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ANALISIS NUMERICO DE PROBLEMAS DE TRANSMISION CON DISCONTINUIDADES

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ANALISIS NUMERICO DE PROBLEMAS DE TRANSMISION CON DISCONTINUIDADES

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Resumen

El presente trabajo consta de dos partes claramente definidas.

En la primera parte usamos una formulación mixta para analizar la resolución numérica de cierta clase de problemas de valores de contorno elípticos no lineales, en un dominio Lipschitz del plano. Más precisamente, consideramos un problema de transmisión exterior no lineal con discontinuidades, cuya formulación variacional discreta se obtiene acoplando el método de elementos finitos mixtos con el método integral de frontera. Mostramos que el esquema discreto está bien propuesto y probamos razones de convergencia optimales. Además, presentamos un análisis de error a-posteriori para esta formulación, tema que no había sido desarrollado aún para problemas de transmisión. Varios ejemplos numéricos confirman nuestros resultados teóricos y proporcionan evidencias empíricas de una eventual eficiencia del estimador de error a-posteriori.

En la segunda parte de esta tesis extendemos la aplicabilidad del método de Galerkin discontinuo local a problemas de valores de contorno no lineales. Primero consideramos un problema de difusión no lineal con condiciones de contorno mixtas, y luego estudiamos una clase de fluidos de Stokes cuasi-newtonianos en regimen estacionario. Probamos que los esquemas discrestos están bien propuestos y proporcionamos las estimaciones de error a-priori correspondientes. Además, desarrollamos también análisis de error a-posteriori que producen estimadores confiables para ambos problemas, y presentamos resultados numéricos que ilustran el comportamiento de los estimadores de error a-posteriori.

Abstract

The present work consists of two parts clearly defined.

In the first part we use a mixed formulation to analyze the numerical resolution of certain class of nonlinear elliptic boundary value problems, on a Lipschitz domain in the plane. More precisely, we consider a nonlinear exterior transmission problem with discontinuities, whose discrete variational formulation is obtained by coupling the mixed finite element method with the boundary integral method. We show that the discrete scheme is well-posed and prove optimal rates of convergence. In addition, we present an a-posteriori error analysis for this formulation, subject that had not been developed yet for transmission problems. Several numerical examples confirm our theoretical results and provide empiric evidences of an eventual efficiency of the a-posteriori error estimator.

In the second part of this thesis, we extend the applicability of the local discontinuous Galerkin method to nonlinear boundary value problems. First, we consider a nonlinear diffusion problem with mixed boundary conditions, and then we study a class of quasi-newtonian Stokes fluids in stationary regime. We prove that the discrete schemes are well-posed and provide the corresponding a-priori error estimates. In addition, we also develop a-posteriori error analyses yielding reliable estimators for both problems, and present several numerical results illustrating the performance of the a-posteriori error estimators.

Contents

Chapter 1

Introducción

Actualmente, la aplicación de formulaciones variacionales mixtas se ha convertido en una herramienta estándar y necesaria para la resolución de problemas de valores de contorno lineales y no lineales en la física y en las ciencias de la ingeniería, ya que permite encontrar, en el caso de la conducción de calor, por ejemplo, una mejor aproximación del gradiente de temperatura que la que uno encontraría a partir de la aproximación de la temperatura misma obtenida al resolver la formulación primal correspondiente. Como es natural, la mayor parte de la literatura que nos proporciona los fundamentos y aplicaciones de los métodos mixtos está aplicada a una gran familia de problemas de valores de contorno lineales (ver, por ejemplo, [25, 64] y las referencias incluidas en ellas). Para el caso no lineal, a saber, hay dos formas de enfrentar el problema. La primera consiste en invertir la ecuación constitutiva "no lineal", haciendo uso del teorema de la función implícita. Trabajos en esta dirección, incluyendo resultados para las versiones h y p de elementos finitos, así como extensiones a problemas parabólicos no lineales, pueden encontrarse, por ejemplo, en [73, 75, 77, 78, 81, 82].

Por otro lado, cuando la ecuación constitutiva en cuestión no es inversible explícitamente, una alternativa es incorporar variables auxiliares, como en [62, 63], en los cuales se realiza un acoplamiento del método de elementos finitos mixtos con el método de elementos de frontera para resolver problemas de transmisión no lineales. Más precisamente, allí se introduce el gradiente (nombre que se ocupa en teoría de potencial y conducción de calor) o el tensor de deformación (en elasticidad) como una incógnita adicional, con lo que la formulación variacional que se deduce se puede ver como un sistema de ecuaciones de operadores con estructura de punta de silla doble, llamada también *dual-dual*. Al respecto podemos mencionar que en [52] se encuentra la generalización de la teoría de Babuška-Brezzi para este tipo de ecuaciones, mientras que en [55, 56, 59] podemos encontrar algunas de las aplicaciones más relevantes.

La presente tesis está dividida en dos partes bien definidas. En la primera parte se estudia un problema de transmisión exterior no lineal con discontinuidades en la frontera que separa (o conecta) un medio (dominio) del otro. Los resultados aquí obtenidos generalizan un trabajo previo (ver [59]), en donde se consideran condiciones de transmisión continuas.

La segunda parte consiste en el empleo de espacios de aproximación, en los cuales no se requiere que haya continuidad inter-elemento, para resolver problemas elípticos no lineales. Puede pensarse entonces que se está frente a un problema de transmisión con discontinuidades inter-elementos. En este caso, la restricción de continuidad se introduce en la formulación variacional en forma adecuada, lo cual hace a este método ser semejante a los métodos estabilizados. Esta técnica corresponde a los llamados métodos de Galerkin discontinuos, los cuales surgieron originalmente para resolver sistemas hiperbólicos no lineales de primer orden, y tiempo después se emplearon también para analizar problemas elípticos y parabólicos. No obstante, con respecto a las ecuaciones elípticas, cabe señalar que hasta el momento se habían tratado sólo problemas lineales. Este hecho constituyó la motivación inicial para extender la aplicabilidad de este tipo de métodos a problemas de difusión no lineal. En particular, aquí se considera el conocido método de Galerkin discontinuo local.

A continuación se presentan mayores detalles sobre cada una de estas dos partes, describiendo lo mejor posible las razones que motivaron a tratar cada uno de los problemas presentados en los capítulos siguientes. Respecto a la segunda parte, se empieza dando una breve reseña de la evolución de los métodos de Galerkin discontinuos para resolver ecuaciones en derivadas parciales elípticas, indicando los parecidos y diferencias con otros métodos de elementos finitos.

1.1 Un problema de transmisión exterior no lineal

En esta parte estamos interesados en algunos *problemas de transmisión exteriores no lineales*, que aparecen en el cálculo de campos magnéticos de aparatos electromagnéticos (ver, por ejemplo, [66, 67]), en problemas de mecánica de fluidos y flujo subsónico (ver, por ejemplo, [49, 50]), y también en la conducción de calor estacionario.

