bmatrix} (\mathbf{t}, \boldsymbol{\sigma}, p) \\ (\mathbf{u}, \boldsymbol{\gamma}, \boldsymbol{\xi}) \end{bmatrix} = \begin{bmatrix} \mathcal{O} \\ \mathcal{G} \end{bmatrix}. \tag{3.6.3}$$

Then, we define the projection of the residual with respect to the usual inner product $\langle \cdot, \cdot \rangle_{\mathbf{X}}$ of $\mathbf{X}$ as the unique element $(\bar{\mathbf{t}}, \bar{\boldsymbol{\sigma}}, \bar{p}) \in \mathbf{X}$ such that

$$\begin{aligned} \langle (\bar{\mathbf{t}}, \bar{\boldsymbol{\sigma}}, \bar{p}), (\mathbf{s}, \boldsymbol{\tau}, q) \rangle_{\mathbf{X}} = {} & [\mathbf{A}(\mathbf{t}, \boldsymbol{\sigma}, p) - \mathbf{A}(\mathbf{t}_h, \boldsymbol{\sigma}_h, p_h), (\mathbf{s}, \boldsymbol{\tau}, q)] \\ & + [\mathbf{B}(\mathbf{s}, \boldsymbol{\tau}, q), (\mathbf{u} - \mathbf{u}_h, \boldsymbol{\gamma} - \boldsymbol{\gamma}_h, \boldsymbol{\xi} - \boldsymbol{\xi}_{\tilde{h}})] \end{aligned} \tag{3.6.4}$$

for all $(\mathbf{s}, \boldsymbol{\tau}, q) \in \mathbf{X}$. The existence of $(\bar{\mathbf{t}}, \bar{\boldsymbol{\sigma}}, \bar{p})$ is guaranted by the Riesz representation theorem and the fact that the right hand side of (3.6.4) constitutes a linear and bounded functional in $\mathbf{X}$ (as a mapping acting on $(\mathbf{s}, \boldsymbol{\tau}, q,))$.

The following theorem provides an upper bound for $\|(\bar{\mathbf{t}}, \bar{\boldsymbol{\sigma}}, \bar{p})\|_{\mathbf{X}}$.

Theorem 3.6.1 *Let $\boldsymbol{\varphi}_h$ be a function in $[H^1(\Omega) \cap W^{1,s}(\Omega)]^2$, with $s > 2$, such that $\boldsymbol{\varphi}_h(\bar{x}) = 0$ for each vertex $\bar{x}$ of $\mathcal{T}_h$ lying on Γ_D, and $\boldsymbol{\varphi}_h(\bar{x}) = -\boldsymbol{\xi}_{\tilde{h}}(\bar{x})$ for each vertex $\bar{x}$ of $\mathcal{T}_h$ lying on Γ_N. Also, given $T \in \mathcal{T}_h$, we let $\hat{\boldsymbol{\sigma}}_T \in H(\mathbf{div}; T)$ be the unique solution of the local problem*

$$\langle \hat{\boldsymbol{\sigma}}_T, \boldsymbol{\tau} \rangle_{H(\mathbf{div};T)} \;=\; F_{h,T}(\boldsymbol{\tau}) \qquad \forall\, \boldsymbol{\tau} \in H(\mathbf{div}; T)\,, \tag{3.6.5}$$

where $F_{h,T} \in H(\mathbf{div}; T)'$ is defined by

$$\begin{aligned} F_{h,T}(\boldsymbol{\tau}) \;:=\;& \int_T (\mathbf{t}_h + \boldsymbol{\gamma}_h) : \boldsymbol{\tau} \;+\; \int_T \mathbf{u}_h \cdot \mathbf{div}\, \boldsymbol{\tau} \;-\; \langle \boldsymbol{\tau}\boldsymbol{\nu}_T, \boldsymbol{\varphi}_h \rangle_{\partial T} \\ &+ \sum_{e \in E(T) \cap E_h(\Gamma_D)} \langle \boldsymbol{\tau}\boldsymbol{\nu}_T, \boldsymbol{\varphi}_h \rangle_e + \sum_{e \in E(T) \cap E_h(\Gamma_N)} \langle \boldsymbol{\tau}\boldsymbol{\nu}_T, \boldsymbol{\varphi}_h + \boldsymbol{\xi}_{\tilde{h}} \rangle_e\,. \end{aligned} \tag{3.6.6}$$

Then, there holds

$$\|(\bar{\mathbf{t}}, \bar{\boldsymbol{\sigma}}, \bar{p})\|^2_{\mathbf{X}} \;\le\; \sum_{T \in \mathcal{T}_h} \|\hat{\boldsymbol{\sigma}}_T\|^2_{H(\mathbf{div};T)} + \|\boldsymbol{\sigma}_h - \mathcal{N}(\mathbf{t}_h) - p_h\,\mathbf{I}\|^2_{[L^2(\Omega)]^{2\times 2}} + \|\mathrm{tr}(\mathbf{t}_h)\|^2_{L^2(\Omega)}\,. \tag{3.6.7}$$

Proof. The proof follows the same arguments of Theorem 5.2 in [48] and Lemma 3.1 in [40]. Indeed, we notice from the first two equations in (3.2.9) (equivalently, first equation in (3.6.3)) that

$$[\mathbf{A}(\mathbf{t}, \boldsymbol{\sigma}, p), (\mathbf{s}, \boldsymbol{\tau}, q)] \;+\; [\mathbf{B}(\mathbf{s}, \boldsymbol{\tau}, q), (\mathbf{u}, \boldsymbol{\gamma}, \boldsymbol{\xi})] \;=\; 0\,,$$

and hence (3.6.4) becomes

$$\begin{aligned} \langle (\bar{\mathbf{t}}, \bar{\boldsymbol{\sigma}}, \bar{p}), (\mathbf{s}, \boldsymbol{\tau}, q) \rangle_{\mathbf{X}} \;=\;& -\,[\mathbf{A}(\mathbf{t}_h, \boldsymbol{\sigma}_h, p_h), (\mathbf{s}, \boldsymbol{\tau}, q)] \;-\; [\mathbf{B}(\mathbf{s}, \boldsymbol{\tau}, q), (\mathbf{u}_h, \boldsymbol{\gamma}_h, \boldsymbol{\xi}_{\tilde{h}})] \\ & \forall\, (\mathbf{s}, \boldsymbol{\tau}, q) \;\in\; \mathbf{X}\,. \end{aligned} \tag{3.6.8}$$

Now, we clearly have that

$$-\,\frac{1}{2}\,\|(\bar{\mathbf{t}}, \bar{\boldsymbol{\sigma}}, \bar{p})\|^2_{\mathbf{X}} \;=\; \inf_{(\mathbf{s}, \boldsymbol{\tau}, q) \in \mathbf{X}} \left\{ \frac{1}{2}\,\|(\mathbf{s}, \boldsymbol{\tau}, q)\|^2_{\mathbf{X}} \;-\; \langle (\bar{\mathbf{t}}, \bar{\boldsymbol{\sigma}}, \bar{p}), (\mathbf{s}, \boldsymbol{\tau}, q) \rangle_{\mathbf{X}} \right\}\,,$$

which, employing (3.6.1), (3.6.2), and (3.6.8), becomes

$$-\frac{1}{2}\,\|(\bar{\mathbf{t}},\bar{\boldsymbol{\sigma}},\bar{p})\|^2_{\mathbf{X}} \;=\; \inf_{\mathbf{s}\in[L^2(\Omega)]^{2\times 2}} \mathbf{J}_1(\mathbf{s}) \;+\; \inf_{\boldsymbol{\tau}\in H(\mathbf{div};\Omega)} \mathbf{J}_2(\boldsymbol{\tau}) \;+\; \inf_{q\in L^2(\Omega)} \mathbf{J}_3(q)\,, \quad (3.6.9)$$

where

$$\mathbf{J}_1(\mathbf{s}) \;:=\; \frac{1}{2}\,\|\mathbf{s}\|^2_{[L^2(\Omega)]^{2\times 2}} \;-\; \int_\Omega (\boldsymbol{\sigma}_h \;-\; \mathcal{N}(\mathbf{t}_h) \;-\; p_h\,\mathbf{I}) : \mathbf{s}\,,$$

$$\mathbf{J}_2(\boldsymbol{\tau}) \;:=\; \frac{1}{2}\,\|\boldsymbol{\tau}\|^2_{H(\mathbf{div};\Omega)} \;-\; \left\{ \int_\Omega (\mathbf{t}_h + \boldsymbol{\gamma}_h) : \boldsymbol{\tau} \;+\; \int_\Omega \mathbf{u}_h \cdot \mathbf{div}\,\boldsymbol{\tau} \;+\; \langle \boldsymbol{\tau\nu}, \boldsymbol{\xi}_{\tilde{h}} \rangle_{\Gamma_N} \right\}\,,$$

and

$$\mathbf{J}_3(q) \;:=\; \frac{1}{2}\,\|q\|^2_{L^2(\Omega)} \;+\; \int_\Omega q\,\mathrm{tr}(\mathbf{t}_h)\,.$$

Next, it is easy to see that

$$\inf_{\mathbf{s}\in[L^2(\Omega)]^{2\times 2}} \mathbf{J}_1(\mathbf{s}) \;=\; -\frac{1}{2}\,\|\boldsymbol{\sigma}_h \;-\; \mathcal{N}(\mathbf{t}_h) \;-\; p_h\,\mathbf{I}\|^2_{[L^2(\Omega)]^{2\times 2}} \quad (3.6.10)$$

and

$$\inf_{q\in L^2(\Omega)} \mathbf{J}_3(q) \;=\; -\frac{1}{2}\,\|\mathrm{tr}(\mathbf{t}_h)\|^2_{L^2(\Omega)}\,. \quad (3.6.11)$$

On the other hand, since $\boldsymbol{\varphi}_h$ belongs to $[H^1(\Omega)]^2$, we apply Gauss' formula and get

$$\langle \boldsymbol{\tau\nu}, \boldsymbol{\varphi}_h \rangle_\Gamma \;-\; \sum_{T\in\mathcal{T}_h} \langle \boldsymbol{\tau\nu}_T, \boldsymbol{\varphi}_h \rangle_{\partial T} \;=\; 0\,, \quad (3.6.12)$$

where $\langle\cdot,\cdot\rangle_\Gamma$ stands for the duality pairing between $[H^{-1/2}(\Gamma)]^2$ and $[H^{1/2}(\Gamma)]^2$. Also, according to the hypotheses on $\boldsymbol{\varphi}_h$, we can write

$$\langle \boldsymbol{\tau\nu}, \boldsymbol{\varphi}_h + \boldsymbol{\xi}_{\tilde{h}} \rangle_\Gamma \;=\; \sum_{e\in E_h(\Gamma_D)} \langle \boldsymbol{\tau\nu}, \boldsymbol{\varphi}_h \rangle_e \;+\; \sum_{e\in E_h(\Gamma_N)} \langle \boldsymbol{\tau\nu}, \boldsymbol{\varphi}_h + \boldsymbol{\xi}_{\tilde{h}} \rangle_e\,. \quad (3.6.13)$$