Con el objeto de comprender mejor el modelo que consideramos, introducimos Ω_0 como un dominio acotado simplemente conexo del plano con frontera Γ_0 Lipschitz-continua. Por otro lado, definimos Ω_1 como el dominio anular acotado por Γ_0 y otra curva cerrada Lipschitz-continua Γ_1, cuya región interior contiene a Ω_0. Además, sean $a_i : \Omega_1 \times \mathbf{R}^2 \to \mathbf{R}$ $(i = 1, 2)$ aplicaciones no lineales que satisfacen ciertas condiciones apropiadas y que se especificarán en el Capítulo 2, y sea $\mathbf{a} : \Omega_1 \times \mathbf{R}^2 \to \mathbf{R}^2$ la función vectorial definida por $\mathbf{a}(x, \boldsymbol{\zeta}) := (a_1(x, \boldsymbol{\zeta}), a_2(x, \boldsymbol{\zeta}))^{\mathbf{T}}$, para todo $(x, \boldsymbol{\zeta}) \in \Omega_1 \times \mathbf{R}^2$. Entonces, dados $g_0 \in H^{1/2}(\Gamma_0)$, $f_1 \in L^2(\Omega_1)$, $g_1 \in H^{1/2}(\Gamma_1)$, y $g_2 \in H^{-1/2}(\Gamma_1)$, se desea encontrar: $u_1 \in H^1(\Omega_1)$, $u_2 \in H^1_{loc}(\mathbf{R}^2 - \bar{\Omega}_0 \cup \bar{\Omega}_1)$ *tal que*

$$u_1 = g_0 \quad \text{en} \quad \Gamma_0\,, \qquad -\operatorname{div}\, \mathbf{a}(\cdot, \nabla u_1(\cdot)) = f_1 \quad \text{en} \quad \Omega_1\,,$$

$$u_1 - u_2 = g_1 \quad \text{y} \quad \mathbf{a}(\cdot, \nabla u_1(\cdot)) \cdot \mathbf{n} - \frac{\partial u_2}{\partial \mathbf{n}} = g_2 \quad \text{en} \quad \Gamma_1\,,$$

$$-\Delta u_2 = 0 \quad \text{en} \quad \mathbf{R}^2 - \bar{\Omega}_0 \cup \bar{\Omega}_1\,, \quad u_2(x) = O(1) \quad \text{cuando} \quad \|x\| \to +\infty\,,$$

donde div es el operador de divergencia usual y $\mathbf{n}$ denota el vector normal unitario exterior a $\partial\Omega_1$.

En el Capítulo 2 del presente trabajo se detalla la aproximación mediante un acoplamiento de elementos finitos mixtos con elementos de frontera para resolver el problema antes expuesto. El contenido de este capítulo corresponde al artículo [31]:

- R. Bustinza, G.C. García and G.N. Gatica: *A mixed finite element method with Lagrange multipliers for nonlinear exterior transmission problems.* Numerische Mathematik, vol. 96, pp. 481-523, (2004).

Introduciendo, además del gradiente de temperatura y los flujos como incógnitas auxiliares, la traza de la solución exterior como un multiplicador de Lagrange, y aplicando la generalización de la teoría de Babuška-Brezzi [52], logramos probar que el esquema dual-dual resultante, tanto a nivel continuo como el esquema de Galerkin correspondiente definido con espacios de Raviart-Thomas, están bien planteados. Se obtienen también estimaciones de error a-priori y se demuestra que el orden de convergencia correspondiente es el óptimo. De esta manera, logramos extender los resultados obtenidos en [16] al caso de condiciones de transmisión discontinuas. Además, combinando la técnica presentada por Bank-Weiser [14] con lo descrito en [3] y [29], obtenemos primero un estimador de error a-posteriori implícito confiable y cuasi-eficiente, y luego otro completamente explícito y confiable. Es relevante señalar que hasta entonces no se contaba con un estimador de error a-posteriori para esta clase de problemas. Se incluye una serie de ejemplos numéricos que validan los resultados obtenidos.

El objetivo siguiente fue tratar de resolver problemas elípticos no lineales usando algún método de elementos finitos discontinuos. A modo de introducir al lector en este tópico, presentamos a continuación una breve reseña histórica sobre los métodos de Galerkin discontinuos, la cual se extrajo principalmente de [36] y [38].

1.2 Métodos de Galerkin discontinuos (DG) para problemas elípticos

El uso de métodos de Galerkin discontinuos para resolver problemas elípticos se remonta a comienzos de los setenta. Desde entonces y hasta ahora, se ha tratado de extender la aplicación de estos métodos en problemas donde el término difusivo no es despreciable, así como en problemas puramente elípticos. Una de las ventajas de esta técnica es el hecho que podemos considerar mallas más generales, por ejemplo con nodos colgantes ("hanging nodes"), debido a que las aproximaciones que se buscan son discontinuas por elemento. Esto hace que este método sea adecuado para la versión hp del método de elementos finitos.

En el contexto de los problemas elípticos, el desarrollo de los métodos de Galerkin discontinuos empezó con la idea de Nitsche [80] de introducir las condiciones de frontera de Dirichlet de manera débil, en vez de imponerlos de manera explícita en el espacio de elementos finitos, lo cual simplifica la implementación computacional. En el método de Nitsche [80], las condiciones de contorno esenciales son impuestas a través de términos de penalización sobre los lados de frontera en la formulación variacional. En [9], Babuška analiza una versión simplificada del método de penalización de Nitsche para resolver el problema de Poisson con condiciones de contorno homogéneas. Sin embargo, este método usa un parámetro de penalización muy grande. Por otro lado, en [47], Douglas y Dupont extendieron el método de penalización de Nitsche para ecuaciones en derivadas parciales elípticas no lineales.

1.2.1 Métodos de Penalización Interior

El método de Nitsche fue rápidamente extendido a los lados interiores como un medio de forzar la continuidad entre elementos que comparten un lado de la malla, permitiendo el uso de un espacio de elementos finitos más natural, el espacio de los polinomios por elemento discontinuos. Más aún, polinomios de diferente grado pueden ser usados en cada elemento, haciendo esta técnica adecuada para la versión *hp*. En [13], Babuška y Zlámal desarrollaron un método discontinuo usando penalizaciones interiores para resolver la ecuación con el operador bi-armónico con condiciones de contorno homogéneas. A mediados de los setenta, una serie de variantes de los métodos de Penalización Interior (Interior Penalty methods (IP)) fueron desarrolladas para resolver problemas elípticos de segundo orden. Esta idea fue originalmente introducida por Douglas y Dupont [48], y analizada para problemas parabólicos lineales y no lineales por Arnold [5, 6]. Por ese entonces, y en forma paralela, un método similar fue también desarrollado por Wheeler [97].

1.2.2 Métodos de Galerkin con penalización interior no simétricos

A fines de los noventa se desarrolló una técnica diferente, que se basa en aproximar un problema auto-adjunto usando una forma bilineal no simétrica. Esto fue motivado por el hecho que al tratar un problema de convección-difusión, el operador asociado resultaba ser no simétrico. Sin embargo, no hay evidencia que esta técnica sea más eficiente que un método DG simétrico cuando resolvemos problemas elípticos. El primero de tales métodos fue propuesto por Baumann y Oden [21], y analizado para el caso unidimensional por Babuška, Baumann y Oden [11]. Recientemente, Rivière, Wheeler y Girault [89] obtuvieron estimaciones de error óptimos para una clase de métodos DG, llamados métodos NIPG (non-symmetric interior penalty Galerkin), los cuales consisten en una generalización del método de Baumann y Oden. Esta nueva formulación se obtiene introduciendo términos de penalización, interior y de frontera, en la forma bilineal del método de Baumann y Oden. Esta clase también incluye el método *hp* de elementos finitos de Houston, Schwab y Süli [71].

1.2.3 Aplicación de los métodos DG en problemas de convección-difusión

Como se acotó antes, los métodos DG surgieron inicialmente para resolver sistemas hiperbólicos no lineales. No obstante, pronto el campo de aplicaciones fue extendiéndose hasta llegar a problemas de convección dominante con parte difusiva no despreciable. Esto motivó el deseo de aplicar los métodos DG para resolver problemas elípticos y parabólicos. Es así que Bassi y Rebay [18] extendieron el método propuesto en [19] para resolver ecuaciones de Navier-Stokes con números de Reynolds altos. Posteriormente, Brezzi, Manzini, Marini, Pietra y Russo [27, 28], mostraron que la formulación original del método de Bassi-Rebay puede ser inestable para el caso estacionario. Por otro lado, ellos mismos obtuvieron estimaciones óptimas para la versión estabilizada del método propuesto por Bassi, Rebay, Mariotti, Pedinotti y Savini [20]. Además, ellos también analizaron una variante estable cuya matriz de rigidez resulta ser más dispersa.