In this way, incorporating (3.6.12) into the expression defining $\mathbf{J}_2$, and using identity (3.6.13), we find that

$$\mathbf{J}_2(\boldsymbol{\tau}) \;:=\; \sum_{T\in\mathcal{T}_h} \left\{ \frac{1}{2}\,\|\boldsymbol{\tau}_T\|^2_{H(\mathbf{div};T)} \;-\; F_{h,T}(\boldsymbol{\tau}_T) \right\} \quad \forall\,\boldsymbol{\tau}\in H(\mathbf{div};\Omega)\,,$$

where $\boldsymbol{\tau}_T$ denotes the restriction of $\boldsymbol{\tau}$ to $T\in\mathcal{T}_h$. Thus, since $H(\mathbf{div};\Omega)$ is contained in the space $\{\boldsymbol{\tau}\in[L^2(\Omega)]^{2\times 2}: \quad \boldsymbol{\tau}_T\in H(\mathbf{div};T) \quad \forall\,T\in\mathcal{T}_h\}$, we deduce that

$$\begin{aligned} \inf_{\boldsymbol{\tau}\in H(\mathbf{div};\Omega)} \mathbf{J}_2(\boldsymbol{\tau}) \;&\geq\; \sum_{T\in\mathcal{T}_h} \inf_{\boldsymbol{\tau}_T\in H(\mathbf{div};T)} \left\{ \frac{1}{2}\,\|\boldsymbol{\tau}_T\|^2_{H(\mathbf{div};T)} - F_{h,T}(\boldsymbol{\tau}_T) \right\} \\ &=\; -\frac{1}{2} \sum_{T\in\mathcal{T}_h} \|\hat{\boldsymbol{\sigma}}_T\|^2_{H(\mathbf{div};T)}\,. \end{aligned}$$

Finally, replacing (3.6.10), (3.6.11), and the above estimate, back into (3.6.9), we obtain (3.6.7) and complete the proof. □

It is important to remark here that the auxiliary function $\boldsymbol{\varphi}_h$ allows us to obtain the identities (3.6.12) and (3.6.13), which decompose the global duality $\langle \cdot, \cdot \rangle_\Gamma$ in terms of the local ones $\langle \cdot, \cdot \rangle_{\partial T}$ and $\langle \cdot, \cdot \rangle_e$. This kind of decomposition is not possible for the expression $\langle \boldsymbol{\tau\nu}, \boldsymbol{\xi}_{\tilde{h}} \rangle_{\Gamma_N}$ appearing in the original definition of the functional $\mathbf{J}_2$. Actually, this latter fact constitutes the main reason for introducing $\boldsymbol{\varphi}_h$ in the present setting.

The following theorem establishes an implicit a-posteriori error estimate for the mixed finite element scheme (3.5.1). It assumes Gâteaux differentiability of the nonlinear operator $\mathcal{A}_1$ (cf. (3.2.12)), and makes use of the a-priori estimate (3.4.8), the projection $(\bar{\mathbf{t}}, \bar{\boldsymbol{\sigma}}, \bar{p})$, and the associated upper bound provided by Theorem 3.6.1. The term *implicit* refers to the dependence on the unknown solutions of the local problems (3.6.5), which hold on the infinite dimensional space $H(\mathbf{div}; T)$.

Theorem 3.6.2 *In addition to the hypotheses of Theorem* 3.6.1*, assume that* $\mathcal{A}_1$ *has a continuous first order Gâteaux derivative* $\mathcal{D}\mathcal{A}_1$*. Then there exists* $C > 0$*, independent of* h *and* $\tilde{h}$*, such that*

$$C \, \|(\mathbf{t}, (\boldsymbol{\sigma}, p), (\mathbf{u}, \boldsymbol{\gamma}, \boldsymbol{\xi})) - (\mathbf{t}_h, (\boldsymbol{\sigma}_h, p_h), (\mathbf{u}_h, \boldsymbol{\gamma}_h, \boldsymbol{\xi}_{\tilde{h}}))\|^2 \,\leq\, \hat{\boldsymbol{\theta}}^2 \,:=\, \sum_{T \in \mathcal{T}_h} \hat{\boldsymbol{\theta}}_T^2$$

$$+ \, \|\boldsymbol{\sigma}_h \boldsymbol{\nu} \,-\, \mathbf{g}\|^2_{[H^{-1/2}(\Gamma_N)]^2} \,,$$

where, for each $T \in \mathcal{T}_h$*,*

$$\hat{\boldsymbol{\theta}}_T^2 \,:=\, \|\hat{\boldsymbol{\sigma}}_T\|^2_{H(\mathbf{div};T)} \,+\, \|\boldsymbol{\sigma}_h \,-\, \mathcal{N}(\mathbf{t}_h) \,-\, p_h \,\mathbf{I}\|^2_{[L^2(T)]^{2\times 2}} \,+\, \|\mathrm{tr}(\mathbf{t}_h)\|^2_{L^2(T)}$$

$$+ \, \|\mathbf{f} + \mathbf{div}\, \boldsymbol{\sigma}_h\|^2_{[L^2(T)]^2} \,+\, \|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_h^{\mathrm{t}}\|^2_{[L^2(T)]^{2\times 2}} \,.$$

Proof. We proceed similarly to the proofs of Theorem 5.3 in [48] and Lemma 3.2 in [40]. To this end, we first recall from Lemma 3.4.4 that $\mathcal{A}_1$ is strongly monotone and Lipschitz continuous, and hence $\mathcal{D}\mathcal{A}_1(\mathbf{r})(\cdot, \cdot)$ becomes a uniformly bounded and elliptic bilinear form on $X_1 \times X_1$ for all $\mathbf{r} \in X_1$. In addition, because of the continuity of $\mathcal{D}\mathcal{A}_1$, there exists $\bar{\mathbf{r}} \in X_1$ such that $\mathcal{D}\mathcal{A}_1(\bar{\mathbf{r}})(\mathbf{t} - \mathbf{t}_h, \mathbf{s}) \,=\, [\mathcal{A}_1(\mathbf{t}) - \mathcal{A}_1(\mathbf{t}_h), \mathbf{s}]$ for all $\mathbf{s} \in X_1$. Thus, applying the global inf-sup condition provided by (3.4.8) (cf. Theorem

3.4.1) to the linear operator obtained from the left-hand side of (3.2.9) (equivalently (3.6.3)) after replacing $\mathcal{A}_1$ by $\mathcal{D}\mathcal{A}_1(\bar{\mathbf{r}})$, and then using (3.6.4), we deduce that

$$C\ \|(\mathbf{t},(\boldsymbol{\sigma},p),(\mathbf{u},\boldsymbol{\gamma},\boldsymbol{\xi}))-(\mathbf{t}_h,(\boldsymbol{\sigma}_h,p_h),(\mathbf{u}_h,\boldsymbol{\gamma}_h,\boldsymbol{\xi}_{\tilde{h}}))\|$$

$$\leq \sup_{\substack{(\mathbf{s},(\boldsymbol{\tau},q),(\mathbf{v},\boldsymbol{\delta},\boldsymbol{\lambda}))\in\mathbf{X}\times M\\ \|(\mathbf{s},(\boldsymbol{\tau},q),(\mathbf{v},\boldsymbol{\delta},\boldsymbol{\lambda}))\|\leq 1}} \left\{ \langle(\bar{\mathbf{t}},\bar{\boldsymbol{\sigma}},\bar{p}),(\mathbf{s},\boldsymbol{\tau},q)\rangle_{\mathbf{X}} + [\mathbf{B}(\mathbf{t}-\mathbf{t}_h,\boldsymbol{\sigma}-\boldsymbol{\sigma}_h,p-p_h),(\mathbf{v},\boldsymbol{\delta},\boldsymbol{\lambda})] \right\},$$

which, according to the third equation of (3.2.9) and the definition of $\mathbf{B}$, yields

$$C\ \|(\mathbf{t},(\boldsymbol{\sigma},p),(\mathbf{u},\boldsymbol{\gamma},\boldsymbol{\xi}))-(\mathbf{t}_h,(\boldsymbol{\sigma}_h,p_h),(\mathbf{u}_h,\boldsymbol{\gamma}_h,\boldsymbol{\xi}_{\tilde{h}}))\|$$

$$\leq \sup_{\substack{(\mathbf{s},(\boldsymbol{\tau},q),(\mathbf{v},\boldsymbol{\delta},\boldsymbol{\lambda}))\in\mathbf{X}\times M\\ \|(\mathbf{s},(\boldsymbol{\tau},q),(\mathbf{v},\boldsymbol{\delta},\boldsymbol{\lambda}))\|\leq 1}} \left\{ \langle(\bar{\mathbf{t}},\bar{\boldsymbol{\sigma}},\bar{p}),(\mathbf{s},\boldsymbol{\tau},q)\rangle_{\mathbf{X}} + \int_\Omega (\mathbf{f}+\mathbf{div}\,\boldsymbol{\sigma}_h)\cdot\mathbf{v} + \int_\Omega \boldsymbol{\sigma}_h : \boldsymbol{\delta} \right.$$

$$\left. + \langle\boldsymbol{\sigma}_h\boldsymbol{\nu}-\mathbf{g},\boldsymbol{\lambda}\rangle_{\Gamma_N} \right\}.$$

Finally, noting that $\int_\Omega \boldsymbol{\sigma}_h : \boldsymbol{\delta} = \frac{1}{2}\int_\Omega (\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_h^{\mathtt{t}}) : \boldsymbol{\delta}$, and applying Cauchy-Schwarz's inequality and Theorem 3.6.1, we derive the required estimate and complete the proof. □

We now establish a-priori estimates for the solutions of the local problems (3.6.5).

Lemma 3.6.1 *There exists $C>0$, independent of h, $\tilde{h}$ and T, such that*

$$\|\hat{\boldsymbol{\sigma}}_T\|_{H(\mathbf{div};T)} \leq C\left\{\|\mathbf{t}_h+\boldsymbol{\gamma}_h-\nabla\boldsymbol{\varphi}_h\|^2_{[L^2(T)]^{2\times 2}} + \|\mathbf{u}_h-\boldsymbol{\varphi}_h\|^2_{[L^2(T)]^2}\right.$$

$$\left. + \sum_{e\in E(T)\cap E_h(\Gamma_D)} \|\boldsymbol{\varphi}_h\|^2_{[H_{00}^{1/2}(e)]^2} + \sum_{e\in E(T)\cap E_h(\Gamma_N)} \|\boldsymbol{\varphi}_h+\boldsymbol{\xi}_{\tilde{h}}\|^2_{[H_{00}^{1/2}(e)]^2}\right\}^{1/2}. \quad (3.6.14)$$

Furthermore, for any $\mathbf{z}\in[H^1(\Omega)\cap W^{1,s}(\Omega)]^2$, with $s>2$, such that $\mathbf{z}=\mathbf{0}$ on Γ_D, there holds

$$\|\hat{\boldsymbol{\sigma}}_T\|_{H(\mathbf{div};T)} \leq C\left\{ \|\mathbf{t}_h+\boldsymbol{\gamma}_h-\nabla\mathbf{z}\|^2_{[L^2(T)]^{2\times 2}} + \|\mathbf{u}_h-\mathbf{z}\|^2_{[L^2(T)]^2}\right.$$

$$\left. + \|\mathcal{J}_{h,T}(\mathbf{z})\|^2_{[H^{1/2}(\partial T)]^2}\right\}^{1/2}, \quad (3.6.15)$$

where $$\mathcal{J}_{h,T}(\mathbf{z}) := \begin{cases} \mathbf{0} & \text{on } \partial T\cap\Gamma_D \\ \mathbf{z}+\boldsymbol{\xi}_{\tilde{h}} & \text{on } \partial T\cap\Gamma_N \\ \mathbf{z}-\boldsymbol{\varphi}_h & \text{otherwise.}\end{cases}$$

Proof. It is similar to the proof of Lemma 5.4 in [48] since it reduces to bounding the norm of the linear functional $F_{h,T} \in H(\mathbf{div}:T)'$. In fact, we first notice, using the integration by parts formula in T, that

$$\langle \boldsymbol{\tau}\boldsymbol{\nu}_T, \boldsymbol{\varphi}_h \rangle_{\partial T} = \int_T \nabla \boldsymbol{\varphi}_h : \boldsymbol{\tau} \; + \; \int_T \boldsymbol{\varphi}_h \cdot \mathbf{div}\, \boldsymbol{\tau}\,,$$

and hence

$$\begin{aligned} &\int_T (\mathbf{t}_h + \boldsymbol{\gamma}_h) : \boldsymbol{\tau} \; + \; \int_T \mathbf{u}_h \cdot \mathbf{div}\, \boldsymbol{\tau} - \langle \boldsymbol{\tau}\boldsymbol{\nu}_T, \boldsymbol{\varphi}_h \rangle_{\partial T} \\ = &\int_T (\mathbf{t}_h + \boldsymbol{\gamma}_h - \nabla \boldsymbol{\varphi}_h) : \boldsymbol{\tau} \; + \; \int_T (\mathbf{u}_h - \boldsymbol{\varphi}_h) \cdot \mathbf{div}\, \boldsymbol{\tau}. \end{aligned} \tag{3.6.16}$$

Next, extending $\boldsymbol{\varphi}_h + \boldsymbol{\xi}_{\tilde{h}}$ by zero on $\partial T - e$, using the duality pairing $\langle \cdot, \cdot \rangle_{\partial T}$, and employing the norm $\|\boldsymbol{\varphi}\|_{[H^{1/2}(\partial T)]^2} := \inf \left\{ \|\mathbf{v}\|_{[H^1(T)]^2} \,:\, \mathbf{v} \in [H^1(T)]^2\,,\; \mathbf{v}|_{\partial T} = \boldsymbol{\varphi} \right\}$, we find that

$$|\langle \boldsymbol{\tau}\boldsymbol{\nu}_T, \boldsymbol{\varphi}_h + \boldsymbol{\xi}_{\tilde{h}} \rangle_e| = \left| \int_T \nabla \mathbf{v}_n : \boldsymbol{\tau} \; + \; \mathbf{v}_n \cdot \mathbf{div}\, \boldsymbol{\tau} \right|, \tag{3.6.17}$$

where $\{\mathbf{v}_n\}_{n \in \mathbb{N}} \subseteq [H^1(T)]^2$ is such that $\mathbf{v}_n|_{\partial T} = \boldsymbol{\varphi}_h + \boldsymbol{\xi}_{\tilde{h}}$ and

$$\|\mathbf{v}_n\|_{[H^1(T)]^2} \to \|\boldsymbol{\varphi}_h + \boldsymbol{\xi}_{\tilde{h}}\|_{[H^{1/2}_{00}(e)]^2} = \|\boldsymbol{\varphi}_h + \boldsymbol{\xi}_{\tilde{h}}\|_{[H^{1/2}(\partial T)]^2}.$$

Then, applying Cauchy-Schwarz inequality in (3.6.17), we obtain

$$|\langle \boldsymbol{\tau}\boldsymbol{\nu}_T, \boldsymbol{\varphi}_h + \boldsymbol{\xi}_{\tilde{h}} \rangle_e| \leq \|\boldsymbol{\tau}\|_{H(\mathbf{div};T)}\, \|\mathbf{v}_n\|_{[H^1(T)]^2},$$

which, taking $\lim_{n\to\infty}$, yields

$$|\langle \boldsymbol{\tau}\boldsymbol{\nu}_T, \boldsymbol{\varphi}_h + \boldsymbol{\xi}_{\tilde{h}} \rangle_e| \leq \|\boldsymbol{\tau}\|_{H(\mathbf{div};T)}\, \|\boldsymbol{\varphi}_h + \boldsymbol{\xi}_{\tilde{h}}\|_{[H^{1/2}_{00}(e)]^2}. \tag{3.6.18}$$

Similarly,

$$|\langle \boldsymbol{\tau}\boldsymbol{\nu}_T, \boldsymbol{\varphi}_h \rangle_e| \leq \|\boldsymbol{\tau}\|_{H(\mathbf{div};T)}\, \|\boldsymbol{\varphi}_h\|_{[H^{1/2}_{00}(e)]^2}. \tag{3.6.19}$$

Thus, the proof of (3.6.14) follows from (3.6.17), (3.6.19), and an application of Cauchy-Schwarz inequality in (3.6.16). The proof of (3.6.15) follows analogously. □

The following result shows that the reliable a-posteriori error estimate $\hat{\boldsymbol{\theta}}$ (cf. Theorem 3.6.2) is globally *quasi-efficient* in the sense that it is efficient up to terms depending on $(\mathbf{u} + \boldsymbol{\xi}_{\tilde{h}})$ on Γ_N and the traces of $(\mathbf{u} - \boldsymbol{\varphi}_h)$ on the element boundaries not contained in Γ.

Lemma 3.6.2 *Assume that* $\mathbf{u} \in [H^1(\Omega) \cap W^{1,s}(\Omega)]^2$, *with* $s > 2$. *Then, there exists* $C > 0$, *independent of* h *and* $\tilde{h}$, *such that for each* $T \in \mathcal{T}_h$ *there holds*

$$\begin{aligned} \hat{\boldsymbol{\theta}}_T^2 \;\le\; C \Big\{ & \|\mathbf{t} - \mathbf{t}_h\|^2_{[L^2(T)]^{2\times 2}} \;+\; \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|^2_{H(\mathbf{div};T)} \;+\; \|p - p_h\|^2_{L^2(T)} \\ + \quad & \|\mathbf{u} - \mathbf{u}_h\|^2_{[L^2(T)]^2} \;+\; \|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|^2_{[L^2(T)]^{2\times 2}} \;+\; \|\mathcal{J}_{h,T}(\mathbf{u})\|^2_{[H^{1/2}(\partial T)]^2} \Big\}, \end{aligned} \tag{3.6.20}$$

and hence, there also exists $\tilde{C} > 0$, *independent of* h *and* $\tilde{h}$, *such that*

$$\begin{aligned} \hat{\boldsymbol{\theta}}^2 \;\le\; \tilde{C} \Big\{ & \|(\mathbf{t}, (\boldsymbol{\sigma}, p), (\mathbf{u}, \boldsymbol{\gamma}, \boldsymbol{\xi})) - (\mathbf{t}_h, (\boldsymbol{\sigma}_h, p_h), (\mathbf{u}_h, \boldsymbol{\gamma}_h, \boldsymbol{\xi}_{\tilde{h}}))\|^2 \\ & + \sum_{T \in \mathcal{T}_h} \|\mathcal{J}_{h,T}(\mathbf{u})\|^2_{[H^{1/2}(\partial T)]^2} \Big\}. \end{aligned} \tag{3.6.21}$$

Proof. It proceeds similarly to the proof of Lemma 5.5 in [48]. In particular, (3.6.20) follows from a straightforward application of (3.6.15) (cf. Lemma 3.6.1) to $\mathbf{z} = \mathbf{u}$, and (3.6.21) arises after summing up (3.6.20) over all $T \in \mathcal{T}_h$, and using that $\|\boldsymbol{\sigma}_h \boldsymbol{\nu} - \mathbf{g}\|_{[H^{-1/2}(\Gamma_N)]^2} = \|(\boldsymbol{\sigma}_h - \boldsymbol{\sigma})\boldsymbol{\nu}\|_{[H^{-1/2}(\Gamma_N)]^2} \le \|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}\|_{H(\mathbf{div};\Omega)}$, by virtue of the trace theorem in $H(\mathbf{div};\Omega)$. $\square$

We are now in a position to establish a fully explicit and reliable a-posteriori error estimate.

Theorem 3.6.3 *In addition to the hypotheses of Theorem* 3.6.2, *assume that the Neumann data* $\mathbf{g} \in [L^2(\Gamma_N)]^2$. *Then there exists* $C > 0$, *independent of* h *and* $\tilde{h}$, *such that*

$$C\; \|(\mathbf{t}, (\boldsymbol{\sigma}, p), (\mathbf{u}, \boldsymbol{\gamma}, \boldsymbol{\xi})) - (\mathbf{t}_h, (\boldsymbol{\sigma}_h, p_h), (\mathbf{u}_h, \boldsymbol{\gamma}_h, \boldsymbol{\xi}_{\tilde{h}}))\|^2 \;\le\; \boldsymbol{\theta}^2 \;:=\; \sum_{T \in \mathcal{T}_h} \boldsymbol{\theta}_T^2\,,$$

where, for each $T \in \mathcal{T}_h$,

$$\begin{aligned} \boldsymbol{\theta}_T^2 \;:=\; & \|\mathbf{t}_h + \boldsymbol{\gamma}_h - \nabla \boldsymbol{\varphi}_h\|^2_{[L^2(T)]^{2\times 2}} + \|\mathbf{u}_h - \boldsymbol{\varphi}_h\|^2_{[L^2(T)]^2} + \sum_{e \in E(T) \cap E_h(\Gamma_D)} \|\boldsymbol{\varphi}_h\|^2_{[H_{00}^{1/2}(e)]^2} \\ + & \sum_{e \in E(T) \cap E_h(\Gamma_N)} \Big\{ \|\boldsymbol{\varphi}_h + \boldsymbol{\xi}_{\tilde{h}}\|^2_{[H_{00}^{1/2}(e)]^2} \;+\; \log\,[1 + C_{\tilde{h}}(\Gamma_N)]\; \tilde{h}_e\; \|\boldsymbol{\sigma}_h \boldsymbol{\nu} - \mathbf{g}\|^2_{[L^2(e)]^2} \Big\} \\ + & \|\boldsymbol{\sigma}_h \,-\, \mathcal{N}(\mathbf{t}_h) \,-\, p_h\,\mathbf{I}\|^2_{[L^2(T)]^{2\times 2}} + \|\mathrm{tr}(\mathbf{t}_h)\|^2_{L^2(T)} + \|\mathbf{f} + \mathbf{div}\,\boldsymbol{\sigma}_h\|^2_{[L^2(T)]^2} \\ + & \|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_h^{\mathtt{t}}\|^2_{[L^2(T)]^{2\times 2}}\,, \end{aligned}$$

and $\quad C_{\tilde{h}}(\Gamma_N) := \max\left\{ \frac{|\tilde{\Gamma}_i|}{|\tilde{\Gamma}_j|} : \quad |i-j| = 1, \quad i,j \in \{1,\cdots,m\} \right\}.$

Proof. We first bound the Neumann residual $\|\boldsymbol{\sigma}_h\boldsymbol{\nu} - \mathbf{g}\|^2_{[H^{-1/2}(\Gamma_N)]^2}$ by local computable quantities on the segments $\tilde{\Gamma}_j$, $j \in \{1,\cdots,m\}$. Because of the definition of the subspace $\mathcal{M}^{\boldsymbol{\sigma}}_{1,h}$, it is easy to see that $\boldsymbol{\sigma}_h\boldsymbol{\nu} - \mathbf{g} \in [L^2(\Gamma_N)]^2$. In addition, taking $\mathbf{v} = \mathbf{0}$ and $\boldsymbol{\delta} = \mathbf{0}$ in the third equation of (3.5.1), we find that $\boldsymbol{\sigma}_h\boldsymbol{\nu} - \mathbf{g}$ is $[L^2(\Gamma_N)]^2$-orthogonal to the finite element subspace $M^{\boldsymbol{\xi}}_{\tilde{h}}$, and then, a straightforward application of Theorem 2 in [25] yields

$$\|\boldsymbol{\sigma}_h\boldsymbol{\nu} - \mathbf{g}\|^2_{[H^{-1/2}(\Gamma_N)]^2} \leq \log\left[1 + C_{\tilde{h}}(\Gamma_N)\right] \sum_{j=1}^{m} |\tilde{\Gamma}_j| \, \|\boldsymbol{\sigma}_h\boldsymbol{\nu} - \mathbf{g}\|^2_{[L^2(\tilde{\Gamma}_j)]^2} \, .$$