1.2.4 El método de Galerkin discontinuo local (LDG)

Este método fue introducido por Cockburn y Shu [45] para resolver sistemas hiperbólicos no lineales. Gracias a sus propiedades de conservación y alto grado de localidad, fue aplicado después a problemas de convección-difusión. En los últimos años, el método LDG ha sido extendido a problemas completamente elípticos. El primer trabajo en esta línea corresponde a [38], en el cual se trata la ecuación de Poisson con condiciones de contorno mixtas, incluyendo el análisis de error a-priori correspondiente. Más recientemente, se han estudiado aplicaciones a las ecuaciones de Stokes, de Maxwell y de Oseen (ver, por ejemplo, [43], [85] y [42]), en los cuales se consideran fluidos incompresibles.

A continuación presentamos las principales características del método LDG, así como su relación con los métodos de elementos finitos mortar y los métodos estabilizados.

- La forma de obtener la formulación variacional usando el método LDG es similar al que se ocupa para obtener formulaciones mixtas. Por ejemplo, si nuestro problema es: *Hallar* $u \in H^1(\Omega)$ *tal que*

  $$-\Delta u = f \quad \text{en} \quad \Omega \,, \quad u = g \quad \text{en} \quad \partial\Omega \,,$$

  donde $f \in L^2(\Omega)$ y $g \in H^{1/2}(\partial\Omega)$, entonces procedemos introduciendo la variable auxiliar $\boldsymbol{\theta} := \nabla u$ y re-escribimos el problema elíptico como un sistema de ecuaciones de primer orden que luego discretizamos. Cabe señalar aquí que la incógnita auxiliar $\boldsymbol{\theta}$ puede ser *eliminada* del sistema de ecuaciones resultante, lo que no es usual en el caso de los métodos mixtos clásicos.

- Como es común en los métodos DG, no se requiere imponer condición de continuidad alguna entre las fronteras de los elementos de la malla, lo cual hace posible tratar con mallas formadas por elementos de distinta formas, y posiblemente con nodos colgantes, así como el uso de espacios de aproximación de distinto grado por elemento. Esta propiedad también la tiene el método mixto mortar, pero a diferencia de éste, el método LDG no requiere multiplicadores de Lagrange para imponer de manera débil la continuidad inter-elemento, ya que esta restricción se encuentra implícitamente en la definición de los flujos numéricos.

- Los flujos numéricos garantizan la estabilidad del método así como la calidad de la aproximación, por la presencia de saltos de la solución aproximada en la definición de éstos. Esta peculiaridad es lo que relaciona al método LDG con los métodos de elementos finitos mixtos estabilizados

- En el método LDG tanto las aproximaciones de u como las de cada componente de $\boldsymbol{\theta}$, sobre cada elemento, pueden pertenecer al mismo espacio de aproximación, lo cual es muy conveniente a la hora de realizar la implementación computacional. Además, el hecho que no se pida *continuidad* a través de las fronteras de los elementos en los espacios de elementos finitos permite que la programación de la versión *hp* del método LDG sea mucho más simple con respecto a los métodos mixtos estándar.

- En [37], Castillo hace un análisis teórico y numérico del número de condición de los métodos DG simétricos. En dicho trabajo se prueba que el comportamiento asintótico del número de condición espectral $\kappa(h)$, como una función del tamaño de la malla (h), es del orden h^{-2}, el mismo que se tiene para los métodos de elementos finitos clásicos.

Los resultados teóricos obtenidos por los métodos de Galerkin discontinuos (en particular el método LDG) para problemas elípticos, son sólo válidos para el caso lineal. Esto nos motivó, primero a entender el método LDG, para luego aplicarlo a una clase de problemas de difusión no lineales definidos en regiones poligonales de $\mathbf{R}^2$. El análisis desarrollado para alcanzar este objetivo está plasmado en el Capítulo 3, el cual corresponde al artículo [32]:

- R. Bustinza and G.N. Gatica: *A local discontinuous Galerkin method for nonlinear diffusion problems with mixed boundary conditions.* SIAM Journal on Scientific Computing, to appear.

En dicho trabajo, asumimos que la aplicación que define el término difusivo no lineal del problema satisface ciertas propiedades (ver, por ejemplo, [59]), de tal modo que haciendo uso del análisis mostrado en [52] y aplicando técnicas similares a las desarrolladas en [83], obtenemos los mismos órdenes de convergencia que en el caso lineal, tanto para el error del potencial u en la norma de energía como en la norma usual de L^2. Se exhiben varios ejemplos numéricos que corroboran el análisis teórico desarrollado, incluso en situaciones no contempladas por la teoría.

El objetivo siguiente fue el encontrar un estimador a-posteriori del error para el problema presentado en [32]. Si bien es cierto que hay una extensa fuente de análisis de error a-priori para problemas elípticos, usando métodos DG en general, a la fecha existen muy pocos trabajos sobre la obtención de estimadores de error a-posteriori. Entre ellos, podemos citar los trabajos de Becker et al. [23] y de Rivière et al. [88], donde se obtienen estimadores del tipo residual para el error en la norma usual de L^2 y estimadores implícitos basados en problemas locales para el error en la norma de energía. En [22] se desarrolla un estimador de error a-posteriori confiable de tipo residual para el error en la norma de la energía, que depende de la malla, para una familia general de métodos DG. Recientemente, Houston, Perugia y Schötzau [69], y Houston, Schötzau y Wihler [70] han obtenido estimadores de error a-posteriori en la norma de energía para aproximaciones DG mixtas del operador de Maxwell y para el problema de Stokes, respectivamente.

En el Capítulo 4 se describe la obtención de un estimador de error a-posteriori en la norma de la energía para el modelo analizado en [32], completando así el análisis que se inició en el Capítulo 3. Cabe destacar que el contenido de este capítulo corresponde al artículo [30]:

- R. Bustinza, B. Cockburn and G.N. Gatica: *An a-posteriori error estimate for the local discontinuous Galerkin method applied to linear and nonlinear diffusion problems.* Journal of Scientific Computing, to appear.

El análisis clave en este capítulo se basa en una descomposición de Helmholtz del gradiente del error, similar a lo desarrollado por Becker et al. [22]. Sin embargo, en vez de requerir un comportamiento polinomial del dato Dirichlet, como en [22], nosotros introducimos una adecuada función auxiliar que lo interpola. Como en los capítulos anteriores, se incluyen resultados numéricos, los cuales validan la teoría, y proporcionan evidencias empíricas de la eficiencia del estimador.

Finalmente, aplicando el método LDG y usando la técnica desarrollada en [70], obtenemos los órdenes de convergencia esperados, así como también un estimador de error a-posteriori confiable, para la clase de fluidos cuasi-Newtonianos estudiados en [53] y [54]. Los detalles de este trabajo se presentan en el Capítulo 5, que corresponde al artículo [33]:

- R. Bustinza and G.N. Gatica: *A mixed local discontinuous finite element method for a class of quasi-Newtonian flows*, submitted. Preprint 2004-12, Departamento de Ingeniería Matemática, Universidad de Concepción, (2004).

Lo particular del análisis que se desarrolla en este capítulo es la introducción de un multiplicador de Lagrange para garantizar la unicidad de la solución de la formulación LDG. Los ensayos numéricos presentados están acordes con la teoría, y proveen evidencias numéricas sobre la eficiencia del estimador de error a-posteriori derivado.

Part I

A nonlinear exterior transmission problem with discontinuities

Chapter 2

A mixed finite element method with Lagrange multipliers for nonlinear exterior transmission problems

In this chapter we apply a mixed finite element method to numerically solve a class of nonlinear exterior transmission problems in $\mathbf{R}^2$ with inhomogeneous interface conditions. Besides the usual unknowns required for the dual-mixed method, which include the gradient of the *temperature* in this nonlinear case, our approach makes use of the trace of the outer solution on the transmission boundary as a suitable Lagrange multiplier. In addition, we utilize a boundary integral operator to reduce the original transmission problem on the unbounded region into a nonlocal one on a bounded domain. In this way, we are lead to a two-fold saddle point operator equation as the resulting variational formulation. We prove that the continuous formulation and the associated Galerkin scheme defined with Raviart-Thomas spaces are well posed, and derive the a-priori estimates and the corresponding rate of convergence. Then, we introduce suitable local problems and deduce first an implicit reliable and quasi-efficient a-posteriori error estimate, and then a fully explicit reliable one. Finally, several numerical results illustrate the effectivity of the explicit estimate for the adaptive computation of the discrete solutions.