Since each edge $e \in E_h(\Gamma_N)$ is contained in a segment $\tilde{\Gamma}_j$, for some $j \in \{1,\cdots,m\}$, we find that $\sum_{j=1}^{m} |\tilde{\Gamma}_j| \, \|\boldsymbol{\sigma}_h\boldsymbol{\nu} - \mathbf{g}\|^2_{[L^2(\tilde{\Gamma}_j)]^2} = \sum_{e\in E_h(\Gamma_N)} \tilde{h}_e \, \|\boldsymbol{\sigma}_h\boldsymbol{\nu} - \mathbf{g}\|^2_{[L^2(e)]^2}$, whence the above inequality becomes

$$\|\boldsymbol{\sigma}_h\boldsymbol{\nu} - \mathbf{g}\|^2_{[H^{-1/2}(\Gamma_N)]^2} \leq \log\left[1 + C_{\tilde{h}}(\Gamma_N)\right] \sum_{e\in E_h(\Gamma_N)} \tilde{h}_e \, \|\boldsymbol{\sigma}_h\boldsymbol{\nu} - \mathbf{g}\|^2_{[L^2(e)]^2} \, . \tag{3.6.22}$$

The rest of the proof follows easily from (3.6.22), Theorem 3.6.2, and the a-priori estimate (3.6.14) provided in Lemma 3.6.1. □

At this point we remark that, although the a-posteriori error estimate $\hat{\boldsymbol{\theta}}$ is reliable and *quasi-efficient*, its applicability is limited by the non-local character of the term $\|\boldsymbol{\sigma}_h\boldsymbol{\nu} - \mathbf{g}\|^2_{[H^{-1/2}(\Gamma_N)]^2}$, and also by the fact that its computation requires the knowledge of the exact solutions of the local problems (3.6.5), which live all in the infinite dimensional space $H(\mathbf{div};T)$. The latter difficulty could be partially overcomed by using h, p or $h-p$ versions of the finite element method to solve (3.6.5) numerically, which yields approximations of $\hat{\boldsymbol{\theta}}_T$ and hence of $\hat{\boldsymbol{\theta}}$.

On the other hand, the advantage of the a-posteriori error estimate $\boldsymbol{\theta}$, which is certainly reliable but not necessarily *quasi-efficient*, is that it does not need the exact or any approximate solution of (3.6.5), but a suitable auxiliary function $\boldsymbol{\varphi}_h$. To this respect, we adopt the criterion of enforcing the *quasi-efficiency* of $\hat{\boldsymbol{\theta}}$ so that it becomes closer to full efficiency. According to Lemma 3.6.2, one should require, in particular, that the local traces of $\boldsymbol{\varphi}_h$ be as close as possible to the corresponding traces of the exact solution $\mathbf{u}$. However, since $\mathbf{u}$ is certainly unknown, the above must be understood only in a heuristic sense. Therefore, taking into account the polynomial degrees of the finite element subspaces, we propose the following procedure to choose $\boldsymbol{\varphi}_h$. We first compute local functions $\tilde{\boldsymbol{\varphi}}_{h,T}$ for each $T \in \mathcal{T}_h$, satisfying the following conditions:

1. $\tilde{\boldsymbol{\varphi}}_{h,T} \in [\tilde{\mathbf{P}}_3(T)]^2$, where $\tilde{\mathbf{P}}_3(T) := \text{span}\{1, x_1, x_2, x_1^2, x_2^2, x_1^3, x_2^3\}$.
2. $\nabla \tilde{\boldsymbol{\varphi}}_{h,T}$ is the $[L^2(T)]^{2\times 2}$-projection of $(\mathbf{t}_h + \boldsymbol{\gamma}_h)|_T$ onto the space $\nabla [\tilde{\mathbf{P}}_3(T)]^2$.
3. $\tilde{\boldsymbol{\varphi}}_{h,T}(\bar{\mathbf{x}}_T) = \mathbf{u}_h|_T$, where $\bar{\mathbf{x}}_T$ is the barycenter of the triangle T.

We remark that each $\tilde{\boldsymbol{\varphi}}_{h,T}$ is uniquely determined by these constraints. Then, we define $\boldsymbol{\varphi}_h$ as the unique function in $[C(\bar{\Omega})]^2$ such that:

1. $\boldsymbol{\varphi}_h|_T \in [\mathbf{P}_2(T)]^2$ for each $T \in \mathcal{T}_h$.
2. For each vertex $\bar{\mathbf{x}}$ of $\mathcal{T}_h$ lying on Γ_N and for each middle point $\bar{\mathbf{x}}$ of the edges $e \in E_h(\Gamma_N)$, $\boldsymbol{\varphi}_h(\bar{\mathbf{x}}) = -\boldsymbol{\xi}_{\tilde{h}}(\bar{\mathbf{x}})$.
3. For each vertex $\bar{\mathbf{x}}$ of $\mathcal{T}_h$ lying on Γ_D and for each middle point $\bar{\mathbf{x}}$ of the edges $e \in E_h(\Gamma_D)$, $\boldsymbol{\varphi}_h(\bar{\mathbf{x}}) = (0,0)^{\mathtt{t}}$.
4. For each vertex $\bar{\mathbf{x}}$ of $\mathcal{T}_h$ lying in Ω and for each middle point $\bar{\mathbf{x}}$ of the edges $e \in E_h(\Omega)$, $\boldsymbol{\varphi}_h(\bar{\mathbf{x}})$ is the average of the values $\tilde{\boldsymbol{\varphi}}_{h,T}(\bar{\mathbf{x}})$ on all the triangles $T \in \mathcal{T}_h$ to which $\bar{\mathbf{x}}$ belongs.

We end this section by observing that the assumption of uniformity of the meshes on Γ_N, which is needed for the inverse inequalities (3.5.17) and (3.5.18), could be violated by an adaptive algorithm using our a-posteriori error estimate, moreover if the singularities hold on or near the Neumann boundary Γ_N. However, the examples shown below in Chapter 4 constitute numerical evidences for conjecturing that the above is more a technical assumption than a real requirement for the good performance of the method.

Chapter 4

Numerical results for a dual-mixed method in nonlinear incompressible elasticity

In this chapter we use the fully explicit and reliable a-posteriori error estimator given in Chapter 3 and define an adaptive algorithm induced by this estimator to compute the mixed finite element solutions of the nonlinear problem studied there. We report several numerical examples that illustrate the good performance of this algorithm and its capability to localize singularities, even in some cases not covered by our theory.

4.1 Introduction

We recall from Chapter 3 the spaces for the continuous and discrete mixed formulations (3.2.9) and (3.5.1), respectively, of our nonlinear boundary value problem of interest (3.1.3), which introduce $\mathbf{t} := \mathbf{e}(\mathbf{u})$, $\boldsymbol{\gamma} := \frac{1}{2}(\nabla\mathbf{u} - (\nabla\mathbf{u})^{\mathtt{t}})$, and $\boldsymbol{\xi} := -\mathbf{u}|_{\Gamma_N}$ as additional unknowns.

For the continuous formulation, the variables and spaces are; $\mathbf{t} \in X_1$, $(\boldsymbol{\sigma}, p) \in \mathcal{M}_1$ and $(\mathbf{u}, \boldsymbol{\gamma}, \boldsymbol{\xi}) \in M$, where $X_1 := [L^2(\Omega)]^{2\times 2}$, $\mathcal{M}_1 := H(\mathbf{div};\Omega) \times L^2(\Omega)$ and $M := [L^2(\Omega)]^2 \times \mathcal{R} \times [H_{00}^{1/2}(\Gamma_N)]^2$, with $\mathcal{R} := \{\boldsymbol{\delta} \in [L^2(\Omega)]^{2\times 2} : \ \boldsymbol{\delta} + \boldsymbol{\delta}^{\mathtt{t}} = \mathbf{0}\}$ and

$$[H_{00}^{1/2}(\Gamma_N)]^2 := \{\mathbf{v}|_{\Gamma_N} \ : \ \mathbf{v} \in [H^1(\Omega)]^2, \ \mathbf{v} = \mathbf{0} \text{ on } \Gamma_D\}.$$

For the discrete formulation corresponding we have; $\mathbf{t}_h \in X_{1,h}$, $(\boldsymbol{\sigma}_h, p_h) \in \mathcal{M}_{1,h}$ and $(\mathbf{u}_h, \boldsymbol{\gamma}_h, \boldsymbol{\xi}_{\tilde{h}}) \in M_{h,\tilde{h}}$, where $X_{1,h} := \{ \mathbf{s} \in [L^2(\Omega)]^{2\times 2} : \mathbf{s}|_T \in \mathcal{X}(T) \quad \forall\, T \in \mathcal{T}_h \}$, $\mathcal{M}_{1,h} := \mathcal{M}_{1,h}^{\boldsymbol{\sigma}} \times \mathcal{M}_{1,h}^{p}$ and $M_{h,\tilde{h}} := M_h^{\mathbf{u}} \times M_h^{\boldsymbol{\gamma}} \times M_{\tilde{h}}^{\boldsymbol{\xi}}$, with

$$
\begin{aligned}
\mathcal{X}(T) &:= [\mathbb{P}_0(T)]^{2\times 2} \oplus [\mathbb{P}_0(T)\,\mathbf{curl}^{\mathtt{t}}\, b_T]^2 \oplus \operatorname{dev}\left([\mathbb{P}_0(T)\,\mathbf{curl}^{\mathtt{t}}\, b_T]^2\right), \\
\mathcal{M}_{1,h}^{\boldsymbol{\sigma}} &:= \left\{ \boldsymbol{\tau}_h \in H(\mathbf{div};\Omega) : \boldsymbol{\tau}_h|_T \in [\mathcal{RT}_0(T)^{\mathtt{t}}]^2 \oplus [\mathbb{P}_0(T)\,\mathbf{curl}^{\mathtt{t}}\, b_T]^2 \ \forall\, T \in \mathcal{T}_h \right\}, \\
\mathcal{M}_{1,h}^{p} &:= \left\{ q_h \in L^2(\Omega) : \quad q_h|_T \in \mathbb{P}_0(T) \qquad \forall\, T \in \mathcal{T}_h \right\}, \\
M_h^{\mathbf{u}} &:= \left\{ \mathbf{v}_h \in [L^2(\Omega)]^2 : \quad \mathbf{v}_h|_T \in [\mathbb{P}_0(T)]^2 \quad \forall\, T \in \mathcal{T}_h \right\}, \\
M_h^{\boldsymbol{\gamma}} &:= \left\{ \begin{pmatrix} 0 & \delta_h \\ -\delta_h & 0 \end{pmatrix} \in [H^1(\Omega)]^{2\times 2} : \quad \delta_h|_T \in \mathbb{P}_1(T) \quad \forall\, T \in \mathcal{T}_h \right\} \quad \text{and} \\
M_{\tilde{h}}^{\boldsymbol{\xi}} &:= \left\{ \boldsymbol{\lambda}_{\tilde{h}} \in [H_{00}^{1/2}(\Gamma_N)]^2 : \quad \boldsymbol{\lambda}_{\tilde{h}}|_{\tilde{\Gamma}_j} \in [\mathbb{P}_1(\tilde{\Gamma}_j)]^2 \quad \forall\, j \in \{1, \cdots, m\} \right\}.
\end{aligned}
$$

Also, we recall from Theorem 3.6.3 the fully explicit and reliable a-posteriori error estimate $\boldsymbol{\theta}$ of the scheme (3.5.1), given by $\boldsymbol{\theta} := \left\{ \sum_{T\in\mathcal{T}_h} \boldsymbol{\theta}_T^2 \right\}^{1/2}$, where for each $T \in \mathcal{T}_h$,

$$
\begin{aligned}
\boldsymbol{\theta}_T^2 :=\ & \|\mathbf{t}_h + \boldsymbol{\gamma}_h - \nabla \boldsymbol{\varphi}_h\|_{[L^2(T)]^{2\times 2}}^2 + \|\mathbf{u}_h - \boldsymbol{\varphi}_h\|_{[L^2(T)]^2}^2 + \sum_{e\in E(T)\cap E_h(\Gamma_D)} \|\boldsymbol{\varphi}_h\|_{[H_{00}^{1/2}(e)]^2}^2 \\
&+ \sum_{e\in E(T)\cap E_h(\Gamma_N)} \left\{ \|\boldsymbol{\varphi}_h + \boldsymbol{\xi}_{\tilde{h}}\|_{[H_{00}^{1/2}(e)]^2}^2 + \log\left[1 + C_{\tilde{h}}(\Gamma_N)\right]\, \tilde{h}_e\, \|\boldsymbol{\sigma}_h \boldsymbol{\nu} - \mathbf{g}\|_{[L^2(e)]^2}^2 \right\} \\
&+ \|\boldsymbol{\sigma}_h - \mathcal{N}(\mathbf{t}_h) - p_h\, \mathbf{I}\|_{[L^2(T)]^{2\times 2}}^2 + \|\operatorname{tr}(\mathbf{t}_h)\|_{L^2(T)}^2 + \|\mathbf{f} + \mathbf{div}\, \boldsymbol{\sigma}_h\|_{[L^2(T)]^2}^2 \\
&+ \|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_h^{\mathtt{t}}\|_{[L^2(T)]^{2\times 2}}^2,
\end{aligned}
$$

and $C_{\tilde{h}}(\Gamma_N) := \max\left\{ \frac{|\tilde{\Gamma}_i|}{|\tilde{\Gamma}_j|} : \quad |i-j| = 1, \quad i,j \in \{1, \cdots, m\} \right\}$. Here, h_e denotes the diameter of the edge $e \in E_h$, and for each $e \in E_h(\Gamma_N)$ we set $\tilde{h}_e := |\tilde{\Gamma}_j|$, where $\tilde{\Gamma}_j$ is the segment containing e.

The purpose of this chapter is to provide several numerical results illustrating the performance of the mixed finite element scheme (3.5.1) and the explicit a posteriori error estimate given by $\boldsymbol{\theta}$, choosing the auxiliary function $\boldsymbol{\varphi}_h$ as described in the Section 3.6 of Chapter 3.

4.2 Notations and adaptive algorithm

In this section we provide some further notations. The variable N stands for the number of degrees of freedom defining the finite element subspaces $X_{1,h}$, $\mathcal{M}_{1,h}$, and $M_{h,\tilde{h}}$, that is $N := 13\,(\text{number of triangles of } \mathcal{T}_h) + 2\,(\text{number of edges of } \mathcal{T}_h) +$ number of nodes of $\mathcal{T}_h \;+\; 2\,(m-1)$, whereas the individual and total errors are denoted by:

$$\mathbf{e}(\mathbf{t}) := \|\mathbf{t}-\mathbf{t}_h\|_{[L^2(\Omega)]^{2\times 2}}\,, \quad \mathbf{e}(\boldsymbol{\sigma}) := \|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{H(\mathbf{div};\Omega)}\,, \quad \mathbf{e}(p) := \|p-p_h\|_{L^2(\Omega)}$$

$$\mathbf{e}(\mathbf{u}) := \|\mathbf{u}-\mathbf{u}_h\|_{[L^2(\Omega)]^2}\,, \quad \mathbf{e}(\boldsymbol{\gamma}) := \|\boldsymbol{\gamma}-\boldsymbol{\gamma}_h\|_{[L^2(\Omega)]^{2\times 2}}\,, \quad \mathbf{e}(\boldsymbol{\xi}) := \|\boldsymbol{\xi}-\boldsymbol{\xi}_{\tilde{h}}\|_{[H_{00}^{1/2}(\Gamma_N)]^2}\,,$$

and

$$\mathbf{e} := \left\{[\mathbf{e}(\mathbf{t})]^2 + [\mathbf{e}(\boldsymbol{\sigma})]^2 + [\mathbf{e}(p)]^2 + [\mathbf{e}(\mathbf{u})]^2 + [\mathbf{e}(\boldsymbol{\gamma})]^2 + [\mathbf{e}(\boldsymbol{\xi})]^2\right\}^{1/2},$$

where $(\mathbf{t},(\boldsymbol{\sigma},p),(\mathbf{u},\boldsymbol{\gamma},\boldsymbol{\xi})) \in X_1 \times \mathcal{M}_1 \times M$ and $(\mathbf{t}_h,(\boldsymbol{\sigma}_h,p_h),(\mathbf{u}_h,\boldsymbol{\gamma}_h,\boldsymbol{\xi}_{\tilde{h}})) \in X_{1,h} \times \mathcal{M}_{1,h} \times M_{h,\tilde{h}}$ are the unique solutions of the continuous and discrete mixed formulations (3.2.9) and (3.5.1), respectively.

Also, given two consecutive triangulations with degrees of freedom N and N' and corresponding total errors $\mathbf{e}$ and $\mathbf{e}'$, respectively, we define the experimental rate of convergence by

$$r(\mathbf{e}) := -2\,\frac{\log(\mathbf{e}/\mathbf{e}')}{\log(N/N')}\,.$$

On the other hand, the adaptive algorithm to be used in the computation of the solutions of (3.5.1) reads as follows (see [71]):

1. Start with a coarse mesh $\mathcal{T}_h$.

2. Solve the Galerkin scheme (3.5.1) for the current mesh $\mathcal{T}_h$.

3. Compute $\boldsymbol{\theta}_T$ for each triangle $T \in \mathcal{T}_h$.

4. Consider stopping criterion and decide to finish or go to next step.

5. Use *blue-green* procedure to refine each element $T' \in \mathcal{T}_h$ whose local indicator $\boldsymbol{\theta}_{T'}$ satisfies $\boldsymbol{\theta}_{T'} \geq \frac{1}{2} \max\{\boldsymbol{\theta}_T : \quad T \in \mathcal{T}_h\}$.

6. Define resulting mesh as the new $\mathcal{T}_h$ and go to step 2.

4.3 Numerical examples

In this section we provide the numerical examples.

It is important to mention here that the $[H_{00}^{1/2}(e)]^2$-norm needed for the computation of $\boldsymbol{\theta}_T^2$ is approximated by the well known interpolation estimate

$$\| \cdot \|^2_{[H_{00}^{1/2}(e)]^2} \leq \| \cdot \|_{[L^2(e)]^2} \, \| \cdot \|_{[H_0^1(e)]^2}.$$

An analogue estimate is used to compute the error $\mathbf{e}(\boldsymbol{\xi}) = \|\boldsymbol{\xi} - \boldsymbol{\xi}_{\tilde{h}}\|_{[H_{00}^{1/2}(\Gamma_N)]^2}$.

In what follows, we consider either a linear or a pure nonlinear version of the boundary value problem (3.1.3). In the first case we take $\mathcal{N}(\mathbf{r}) = 2\,\mathbf{r}$, while in the second one we let $\mathcal{N}(\mathbf{r}) := \left(\frac{3}{2} - \mu(\|\mathrm{dev}\,\mathbf{r}\|^2)\right) \mathrm{tr}(\mathbf{r})\,\mathbf{I} + 2\,\mu(\|\mathrm{dev}\,\mathbf{r}\|^2)\,\mathbf{r}$ for all $\mathbf{r} \in [L^2(\Omega)]^{2\times 2}$, with $\mu : \mathbb{R}^+ \to \mathbb{R}$ being the nonlinear Lamé function defined by $\mu(t) = 1 + \dfrac{1}{2(1+t)}$ for all $t \in \mathbb{R}^+$, which corresponds to a hyperelastic material whose constitutive equation is given by the Hencky-von Mises stress-strain relation (see [15], [41], [48]). Since $\mu \in C^1(\mathbb{R}^+)$ we find that the corresponding nonlinear operator $\mathcal{A}_1$ has a continuous first order Gâteaux derivative $\mathcal{D}\mathcal{A}_1$. In addition, it is easy to see that there exists constants μ_1, μ_2 such that $0 < \mu_1 \leq \mu(t) < \frac{3}{2}$ and $0 < \mu_1 \leq \mu(t) + 2t\mu'(t) \leq \mu_2$ for all $t \in \mathbb{R}^+$, which allows to show (see, e.g. Lemma 5.1 in [48]) that $\mathcal{A}_1$ becomes strongly monotone and Lipschitz continuous.

We now specify the examples to be considered. In the first one we consider the linear version of the boundary value problem (3.1.3) in the domain $\Omega :=]0,1[^2$, with $\Gamma_D := [0,1] \times \{0\} \cup \{0\} \times [0,1]$ and $\Gamma_N := \Gamma - \Gamma_D$, and choose the data $\mathbf{f}$ and $\mathbf{g}$ so that the exact solution is given by

$$\begin{cases} \mathbf{u}(\mathbf{x}) := & x_1\, x_2\, (2.1 - x_1 - x_2)^{-2}\, (x_1\, (4.2 - 2x_1 - x_2), -x_2\, (4.2 - x_1 - 2x_2))\,, \\ p(\mathbf{x}) := & (x_1^2 + x_2^2 + 0.001)^{-1}\,, \end{cases}$$

for all $\mathbf{x} := (x_1, x_2) \in \Omega$. We notice that $\mathbf{u}$ is divergence free in Ω and has a singular behaviour in an exterior neighborhood of the point $(1,1) \in \Omega$. Also, the function p has a singular behaviour in an exterior neighborhood of $(0,0) \in \Omega$.

The second example deals with the pure nonlinear version of the boundary value problem (3.1.3) in the same domain of example 1, that is $\Omega :=]0,1[^2$, with $\Gamma_D := [0,1] \times \{0\} \cup \{0\} \times [0,1]$ and $\Gamma_N := \Gamma - \Gamma_D$. We choose the data $\mathbf{f}$ and $\mathbf{g}$ so that the exact solution is given by

$$\begin{cases} \mathbf{u}(\mathbf{x}) := & x_1\, x_2\, (x_2 - 1.1)^{-2}\, (x_1\, (x_2 - 2.2), -2\, x_2\, (x_2 - 1.1))\,, \\ p(\mathbf{x}) := & x_1 + x_2\,, \end{cases}$$

for all $\mathbf{x} := (x_1, x_2) \in \Omega$. We observe here that $\mathbf{u}$ is divergence free in Ω and has a singular behaviour in an exterior neighborhood of the segment $]0,1[\times\{1\}$.