2.1 Introduction

In [59] we combined a dual-mixed finite element method with a Dirichlet-to-Neumann mapping (derived by the boundary integral equation method) to study the solvability and Galerkin approximations of a class of nonlinear exterior transmission problems in $\mathbf{R}^2$. As a model, we considered there a nonlinear elliptic equation in a bounded annular domain, coupled with the Laplace equation in the corresponding unbounded region of the plane. This approach, which leads to a two-fold saddle point operator equation as the resulting mixed variational formulation, has been recently extended to other linear and nonlinear boundary value problems (see, e.g. [17], [55], [56], and the references therein). The corresponding abstract framework, which is an extension of the classical Babuška-Brezzi theory, and which includes the numerical analysis of fully discrete Galerkin schemes, can be seen in [51] and [57].

Nevertheless, the drawback of the analysis in [59] lies mainly on two facts. First, the model problem considers only homogeneous jumps of both the *temperature* and the *flux* on the transmission boundary, and hence it does not allow any discontinuity through it. Further, in order to prove the unique solvability results and the a-priori estimates for the continuous and discrete formulations, we needed to restrict some of the unknowns so that they fall into certain quotient spaces. However, this restriction complicates not only the definition of suitable finite element subspaces but also the implementation of the associated Galerkin scheme.

On the other hand, we recall that the utilization of adaptive algorithms, based on a-posteriori error estimates, guarantees a good rate of convergence of the finite element solution to boundary value problems. Moreover, this adaptivity is particularly necessary for nonlinear problems where no a-priori hints on how to build convenient meshes are available. To this respect, we have showed recently that the combination of the usual Bank-Weiser procedure from [14] with the analysis from [3] and [29] allows to derive fully explicit and reliable a-posteriori error estimates for single and two-fold saddle point formulations (see, e.g. [4], [16], [17], [58], and [61]). Nevertheless, no a-posteriori error analysis has been developed yet for the class of nonlinear exterior problem studied in [59].

Consequently, the purpose of this chapter is to extend the results in [59] to the case of inhomogeneous transmission conditions, and to improve our approach there so that no restriction on any unknown is needed. Further, we plan to follow the Bank-Weiser type error analysis from the above mentioned references and derive implicit and fully explicit a-posteriori estimates for the associated discrete scheme. In order to describe the present model problem, which is the same as in [59] but with discontinuous jumps through the coupling boundary, we let Ω_0 be a bounded and simply connected domain in $\mathbf{R}^2$ with Lipschitz-continuous boundary Γ_0. Also, let Ω_1 be the annular domain bounded by Γ_0 and another Lipschitz-continuous closed curve Γ_1 whose interior region contains Γ_0. In addition, let $a_i : \Omega_1 \times \mathbf{R}^2 \to \mathbf{R},\ i = 1, 2,$ be nonlinear mappings satisfying certain conditions to be specified later on, and let $\mathbf{a} : \Omega_1 \times \mathbf{R}^2 \to \mathbf{R}^2$ be the vector function defined by $\mathbf{a}(x, \boldsymbol{\zeta}) := (a_1(x, \boldsymbol{\zeta}), a_2(x, \boldsymbol{\zeta}))^T$ for all $(x, \boldsymbol{\zeta}) \in \Omega_1 \times \mathbf{R}^2$. Then, given $g_0 \in H^{1/2}(\Gamma_0)$, $f_1 \in L^2(\Omega_1)$, $g_1 \in H^{1/2}(\Gamma_1)$, and $g_2 \in H^{-1/2}(\Gamma_1)$, we look for: $u_1 \in H^1(\Omega_1)$ *and* $u_2 \in H^1_{loc}(\mathbf{R}^2 - \bar{\Omega}_0 \cup \bar{\Omega}_1)$ *such that*

$$
\begin{gathered}
u_1 = g_0 \quad \text{on} \quad \Gamma_0 \,, \qquad - \operatorname{div} \mathbf{a}(\cdot, \nabla u_1(\cdot)) = f_1 \quad \text{in} \quad \Omega_1 \,, \\
u_1 - u_2 = g_1 \quad \text{and} \quad \mathbf{a}(\cdot, \nabla u_1(\cdot)) \cdot \mathbf{n} - \frac{\partial u_2}{\partial \mathbf{n}} = g_2 \quad \text{on} \quad \Gamma_1 \,, \\
-\Delta u_2 = 0 \quad \text{in} \quad \mathbf{R}^2 - \bar{\Omega}_0 \cup \bar{\Omega}_1 \,, \quad u_2(x) = O(1) \quad \text{as} \quad \|x\| \to +\infty \,,
\end{gathered} \tag{2.1.1}
$$

where div is the usual divergence operator and $\mathbf{n}$ denotes the unit outward normal to $\partial\Omega_1$.

The rest of the present chapter is organized as follows. In Section 2.2 we proceed as in [59] and utilize a Dirichlet-to-Neumann mapping (based on the boundary integral equation method) to transform (2.1.1) into an equivalent nonlocal boundary value problem on a bounded domain. Then we derive an equivalent mixed variational formulation of this problem, show that it can be written as a two-fold saddle point system, and prove the corresponding solvability and a-priori estimate results. The trace of the outer solution u_2 on the transmission boundary Γ_1 is introduced here as a further unknown that acts also as a suitable Lagrange multiplier. Next, in Section 2.3 we define the Galerkin scheme by using Raviart-Thomas subspaces, show that it is stable and uniquely solvable, derive the Céa estimate, and establish the associated rate of convergence. In Section 2.4 we develop the Bank-Weiser type a-posteriori error analysis and provide a reliable and quasi-efficient implicit estimate, and also a reliable fully explicit one. Several numerical results illustrating the effectivity of the adaptive algorithm induced by the a-posteriori error estimates are reported in Section 2.5.

2.2 The mixed variational formulation

We follow [59] and introduce a sufficiently large circle Γ with center at the origin and radius r such that its interior region contains $\bar{\Omega}_0 \cup \bar{\Omega}_1$, denote by Ω_2 the annular region bounded by Γ_1 and Γ, and put $\Omega := \Omega_1 \cup \Gamma_1 \cup \Omega_2$.

Then, by applying the boundary integral equation method in the region exterior to the circle Γ, we obtain as in [59] the Dirichlet-to-Neumann mapping: $\dfrac{\partial u_2}{\partial \boldsymbol{\nu}} = -2\,\mathbf{W}(u_2|_\Gamma)$ on Γ, where $\boldsymbol{\nu}$ is the unit outward normal to Γ and $\mathbf{W}$ is the hypersingular boundary integral operator associated with the Laplacian.

It is well known (see, e.g., [46]) that $\mathbf{W} : H^{1/2}(\Gamma) \rightarrow H^{-1/2}(\Gamma)$ is linear and bounded and that there exists $C_0 > 0$ such that

$$\langle \mathbf{W}(\xi), \xi \rangle_\Gamma \;\geq\; C_0\, \|\xi\|^2_{H^{1/2}(\Gamma)} \quad \forall\, \xi \in H^{1/2}_0(\Gamma)\,, \tag{2.2.1}$$

where $\langle \cdot, \cdot \rangle_\Gamma$ denotes the duality pairing of $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$ with respect to the $L^2(\Gamma)$-inner product, and $H^{1/2}_0(\Gamma) := \{\xi \in H^{1/2}(\Gamma) : \quad \langle 1, \xi \rangle_\Gamma = 0\,\}$. In addition, $\mathbf{W}(1) = 0$ and $\mathbf{W}$ is symmetric in the sense that $\langle \mathbf{W}(\xi), \hat{\xi} \rangle_\Gamma = \langle \mathbf{W}(\hat{\xi}), \xi \rangle_\Gamma$ for all ξ, $\hat{\xi} \in H^{1/2}(\Gamma)$.

Since we are interested in a mixed variational formulation of (2.1.1), we define the fluxes $\boldsymbol{\sigma}_1 := \mathbf{a}(\cdot, \nabla u_1)$ in Ω_1 and $\boldsymbol{\sigma}_2 := \nabla u_2$ in Ω_2 as further unknowns. Also, we need to introduce the gradients $\boldsymbol{\theta}_i := \nabla u_i$ in Ω_i, for all $i \in \{1, 2\}$, and the traces $\eta := -u_2|_{\Gamma_1}$ and $\lambda := u_2|_\Gamma$, to deal with the nonlinearity $\mathbf{a}$, the jump of the fluxes on Γ_1, and the Dirichlet-to-Neumann mapping, respectively. In order to simplify our notations, we set from now on $\tilde{\boldsymbol{\theta}} := (\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)$, $\tilde{\boldsymbol{\sigma}} := (\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2)$, and $\mathbf{u} := (u_1, u_2)$.

According to the above, the exterior transmission problem (2.1.1) can be reformulated as the following nonlocal boundary value problem in $\bar{\Omega}$ with inhomogeneous transmission conditions on Γ_1: *Find* $(\tilde{\boldsymbol{\theta}}, \lambda, \tilde{\boldsymbol{\sigma}}, \mathbf{u}, \eta)$ *in appropriate spaces such that,* *in the distributional sense,*

$$u_1 = g_0 \quad \text{on} \quad \Gamma_0\,, \quad \boldsymbol{\theta}_1 = \nabla u_1 \quad \text{in} \quad \Omega_1\,, \quad \boldsymbol{\sigma}_1 = \mathbf{a}(\cdot,\boldsymbol{\theta}_1) \quad \text{in} \quad \Omega_1\,,$$

$$-\operatorname{div}\boldsymbol{\sigma}_1 = f_1 \quad \text{in} \quad \Omega_1\,, \quad u_1 - u_2 = g_1 \quad \text{on} \quad \Gamma_1\,, \quad \eta = -u_2 \quad \text{on} \quad \Gamma_1\,,$$

$$\boldsymbol{\theta}_2 = \nabla u_2 \quad \text{in} \quad \Omega_2\,, \quad \boldsymbol{\sigma}_2 = \boldsymbol{\theta}_2 \quad \text{in} \quad \Omega_2\,, \quad \boldsymbol{\sigma}_1\cdot\mathbf{n} - \boldsymbol{\sigma}_2\cdot\mathbf{n} = g_2 \quad \text{on} \quad \Gamma_1\,,$$

$$-\operatorname{div}\boldsymbol{\sigma}_2 = 0 \quad \text{in} \quad \Omega_2\,, \quad \lambda = u_2 \quad \text{on} \quad \Gamma\,, \quad \text{and} \quad \boldsymbol{\sigma}_2\cdot\boldsymbol{\nu} = -2\,\mathbf{W}(\lambda) \quad \text{on} \quad \Gamma\,. \tag{2.2.2}$$

The mappings a_i are supposed to satisfy the following hypotheses:
(H.1) *Carathéodory condition.* The function $a_i(\cdot,\boldsymbol{\theta})$, $i=1,2$, is measurable in Ω_1 for all $\boldsymbol{\theta}\in\mathbf{R}^2$, and $a_i(x,\cdot)$ is continuous in $\mathbf{R}^2$ for almost all $x\in\Omega_1$.
(H.2) *Growth condition.* There exists $\phi_i\in L^2(\Omega_1)$, $i=1,2$, such that

$$|a_i(x,\boldsymbol{\theta})| \;\le\; C\,\{1+|\boldsymbol{\theta}|\} + |\phi_i(x)|\,,$$

for all $\boldsymbol{\theta}\in\mathbf{R}^2$ and for almost all $x\in\Omega_1$.
(H.3) The function $a_i(x,\cdot)$, $i=1,2$, has continuous first order partial derivatives in $\mathbf{R}^2$ for almost all $x\in\Omega_1$. In addition, there exists $C>0$ such that

$$\sum_{i,j=1}^{2}\frac{\partial}{\partial\theta_j}\,a_i(x,\boldsymbol{\theta})\,\zeta_i\,\zeta_j \;\ge\; C\sum_{i=1}^{2}\zeta_i^2\,,$$

for all $\boldsymbol{\theta} := (\theta_1,\theta_2)^{\mathbf{T}}$, $\boldsymbol{\zeta} := (\zeta_1,\zeta_2)^{\mathbf{T}} \in \mathbf{R}^2$ and for almost all $x\in\Omega_1$.
(H.4) The function $a_i(x,\cdot)$, $i=1,2$, has continuous first order partial derivatives in $\mathbf{R}^2$ for almost all $x\in\Omega_1$. In addition, there exists $C>0$ such that for each $i,j\in\{1,2\}$, $\frac{\partial}{\partial\theta_j}\,a_i(x,\boldsymbol{\theta})$ satisfies the Carathéodory condition (H.1), and $\left|\frac{\partial}{\partial\theta_j}\,a_i(x,\boldsymbol{\theta})\right| \le C$, for all $\boldsymbol{\theta}\in\mathbf{R}^2$ and for almost all $x\in\Omega_1$.

We now derive the variational formulation of (2.2.2). We begin with the equations $\boldsymbol{\theta}_i = \nabla u_i$ in Ω_i, $i\in\{1,2\}$, and deduce after integrating by parts that

$$-\int_{\Omega_1}\boldsymbol{\theta}_1\cdot\boldsymbol{\tau}_1\,dx - \int_{\Omega_1}u_1\operatorname{div}\boldsymbol{\tau}_1\,dx + \langle\boldsymbol{\tau}_1\cdot\mathbf{n},u_1\rangle_{\Gamma_1} \;=\; -\,\langle\boldsymbol{\tau}_1\cdot\mathbf{n},g_0\rangle_{\Gamma_0}\,,$$

and

$$-\int_{\Omega_2}\boldsymbol{\theta}_2\cdot\boldsymbol{\tau}_2\,dx - \int_{\Omega_2}u_2\operatorname{div}\boldsymbol{\tau}_2\,dx - \langle\boldsymbol{\tau}_2\cdot\mathbf{n},u_2\rangle_{\Gamma_1} \;+\; \langle\boldsymbol{\tau}_2\cdot\boldsymbol{\nu},u_2\rangle_{\Gamma} \;=\; 0\,,$$

for all $\tilde{\boldsymbol{\tau}} := (\boldsymbol{\tau}_1, \boldsymbol{\tau}_2) \in H(\mathrm{div}\,;\Omega_1) \times H(\mathrm{div}\,;\Omega_2)$, where $\langle\cdot,\cdot\rangle_{\Gamma_0}$ denotes the duality pairing of $H^{-1/2}(\Gamma_0)$ and $H^{1/2}(\Gamma_0)$ with respect to the $L^2(\Gamma_0)$-inner product, and analogously for $\langle\cdot,\cdot\rangle_{\Gamma_1}$. Then, using that $u_1 = u_2 + g_1$ and $u_2 = -\eta$ on Γ_1, and that $u_2 = \lambda$ on Γ, the above equations yield

$$-\sum_{i=1}^{2} \int_{\Omega_i} \boldsymbol{\theta}_i \cdot \boldsymbol{\tau}_i \, dx \;-\; \sum_{i=1}^{2} \int_{\Omega_i} u_i \,\mathrm{div}\, \boldsymbol{\tau}_i \, dx \;-\; \langle \boldsymbol{\tau}_1 \cdot \mathbf{n} - \boldsymbol{\tau}_2 \cdot \mathbf{n}, \eta \rangle_{\Gamma_1} \;+\; \langle \boldsymbol{\tau}_2 \cdot \boldsymbol{\nu}, \lambda \rangle_{\Gamma}$$

$$= \;-\, \langle \boldsymbol{\tau}_1 \cdot \mathbf{n}, g_0 \rangle_{\Gamma_0} \;-\; \langle \boldsymbol{\tau}_1 \cdot \mathbf{n}, g_1 \rangle_{\Gamma_1} \,. \tag{2.2.3}$$