The third example corresponds to a benchmark problem taken from Chapter 4 in [71], which refers to the pure nonlinear version of the boundary value problem (3.1.3) in the L-shaped domain $\Omega :=]-1,1[^2 - [0,1] \times [-1,0]$, with $\Gamma_D := [0,1] \times \{0\} \cup \{0\} \times [-1,0]$ and $\Gamma_N := \Gamma - \Gamma_D$. The data $\mathbf{f}$ and $\mathbf{g}$ are chosen so that $(\mathbf{u}, p)$ is given in polar coordinates by

$$\begin{cases} \mathbf{u}(r,\theta) := & r^\alpha \Big\{ (1+\alpha)\, z(\theta)\, (\sin(\theta), -cos(\theta)) \;+\; z'(\theta)\, (\cos(\theta), \sin(\theta)) \Big\}\,, \\ p(r,\theta) := & \dfrac{r^{\alpha-1}}{(1-\alpha)} \Big\{ (1+\alpha)^2\, z'(\theta) \;+\; z'''(\theta) \Big\}\,, \end{cases}$$

where $\alpha = 856399/1572864 \approx 0.54448$, and the function z is defined, with $\omega = 3\pi/2$, as follows

$$z(\theta) \;:=\; \frac{\sin((1+\alpha)\theta)\, \cos(\alpha\omega)}{(1+\alpha)} - \cos((1+\alpha)\theta) - \frac{\sin((1-\alpha)\theta)\, \cos(\alpha\omega)}{(1-\alpha)} + \cos((1-\alpha)\theta)\,.$$

It is not difficult to see that $\mathbf{u}$ is divergence free in Ω and that p and the partial derivatives of $\mathbf{u}$ have a singular behaviour at the re-entrant corner $(0,0) \in \Omega$. Also, we note that $\mathbf{u}$ vanishes at Γ_D, which holds for $\theta = 0$ and $\theta = 3\pi/2$.

Finally, the fourth example to be considered is given by the computational domain $\Omega := \big\{(x_1, x_2) \in \mathbb{R}^2 \,:\, 0 < x_1 < 2\,, 0 < x_2 < 2\,,\; x_1^2 + x_2^2 > 1\big\}$, with $\Gamma_D := \{\, x_1 > 0\,, x_2 > 0 \;:\; x_1^2 + x_2^2 = 1\,\}$ and $\Gamma_N := \Gamma - \Gamma_D$, representing a fourth of the total geometry of a square membrane containing a nonlinear incompressible material with a circular hole in the centre, which refers to the pure nonlinear version of the boundary value problem (3.1.3). The data $\mathbf{f}$ and $\mathbf{g}$ are chosen so that the exact solution is given by

$$\begin{cases} \mathbf{u}(\mathbf{x}) := & 5\,(1 - x_1^2 - x_2^2)\, e^{-5\,(1-x_1^2-x_2^2)^2}\, (x_1, -x_2)\,, \\ p(\mathbf{x}) := & \sin(x_1\, x_2)\,, \end{cases}$$

for all $\mathbf{x} := (x_1, x_2) \in \Omega$. We observe here that $\mathbf{u}$ is divergence free in Ω and has an inner layer around the unit circle.

The numerical results given below were obtained in a *Compaq Alpha ES40 Parallel Computer* using a Matlab code. We remark that in the pure nonlinear case, the mixed finite element scheme (3.5.1), which becomes a nonlinear algebraic system with N unknowns, is solved by Newton's method with the initial guess given by the solution of the associated linear problem, and setting 10^{-3} as the tolerance for the relative error. Also, according to the requirement established in Theorem 3.5.2 for the mesh sizes h and $\tilde{h}$, and since the constant C_0 mentioned there is not explicitly known, we simply adopt the criterion of setting a vertex of the independent partition $\left\{\tilde{\Gamma}_1, \tilde{\Gamma}_2, \cdots, \tilde{\Gamma}_m\right\}$ every two vertices of $\mathcal{T}_h$ on Γ_N. We note that this choice works out well in each one of the examples shown here. In addition, there is no need of taking sufficiently small values of $\tilde{h}$ (as technically suggested by the inequality $\tilde{h} \leq h_0$ in Theorem 3.5.2) since the resulting discrete schemes are all well posed.

In Tables 4.1-4.8 we give the individual errors (computed on each triangle using a 7 points Gaussian quadrature rule (see [70])), the effectivity index $\boldsymbol{\theta}/\mathbf{e}$, and the corresponding experimental rates of convergence for the uniform and adaptive (red-blue-green) refinements as applied to the four examples. We remark that the errors of the adaptive procedure decrease much faster than those obtained by the uniform one. Furthermore, the effectivity indexes all remain bounded above and below, which confirms the reliability of $\boldsymbol{\theta}$ and provides numerical evidences for its eventual efficiency. It is also interesting to observe that the dominant component of the total error $\mathbf{e}$ is given by $\mathbf{e}(\boldsymbol{\sigma})$, which is particularly notorious in Examples 1 and 2. Next, in Figures 4.1, 4.3, 4.5 and 4.7 we display the total error $\mathbf{e}$ vs. the degrees of freedom N for both refinements. The faster decreasing of $\mathbf{e}$ observed in the adaptive algorithm is certainly in agreement with the individual errors and the experimental rates of convergence provided in the tables. As shown by the values of $r(\mathbf{e})$ in each example, the adaptive method is able to recover the quasi-optimal rate of convergence $O(h)$ for the global error $\mathbf{e}$.

On the other hand, Figures 4.2, 4.4, 4.6 and 4.8 show some intermediate meshes obtained with the adaptive refinement. It is interesting to confirm, as expected, that the method is able to recognize all the singularities of the solution pair $(\mathbf{u}, p)$. In particular, this fact is observed in Example 1 (cf. Figure 4.2) where the adapted meshes are highly refined around the boundary points $(0,0)$ and $(1,1)$, in whose outer neighborhoods the singularities live. Similarly, the adapted meshes obtained in Example 2 (cf. Figure 4.4) concentrate the corresponding refinements mainly around the segment $]1/4,1[\times\{1\}$. The lack of finer refinements near the point $(0,1)$ is due to the fact that $\mathbf{u}$ vanishes precisely at $x_1 = 0$, which attenuates there the effect of the singular term $(x_2 - 1.1)^{-2}$. Also, it is clear from Figure 4.6 that the adapted meshes obtained in Example 3 are highly refined around the singularity of p. Finally, as expected, the corresponding adaptive refinement algorithm is able to recognize the boundary layer in Example 4. Indeed, as can be seen in Figure 4.8, the adapted meshes are highly refined around the unit circle. We also notice here that the refinement identifies a thin band on a interior neighborhood of the boundary Γ_D, which corresponds to the flat behaviour of the solution caused by the power 2 in the exponent of the exponential function.

Summarizing, the examples presented in this chapter strongly demonstrate that the adaptive algorithm is much more efficient than a uniform discretization procedure when solving the dual-mixed finite element scheme (3.5.1).

Table 4.1: Individual errors, effectivity index, and rate of convergence for the **uniform** refinement (Example 1).

N	$\mathbf{e}(\mathbf{t})$	$\mathbf{e}(\boldsymbol{\sigma})$	$\mathbf{e}(p)$	$\mathbf{e}(\mathbf{u})$	$\mathbf{e}(\gamma)$	$\mathbf{e}(\boldsymbol{\xi})$	$\boldsymbol{\theta}/\mathbf{e}$	$r(\mathbf{e})$
147	162.6877	1516.3625	124.0801	68.8115	152.7639	627.6659	0.9663	—
559	200.0505	2343.6924	260.6550	71.1618	256.7651	405.4395	0.9744	—
2175	177.8327	2172.4493	225.8487	53.1257	282.5848	262.8864	0.9702	0.1206
8575	106.8398	1424.7597	136.3600	26.2823	184.0981	129.5243	0.9725	0.6217
34047	46.6777	779.9958	59.1873	9.3127	83.7380	47.2525	0.9824	0.8846
135679	16.6691	394.2527	20.3800	2.6694	29.6561	13.8300	0.9919	0.9964

Table 4.2: Individual errors, effectivity index, and rate of convergence for the **adaptive** refinement (Example 1).

N	$\mathbf{e}(\mathbf{t})$	$\mathbf{e}(\boldsymbol{\sigma})$	$\mathbf{e}(p)$	$\mathbf{e}(\mathbf{u})$	$\mathbf{e}(\gamma)$	$\mathbf{e}(\boldsymbol{\xi})$	$\boldsymbol{\theta}/\mathbf{e}$	$r(\mathbf{e})$
147	162.6877	1516.3625	124.0801	68.8115	152.7639	627.6659	0.9663	—
289	182.6072	2331.9482	233.1652	70.9218	283.8341	414.0999	0.9900	—
427	192.0252	2180.0027	236.1894	67.1257	320.1976	316.3344	0.9846	0.3460
569	100.2300	1431.3577	115.7613	30.3657	229.8283	151.8867	0.9833	2.9773
1275	44.7587	801.5730	52.1878	13.2393	72.1807	58.1603	0.9938	1.4707
2855	20.6043	430.6968	24.1273	5.1557	32.9039	22.1227	0.9935	1.5496
6768	8.3688	227.8662	8.5795	2.1178	9.0928	8.5898	0.9984	1.4845
21257	3.9282	121.0815	3.6799	0.8114	3.5828	2.7914	0.9994	1.1070
76251	1.8294	62.1376	1.6067	0.2588	1.2742	0.7694	0.9996	1.0455

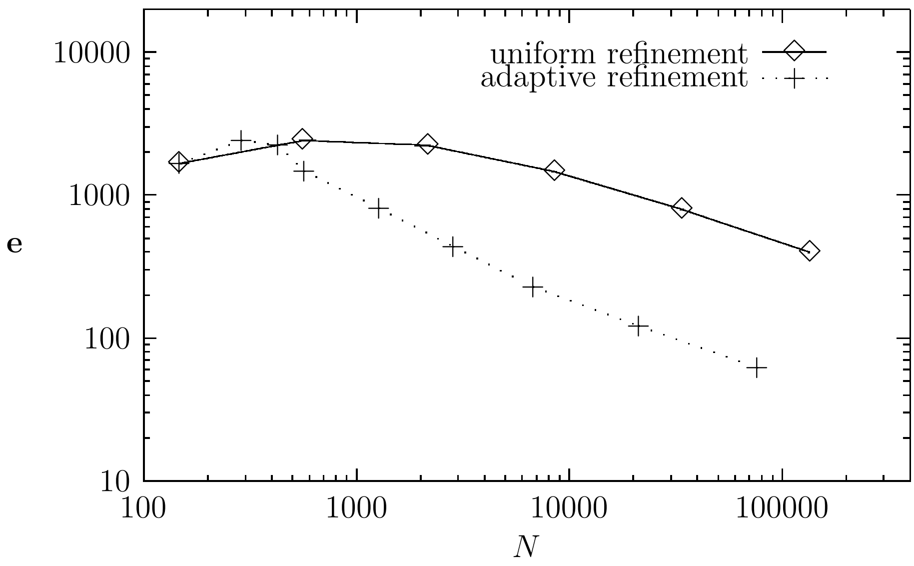

Figure 4.1: **e** vs. degrees of freedom N for both refinements (Example 1).