Similarly, using the equations relating $\boldsymbol{\sigma}_i$ and $\boldsymbol{\theta}_i$, we obtain

$$\int_{\Omega_1} \mathbf{a}(\cdot, \boldsymbol{\theta}_1) \cdot \boldsymbol{\zeta}_1 \, dx \;+\; \int_{\Omega_2} \boldsymbol{\theta}_2 \cdot \boldsymbol{\zeta}_2 \, dx \;-\; \sum_{i=1}^{2} \int_{\Omega_i} \boldsymbol{\sigma}_i \cdot \boldsymbol{\zeta}_i \, dx \;=\; 0 \,, \tag{2.2.4}$$

for all $\tilde{\boldsymbol{\zeta}} := (\boldsymbol{\zeta}_1, \boldsymbol{\zeta}_2) \in [L^2(\Omega_1)]^2 \times [L^2(\Omega_2)]^2$. Also, the equilibrium equations give

$$-\sum_{i=1}^{2} \int_{\Omega_i} v_i \,\mathrm{div}\, \boldsymbol{\sigma}_i \;=\; \int_{\Omega_1} f_1 \; v_1 \, dx \,, \tag{2.2.5}$$

for all $\mathbf{v} := (v_1, v_2) \in L^2(\Omega_1) \times L^2(\Omega_2)$.

Further, testing the jump of the fluxes on Γ_1 and the Dirichlet-to-Neumann mapping on Γ, we find that

$$\langle \boldsymbol{\sigma}_1 \cdot \mathbf{n} \;-\; \boldsymbol{\sigma}_2 \cdot \mathbf{n}, \mu \rangle_{\Gamma_1} \;=\; \langle g_2, \mu \rangle_{\Gamma_1} \qquad \forall\, \mu \in H^{1/2}(\Gamma_1) \,, \tag{2.2.6}$$

and

$$\langle \boldsymbol{\sigma}_2 \cdot \boldsymbol{\nu}, \xi \rangle_{\Gamma} \;+\; 2\, \langle \mathbf{W}(\lambda), \xi \rangle_{\Gamma} \;=\; 0 \qquad \forall\, \xi \in H^{1/2}(\Gamma) \,. \tag{2.2.7}$$

Next, we introduce the spaces $X_1 := [L^2(\Omega_1)]^2 \times [L^2(\Omega_2)]^2 \times H^{1/2}(\Gamma)$, $M_1 := H(\mathrm{div}\,;\Omega_1) \times H(\mathrm{div}\,;\Omega_2)$, $M := L^2(\Omega_1) \times L^2(\Omega_2) \times H^{1/2}(\Gamma_1)$, and define the nonlinear operator $A_1 : X_1 \to X_1'$, the linear and bounded operators $B_1 : X_1 \to M_1'$ and $B : M_1 \to M'$, and the functionals $(F_1, G_1, G) \in (X_1', M_1', M')$, as follows

$$[A_1(\tilde{\boldsymbol{\theta}}, \lambda), (\tilde{\boldsymbol{\zeta}}, \xi)] \;:=\; \int_{\Omega_1} \mathbf{a}(\cdot, \boldsymbol{\theta}_1) \cdot \boldsymbol{\zeta}_1 \, dx \;+\; \int_{\Omega_2} \boldsymbol{\theta}_2 \cdot \boldsymbol{\zeta}_2 \, dx \;+\; 2\, \langle \mathbf{W}(\lambda), \xi \rangle_{\Gamma} \,,$$

$$[B_1(\tilde{\boldsymbol{\theta}}, \lambda), \tilde{\boldsymbol{\tau}}] \;:=\; -\sum_{i=1}^{2} \int_{\Omega_i} \boldsymbol{\theta}_i \cdot \boldsymbol{\tau}_i \, dx \;+\; \langle \boldsymbol{\tau}_2 \cdot \boldsymbol{\nu}, \lambda \rangle_{\Gamma} \,,$$

$$[B(\tilde{\boldsymbol{\sigma}}), (\mathbf{v}, \mu)] \;:=\; -\sum_{i=1}^{2} \int_{\Omega_i} v_i \,\mathrm{div}\, \boldsymbol{\sigma}_i \, dx \;-\; \langle \boldsymbol{\sigma}_1 \cdot \mathbf{n} - \boldsymbol{\sigma}_2 \cdot \mathbf{n}, \mu \rangle_{\Gamma_1} \,,$$

$$[F_1, (\tilde{\boldsymbol{\zeta}}, \xi)] \,=\, 0\,, \quad [G_1, \tilde{\boldsymbol{\tau}}] \,:=\, -\langle \boldsymbol{\tau}_1 \cdot \mathbf{n}, g_0 \rangle_{\Gamma_0} \,-\, \langle \boldsymbol{\tau}_1 \cdot \mathbf{n}, g_1 \rangle_{\Gamma_1}\,,$$

and

$$[G, (\mathbf{v}, \mu)] := \int_{\Omega_1} f_1 v_1 \, dx - \langle g_2, \mu \rangle_{\Gamma_1}\,,$$

for all $(\tilde{\boldsymbol{\theta}}, \lambda)$, $(\tilde{\boldsymbol{\zeta}}, \xi) \in X_1$, $\tilde{\boldsymbol{\sigma}}$, $\tilde{\boldsymbol{\tau}} \in M_1$, $(\mathbf{v}, \mu) \in M$. Hereafter, $[\cdot, \cdot]$ stands for the duality pairing induced by the operators and functionals used in each case. We also remark that (H.1) and (H.2) guarantee that A_1 is well defined.

Therefore, the variational formulation of the nonlocal problem (2.2.2), which is given by (2.2.3) up to (2.2.7), can be rearranged so that it can be written as the following two-fold saddle point system: *Find* $((\tilde{\boldsymbol{\theta}}, \lambda), \tilde{\boldsymbol{\sigma}}, (\mathbf{u}, \eta)) \in X_1 \times M_1 \times M$ *such that*

$$\begin{aligned}
&[A_1(\tilde{\boldsymbol{\theta}}, \lambda), (\tilde{\boldsymbol{\zeta}}, \xi)] + [B_1(\tilde{\boldsymbol{\zeta}}, \xi), \tilde{\boldsymbol{\sigma}}] && = [F_1, (\tilde{\boldsymbol{\zeta}}, \xi)]\,, \\
&[B_1(\tilde{\boldsymbol{\theta}}, \lambda), \tilde{\boldsymbol{\tau}}] \qquad\qquad + \; [B(\tilde{\boldsymbol{\tau}}), (\mathbf{u}, \eta)] && = [G_1, \tilde{\boldsymbol{\tau}}]\,, \\
&\qquad [B(\tilde{\boldsymbol{\sigma}}), (\mathbf{v}, \mu)] && = [G, (\mathbf{v}, \mu)]\,,
\end{aligned} \tag{2.2.8}$$

for all $((\tilde{\boldsymbol{\zeta}}, \xi), \tilde{\boldsymbol{\tau}}, (\mathbf{v}, \mu)) \in X_1 \times M_1 \times M$.