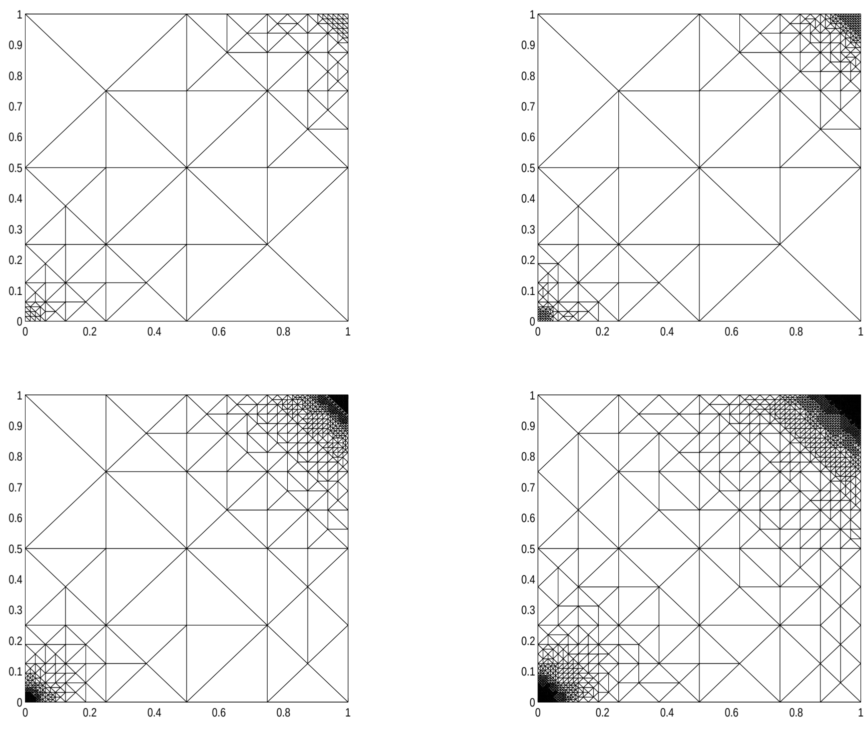

Figure 4.2: Adapted intermediate meshes with 2855, 6768, 21257 and 76251 degrees of freedom, respectively (Example 1).

Table 4.3: Individual errors, effectivity index, and rate of convergence for the **uniform** refinement (Example 2).

N	$\mathbf{e}(\mathbf{t})$	$\mathbf{e}(\boldsymbol{\sigma})$	$\mathbf{e}(p)$	$\mathbf{e}(\mathbf{u})$	$\mathbf{e}(\gamma)$	$\mathbf{e}(\boldsymbol{\xi})$	$\boldsymbol{\theta}/\mathbf{e}$	$r(\mathbf{e})$
147	112.3410	2127.0888	251.8113	31.1637	120.7453	287. 0555	1.0338	—
559	141.7032	2531.2072	171.6559	20.6035	204.4473	150.8298	1.0050	—
2175	107.5653	2206.0114	106.7205	15.3863	165.1150	81.5072	0.9991	0.1966
8575	61.4614	1542.6228	55.3157	7.0323	92.0422	34.7034	0.9988	0.5249
34047	29.3789	906.2548	23.6549	2.5061	40.9853	11.7017	0.9993	0.7737
135679	13.1622	480.2463	9.5210	0.8287	15.5862	3.3108	0.9995	0.9199

Table 4.4: Individual errors, effectivity index, and rate of convergence for the **adaptive** refinement (Example 2).

N	$\mathbf{e}(\mathbf{t})$	$\mathbf{e}(\boldsymbol{\sigma})$	$\mathbf{e}(p)$	$\mathbf{e}(\mathbf{u})$	$\mathbf{e}(\gamma)$	$\mathbf{e}(\boldsymbol{\xi})$	$\boldsymbol{\theta}/\mathbf{e}$	$r(\mathbf{e})$
147	112.3410	2127.0888	251.8113	31.1637	120.7453	287.0555	1.0338	—
220	145.1729	2496.9907	170.3271	28.0330	233.9385	174.1636	1.0059	—
441	113.1879	2228.2703	110.5418	19.9781	181.9071	99.0489	1.0059	0.3388
682	62.3589	1658.0919	64.3475	10.0766	86.1261	35.9772	1.0052	1.3732
1644	38.8577	1110.7994	40.3418	4.3731	50.0285	20.3163	1.0001	0.9119
2527	26.6710	789.3284	27.6401	3.4807	34.6764	14.5043	0.9989	1.5901
4103	21.5463	597.8025	24.5800	3.4935	30.6850	15.2468	0.9971	1.1434
8286	17.3313	407.4328	22.8120	3.4549	25.0135	14.7974	0.9950	1.0856
13494	9.5443	313.3967	9.4886	1.2717	10.1316	4.8156	0.9984	1.0903
28819	6.9402	212.5386	7.3771	1.1026	6.8368	4.0400	0.9980	1.0229
55923	4.7947	155.4689	4.6224	0.6933	3.9870	2.5208	0.9988	0.9447
115799	3.2571	106.9606	2.9979	0.3642	2.1975	1.0497	0.9988	1.0283

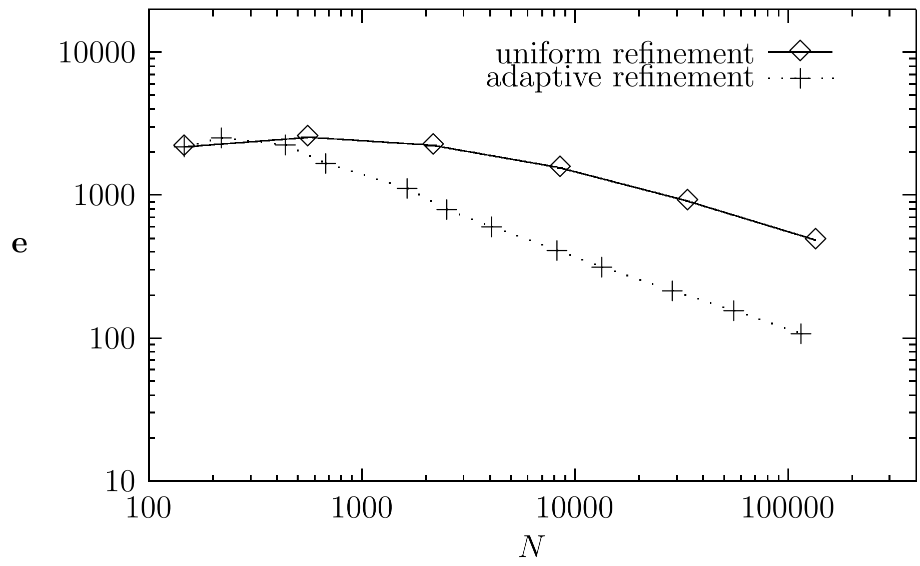

Figure 4.3: **e** vs. degrees of freedom N for both refinements (Example 2).

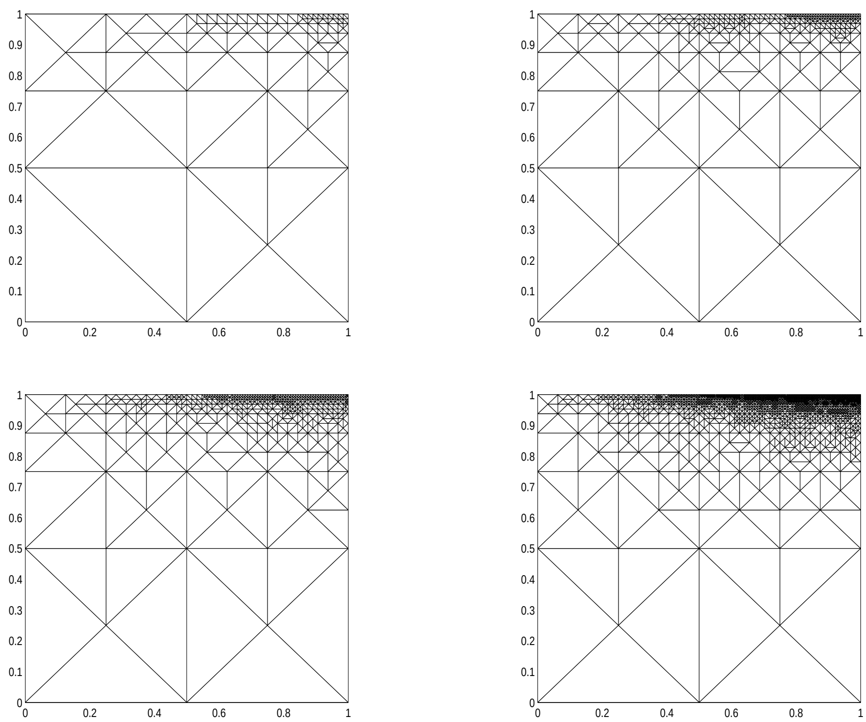

Figure 4.4: Adapted intermediate meshes with 2527, 8286, 13494 and 55923 degrees of freedom, respectively (Example 2).

Table 4.5: Individual errors, effectivity index, and rate of convergence for the **uniform** refinement (EXAMPLE 3).

N	$\mathbf{e}(\mathbf{t})$	$\mathbf{e}(\boldsymbol{\sigma})$	$\mathbf{e}(p)$	$\mathbf{e}(\mathbf{u})$	$\mathbf{e}(\gamma)$	$\mathbf{e}(\boldsymbol{\xi})$	$\boldsymbol{\theta}/\mathbf{e}$	$r(\mathbf{e})$
431	1.6827	4.3206	1.8347	1.7801	2.5933	3.0995	0.6791	—
1655	1.0580	2.8794	1.2782	0.7783	1.2534	1.2072	0.6718	0.8221
6479	0.7035	1.9895	0.8866	0.3350	0.6894	0.4757	0.6894	0.6491
25631	0.4791	1.3994	0.6169	0.1518	0.4293	0.2036	0.7033	0.5561
101951	0.3280	0.9885	0.4265	0.0710	0.2823	0.0915	0.7170	0.5276

Table 4.6: Individual errors, effectivity index, and rate of convergence for the **adaptive** refinement (EXAMPLE 3).

N	$\mathbf{e}(\mathbf{t})$	$\mathbf{e}(\boldsymbol{\sigma})$	$\mathbf{e}(p)$	$\mathbf{e}(\mathbf{u})$	$\mathbf{e}(\gamma)$	$\mathbf{e}(\boldsymbol{\xi})$	$\boldsymbol{\theta}/\mathbf{e}$	$r(\mathbf{e})$
431	1.6827	4.3206	1.8347	1.7801	2.5933	3.0995	0.6791	—
896	1.2404	3.2466	1.3897	1.1565	1.5600	2.0345	0.6790	0.9634
1701	0.9301	2.6065	1.1770	0.7441	1.0234	1.2455	0.7749	0.9146
2981	0.7295	2.0713	0.9130	0.4964	0.6816	0.6413	0.6580	1.0471
3314	0.6772	1.9335	0.8484	0.4688	0.6063	0.5781	0.6490	1.4093
4043	0.5961	1.6780	0.7260	0.4167	0.4984	0.5330	0.6464	1.4175
4815	0.5624	1.5887	0.6974	0.3704	0.4175	0.4211	0.6884	0.8211
8679	0.4657	1.3089	0.5710	0.2701	0.3094	0.2280	0.6425	0.7333
12382	0.3793	1.0490	0.4549	02076	0.2491	0.1895	0.6409	1.2435
13576	0.3628	1.0014	0.4346	0.1996	0.2252	0.1564	0.6277	1.1054
15984	0.3381	0.9251	0.4000	0.1886	0.2001	0.1410	0.6348	0.9798
28635	0.2499	0.6874	0.2905	0.1425	0.1345	0.1011	0.6568	1.0406
37977	0.2204	0.5974	0.2530	0.1218	0.1139	0.0623	0.6489	1.0178
46571	0.1989	0.5399	0.2280	0.1062	0.0988	0.0452	0.6393	1.0314

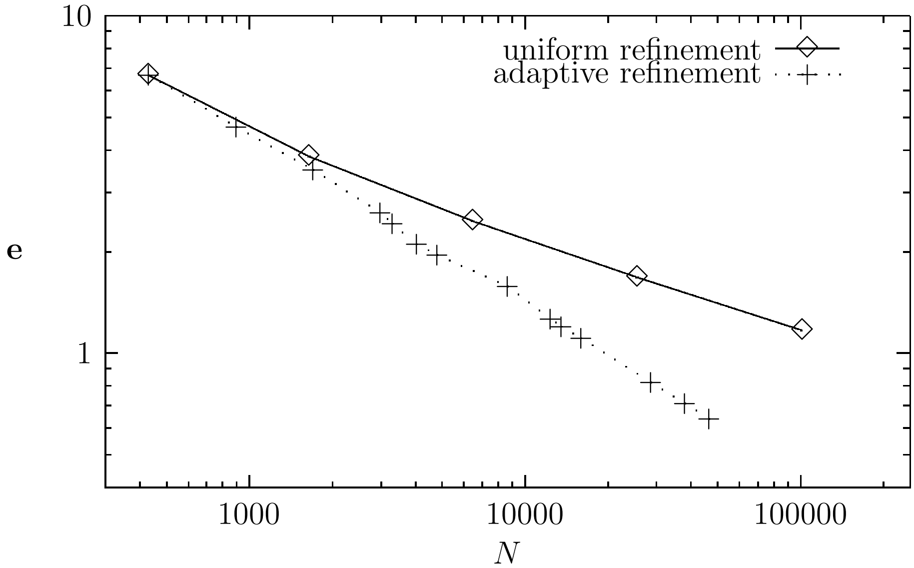

Figure 4.5: **e** vs. degrees of freedom N for both refinements (Example 3).

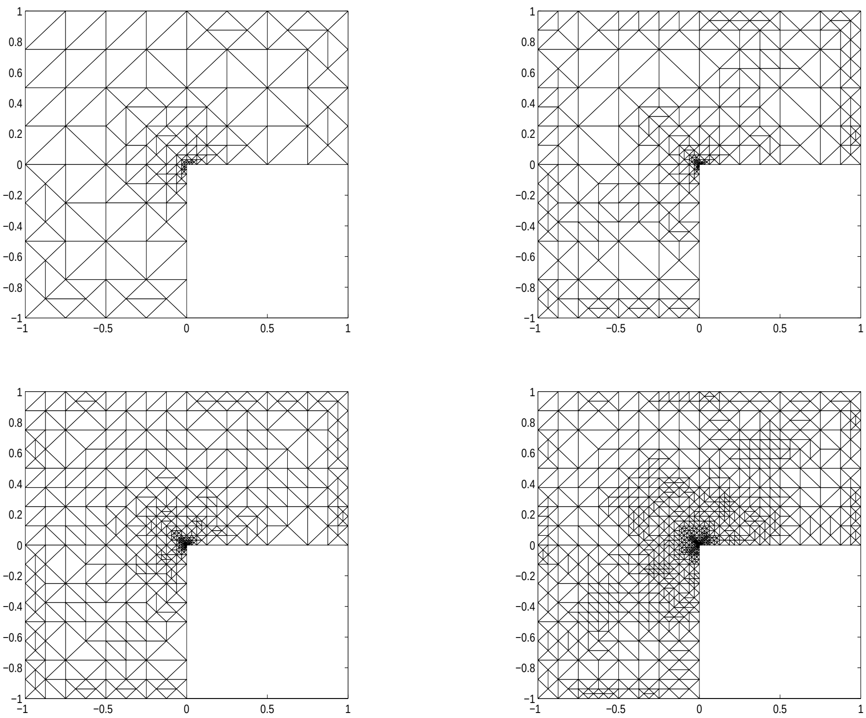

Figure 4.6: Adapted intermediate meshes with 4043, 8679, 13576 and 28635 degrees of freedom, respectively (Example 3).

Table 4.7: Individual errors, effectivity index, and rate of convergence for the **uniform** refinement (EXAMPLE 4).

N	$\mathbf{e}(\mathbf{t})$	$\mathbf{e}(\boldsymbol{\sigma})$	$\mathbf{e}(p)$	$\mathbf{e}(\mathbf{u})$	$\mathbf{e}(\gamma)$	$\mathbf{e}(\boldsymbol{\xi})$	$\boldsymbol{\theta}/\mathbf{e}$	$r(\mathbf{e})$
220	4.2863	52.7757	1.6463	0.9703	4.0542	2.3792	0.9882	—
837	4.1703	46.8548	4.3708	1.8337	5.6152	3.6741	0.9974	0.1614
3259	2.4477	34.0130	1.4432	0.9578	2.9138	1.9283	1.0008	0.4858
12855	1.0550	18.2077	0.4385	0.1136	0.5552	0.1741	0.9996	0.9203
51055	0.5236	10.4560	0.2069	0.0536	0.1448	0.0387	0.9993	0.8056
203487	0.2611	5.9983	0.1015	0.0261	0.0628	0.0096	0.9993	0.8043

Table 4.8: Individual errors, effectivity index, and rate of convergence for the **adaptive** refinement (EXAMPLE 4).

N	$\mathbf{e}(\mathbf{t})$	$\mathbf{e}(\boldsymbol{\sigma})$	$\mathbf{e}(p)$	$\mathbf{e}(\mathbf{u})$	$\mathbf{e}(\gamma)$	$\mathbf{e}(\boldsymbol{\xi})$	$\boldsymbol{\theta}/\mathbf{e}$	$r(\mathbf{e})$
220	4.2863	52.7757	1.6463	0.9703	4.0542	2.3792	0.9882	—
626	4.2099	46.7481	3.7205	1.1941	4.3224	2.6518	1.0070	0.2216
1328	2.7281	34.2342	1.7934	1.3762	3.4172	3.1041	1.0089	0.8258
3482	1.3944	19.0088	1.2002	0.3696	0.8691	0.9548	1.0199	1.2354
10704	0.6876	11.5406	0.6672	0.1657	0.4562	0.4021	1.0032	0.8926
19068	0.5262	9.3770	0.5811	0.1683	0.4357	0.3671	0.9995	0.7172
46203	0.3399	5.8507	0.3321	0.0873	0.2519	0.1525	0.9964	1.0677
67564	0.2850	4.9443	0.2790	0.0717	0.2003	0.1266	0.9968	0.8866
150350	0.1999	3.2427	0.2102	0.0316	0.1297	0.0564	0.9956	1.0535
210532	0.1644	2.7240	0.1643	0.0286	0.1036	0.0461	0.9945	1.0380

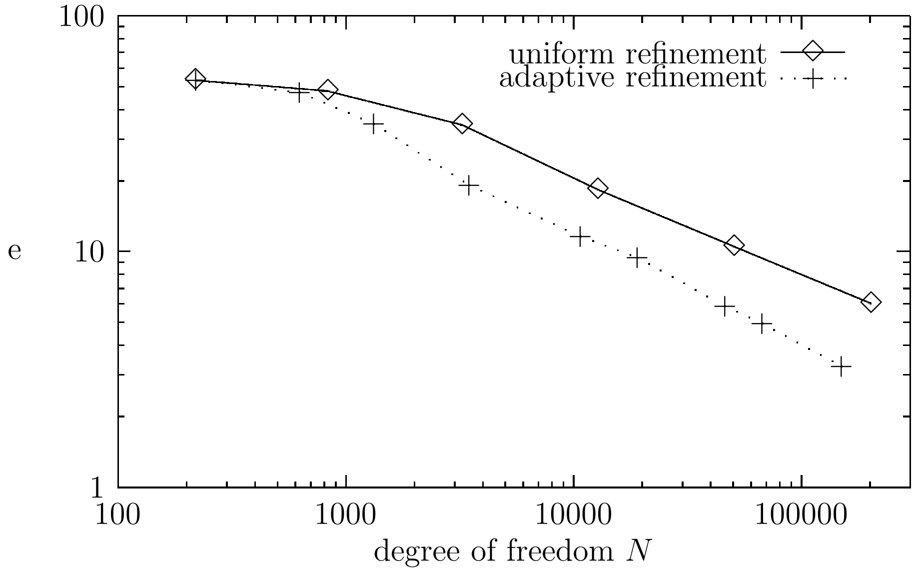

Figure 4.7: **e** vs. degrees of freedom N for both refinements (Example 4).

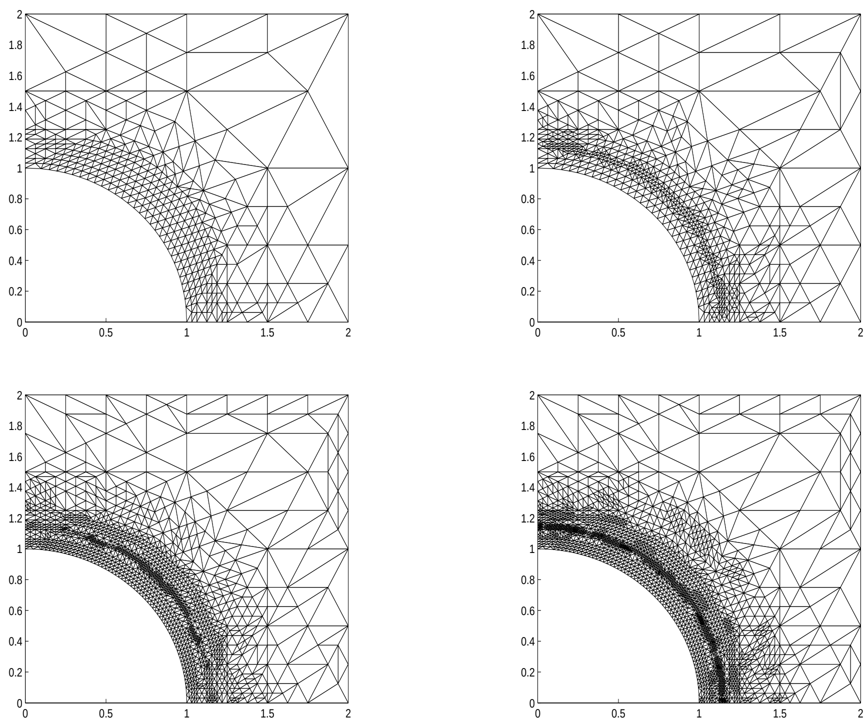

Figure 4.8: Adapted intermediate meshes with 10704, 19068, 46203 and 67564 degrees of freedom, respectively (Example 4).

Chapter 5

Conclusions and future work

In this thesis we have developed new mixed finite elements methods for nonlinear problems in incompressible elasticity on Lipschitz domains.

5.1 Conclusions

The major accomplishments of this study can be categorized into two groups, theoretical and numerical contributions, which are summarized below:

Theoretical contributions

- A slight generalization of the classical Babuška-Brezzi theory to nonlinear three-fold saddle point problems.
- Convergence results that are optimal in the mesh size for two mixed finite element methods in nonlinear incompressible elasticity.
- A-posteriori error estimates for two mixed finite element methods in nonlinear incompressible elasticity.

Numerical contributions

- A code with fully automatic mesh adaptation for the solution of a nonlinear incompressible elasticity problem with mixed boundary conditions.

5.2 Future work

The techniques analyzed and implemented in this study show that mixed finite elements methods are also well-suited for nonlinear incompressible elasticity and that they show great potential for the solution of more complex problems.

Some of the future directions to be explored are the following:

- A-priori and a-posteriori error analysis for the case of nonconforming mixed finite element schemes.
- Coupling of mixed-FEM and BEM for interior and exterior problems in nonlinear incompressible elasticity.

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