In order to prove the unique solvability and a-priori estimate for (2.2.8), we introduce next an equivalent formulation. To this end we observe that each $\xi \in H^{1/2}(\Gamma)$ can be uniquely decomposed as $\xi = \tilde{\xi} + q$, with

$$\tilde{\xi} := \left(\xi - \frac{1}{|\Gamma|} \int_\Gamma \xi ds \right) \in H_0^{1/2}(\Gamma) \quad \text{and} \quad q := \frac{1}{|\Gamma|} \int_\Gamma \xi ds\,,$$

which means that $H^{1/2}(\Gamma) = H_0^{1/2}(\Gamma) \oplus \mathbf{R}$. Moreover, it is easy to see that $\|\xi\|^2_{H^{1/2}(\Gamma)} = \|\tilde{\xi}\|^2_{H^{1/2}(\Gamma)} + |\Gamma|\,|q|^2$.

Then, we consider the alternative formulation: *Find* $((\tilde{\boldsymbol{\theta}}, \tilde{\lambda}), \tilde{\boldsymbol{\sigma}}, (\mathbf{u}, \eta, p)) \in \tilde{X}_1 \times M_1 \times \tilde{M}$ *such that*

$$\begin{aligned}
&[A_1(\tilde{\boldsymbol{\theta}}, \tilde{\lambda}), (\tilde{\boldsymbol{\zeta}}, \tilde{\xi})] + [B_1(\tilde{\boldsymbol{\zeta}}, \tilde{\xi}), \tilde{\boldsymbol{\sigma}}] && = [F_1, (\tilde{\boldsymbol{\zeta}}, \tilde{\xi})]\,, \\
&[B_1(\tilde{\boldsymbol{\theta}}, \tilde{\lambda}), \tilde{\boldsymbol{\tau}}] \qquad\qquad + \; [\tilde{B}(\tilde{\boldsymbol{\tau}}), (\mathbf{u}, \eta, p)] && = [G_1, \tilde{\boldsymbol{\tau}}]\,, \\
&\qquad [\tilde{B}(\tilde{\boldsymbol{\sigma}}), (\mathbf{v}, \mu, q)] && = [\tilde{G}, (\mathbf{v}, \mu, q)]\,,
\end{aligned} \tag{2.2.9}$$

for all $((\tilde{\boldsymbol{\zeta}}, \tilde{\xi}), \tilde{\boldsymbol{\tau}}, (\mathbf{v}, \mu, q)) \in \tilde{X}_1 \times M_1 \times \tilde{M}$, where $\tilde{X}_1 := [L^2(\Omega_1)]^2 \times [L^2(\Omega_2)]^2 \times H_0^{1/2}(\Gamma)$, $\tilde{M} := L^2(\Omega_1) \times L^2(\Omega_2) \times H^{1/2}(\Gamma_1) \times \mathbf{R}$, and the bounded linear operator $\tilde{B} : M_1 \to \tilde{M}'$, and the functional $\tilde{G} \in \tilde{M}'$, are defined by

$$[\tilde{B}(\tilde{\boldsymbol{\sigma}}), (\mathbf{v}, \mu, q)] \,:=\, -\sum_{i=1}^{2} \int_{\Omega_i} v_i \operatorname{div} \boldsymbol{\sigma}_i \, dx \,-\, \langle \boldsymbol{\sigma}_1 \cdot \mathbf{n} - \boldsymbol{\sigma}_2 \cdot \mathbf{n}, \mu \rangle_{\Gamma_1} \,+\, q\, \langle \boldsymbol{\sigma}_2 \cdot \boldsymbol{\nu}, 1 \rangle_\Gamma\,,$$

and

$$[\tilde{G}, (\mathbf{v}, \mu, q)] \; := \; \int_{\Omega_1} f_1 v_1 \, dx - \langle g_2, \mu \rangle_{\Gamma_1} \, .$$

The following theorem establishes the equivalence between (2.2.8) and (2.2.9).

THEOREM 2.2.1 *If* $((\tilde{\boldsymbol{\theta}}, \lambda), \tilde{\boldsymbol{\sigma}}, (\mathbf{u}, \eta)) \in X_1 \times M_1 \times M$ *is a solution of* (2.2.8)*, where* $\lambda := \tilde{\lambda} + p$*, with* $\tilde{\lambda} \in H_0^{1/2}(\Gamma)$ *and* $p \in \mathbf{R}$*, then* $((\tilde{\boldsymbol{\theta}}, \tilde{\lambda}), \tilde{\boldsymbol{\sigma}}, (\mathbf{u}, \eta, p)) \in \tilde{X}_1 \times M_1 \times \tilde{M}$ *is a solution of* (2.2.9)*. Conversely, if* $((\tilde{\boldsymbol{\theta}}, \tilde{\lambda}), \tilde{\boldsymbol{\sigma}}, (\mathbf{u}, \eta, p)) \in \tilde{X}_1 \times M_1 \times \tilde{M}$ *is a solution of* (2.2.9)*, then* $((\tilde{\boldsymbol{\theta}}, \lambda), \tilde{\boldsymbol{\sigma}}, (\mathbf{u}, \eta)) \in X_1 \times M_1 \times M$ *is a solution of* (2.2.8)*, with* $\lambda := \tilde{\lambda} + p$*.*

PROOF. It is based on the decomposition $H^{1/2}(\Gamma) := H_0^{1/2}(\Gamma) \oplus \mathbf{R}$, the symmetry of $\mathbf{W}$, and the fact that $\mathbf{W}(1) = 0$ on Γ. We omit further details and refer the interested reader to the similar result given by Theorem 2.1 in [16]. □

According to Theorem 2.2.1, we now concentrate on the equivalent problem (2.2.9). The corresponding continuous and discrete analyses are based on the abstract theory developed in [51] and [57] (see also Section 4 in [59]).

We first recall the following result from [59].

LEMMA 2.2.1 *The nonlinear operator* $A_1 : \tilde{X}_1 \to \tilde{X}_1'$ *is Lipschitz continuous and strongly monotone, that is there exist* κ, $\alpha > 0$*, such that*

$$\| A_1(\tilde{\boldsymbol{\theta}}, \tilde{\lambda}) \; - \; A_1(\tilde{\boldsymbol{\zeta}}, \tilde{\xi}) \|_{\tilde{X}_1'} \; \leq \; \kappa \, \| (\tilde{\boldsymbol{\theta}}, \tilde{\lambda}) - (\tilde{\boldsymbol{\zeta}}, \tilde{\xi}) \|_{X_1} \, ,$$

and

$$[A_1(\tilde{\boldsymbol{\theta}}, \tilde{\lambda}) \; - \; A_1(\tilde{\boldsymbol{\zeta}}, \tilde{\xi}), (\tilde{\boldsymbol{\theta}}, \tilde{\lambda}) \; - \; (\tilde{\boldsymbol{\zeta}}, \tilde{\xi})] \; \geq \; \alpha \, \| (\tilde{\boldsymbol{\theta}}, \tilde{\lambda}) \; - \; (\tilde{\boldsymbol{\zeta}}, \tilde{\xi}) \|_{X_1}^2 \, ,$$

for all $(\tilde{\boldsymbol{\theta}}, \tilde{\lambda})$*,* $(\tilde{\boldsymbol{\zeta}}, \tilde{\xi}) \in \tilde{X}_1$*.*

PROOF. It follows from the assumptions (H.3) and (H.4), and the strong coerciveness of the operator $\mathbf{W}$ (cf. (2.2.1)). We refer to Lemma 5.3 in [59] for details. □

We now state the continuous inf-sup condition for $\tilde{B}$.

LEMMA 2.2.2 *There exists* $\beta > 0$ *such that for all* $(\mathbf{v}, \mu, q) \in \tilde{M}$ *there holds*

$$\sup_{\substack{\tilde{\boldsymbol{\tau}} \in M_1 \\ \tilde{\boldsymbol{\tau}} \neq 0}} \frac{[\tilde{B}(\tilde{\boldsymbol{\tau}}), (\mathbf{v}, \mu, q)]}{\| \tilde{\boldsymbol{\tau}} \|_{M_1}} \; \geq \; \beta \, \| (\mathbf{v}, \mu, q) \|_{\tilde{M}} \, .$$

PROOF. Given $(\mathbf{v}, \mu, q) \in \tilde{M} := L^2(\Omega_1) \times L^2(\Omega_2) \times H^{1/2}(\Gamma_1) \times \mathbf{R}$, we put $\tilde{v} := \begin{cases} v_1 & \text{in} \quad \Omega_1 \\ v_2 & \text{in} \quad \Omega_2 \end{cases}$, and let $z \in H^1(\Omega)$ be the solution of the mixed boundary value problem: $-\Delta z = \tilde{v}$ in Ω, $z = 0$ on Γ_0, $\dfrac{\partial z}{\partial \boldsymbol{\nu}} = q$ on Γ, whose a-priori estimate gives $\|z\|_{H^1(\Omega)} \leq C \left\{ \|\tilde{v}\|_{L^2(\Omega)} + |q| \right\}$. Then we set $\hat{\boldsymbol{\tau}} := (\hat{\boldsymbol{\tau}}_1, \hat{\boldsymbol{\tau}}_2)$, with $\hat{\boldsymbol{\tau}}_i = \nabla z|_{\Omega_i}$, for all $i \in \{1, 2\}$, and observe that $\operatorname{div} \hat{\boldsymbol{\tau}}_i = -v_i$ in Ω_i, $\hat{\boldsymbol{\tau}}_1 \cdot \mathbf{n} = \hat{\boldsymbol{\tau}}_2 \cdot \mathbf{n}$ on Γ_1, $\hat{\boldsymbol{\tau}}_2 \cdot \boldsymbol{\nu} = q$ on Γ, and $\|\hat{\boldsymbol{\tau}}_i\|_{H(\operatorname{div};\Omega_i)} \leq \hat{C} \{ \|\tilde{v}\|_{L^2(\Omega)} + |q| \}$. It follows that

$$\sup_{\substack{\tilde{\boldsymbol{\tau}} \in M_1 \\ \tilde{\boldsymbol{\tau}} \neq 0}} \frac{[\tilde{B}(\tilde{\boldsymbol{\tau}}), (\mathbf{v}, \mu, q)]}{\|\tilde{\boldsymbol{\tau}}\|_{M_1}} \geq \frac{[\tilde{B}(\hat{\boldsymbol{\tau}}), (\mathbf{v}, \mu, q)]}{\|\hat{\boldsymbol{\tau}}\|_{M_1}}$$

$$= \frac{\sum_{i=1}^{2} \|v_i\|^2_{L^2(\Omega_i)} + |\Gamma| \, q^2}{\|\hat{\boldsymbol{\tau}}\|_{M_1}} \geq C \, \|(\mathbf{v}, q)\|_{L^2(\Omega_1) \times L^2(\Omega_2) \times \mathbf{R}} \,,$$

where $C > 0$ depends on $|\Gamma|$ and $\hat{C}$.

Similarly, given $\rho \in H^{-1/2}(\Gamma_1)$, $\rho \neq 0$, we let $w \in H^1(\Omega_1)$ be the solution of the mixed boundary value problem: $-\Delta w = 0$ in Ω_1, $w = 0$ on Γ_0, $\dfrac{\partial w}{\partial \mathbf{n}} = -\rho$ on Γ_1, whose a-priori estimate yields $\|w\|_{H^1(\Omega_1)} \leq C \, \|\rho\|_{H^{-1/2}(\Gamma_1)}$. Then we set $\hat{\boldsymbol{\tau}} := (\hat{\boldsymbol{\tau}}_1, \hat{\boldsymbol{\tau}}_2)$, with $\hat{\boldsymbol{\tau}}_1 = \nabla w$ in Ω_1, $\hat{\boldsymbol{\tau}}_2 = 0$ in Ω_2, and observe that $\operatorname{div} \hat{\boldsymbol{\tau}}_i = 0$ in Ω_i, $\hat{\boldsymbol{\tau}}_1 \cdot \mathbf{n} = -\rho$ on Γ_1, and $\|\hat{\boldsymbol{\tau}}_1\|_{H(\operatorname{div};\Omega_1)} \leq \hat{C} \, \|\rho\|_{H^{-1/2}(\Gamma_1)}$. It follows that

$$\sup_{\substack{\tilde{\boldsymbol{\tau}} \in M_1 \\ \tilde{\boldsymbol{\tau}} \neq 0}} \frac{[\tilde{B}(\tilde{\boldsymbol{\tau}}), (\mathbf{v}, \mu, q)]}{\|\tilde{\boldsymbol{\tau}}\|_{M_1}} \geq \frac{[\tilde{B}(\hat{\boldsymbol{\tau}}), (\mathbf{v}, \mu, q)]}{\|\hat{\boldsymbol{\tau}}\|_{M_1}} = \frac{\langle \rho, \mu \rangle_{\Gamma_1}}{\|\hat{\boldsymbol{\tau}}_1\|_{H(\operatorname{div};\Omega_1)}} \geq C \, \frac{\langle \rho, \mu \rangle_{\Gamma_1}}{\|\rho\|_{H^{-1/2}(\Gamma_1)}} \,,$$

with $C := \dfrac{1}{\hat{C}}$. Since the above estimate holds for arbitrary $\rho \in H^{-1/2}(\Gamma_1)$, $\rho \neq 0$, we conclude that $\displaystyle\sup_{\substack{\tilde{\boldsymbol{\tau}} \in M_1 \\ \tilde{\boldsymbol{\tau}} \neq 0}} \frac{[\tilde{B}(\tilde{\boldsymbol{\tau}}), (\mathbf{v}, \mu, q)]}{\|\tilde{\boldsymbol{\tau}}\|_{M_1}} \geq \hat{\beta} \, \|\mu\|_{H^{1/2}(\Gamma_1)}$, which finishes the proof of the lemma. □

The continuous inf-sup condition for B_1 is proved next. For this purpose, we need the kernel of the operator $\tilde{B}$, that is $\tilde{V} := \{ \tilde{\boldsymbol{\tau}} \in M_1 : \quad [\tilde{B}(\tilde{\boldsymbol{\tau}}), (\mathbf{v}, \mu, q)] = 0 \quad \forall \, (\mathbf{v}, \mu, q) \in \tilde{M} \}$, which yields $\tilde{V} := \{ \tilde{\boldsymbol{\tau}} := (\boldsymbol{\tau}_1, \boldsymbol{\tau}_2) \in M_1 : \ \operatorname{div} \boldsymbol{\tau}_i = 0 \text{ in } \Omega_i \,, \ \boldsymbol{\tau}_1 \cdot \mathbf{n} = \boldsymbol{\tau}_2 \cdot \mathbf{n} \text{ on } \Gamma_1 \,, \ \langle \boldsymbol{\tau}_2 \cdot \boldsymbol{\nu}, 1 \rangle_\Gamma = 0 \}$.

LEMMA 2.2.3 *There exists $\beta_1 > 0$ such that for all $\tilde{\boldsymbol{\tau}} \in \tilde{V}$ there holds*

$$\sup_{\substack{(\tilde{\boldsymbol{\zeta}},\tilde{\xi}) \in \tilde{X}_1 \\ (\tilde{\boldsymbol{\zeta}},\tilde{\xi}) \neq 0}} \frac{[B_1(\tilde{\boldsymbol{\zeta}},\tilde{\xi}),\tilde{\boldsymbol{\tau}}]}{\|(\tilde{\boldsymbol{\zeta}},\tilde{\xi})\|_{X_1}} \geq \beta_1 \|\tilde{\boldsymbol{\tau}}\|_{M_1} .$$

PROOF. Let $\tilde{\boldsymbol{\tau}} := (\boldsymbol{\tau}_1, \boldsymbol{\tau}_2) \in \tilde{V}$. Then, using that $\|\boldsymbol{\tau}_i\|_{[L^2(\Omega_i)]^2} = \|\boldsymbol{\tau}_i\|_{H(\mathrm{div}\,;\Omega_i)}$, we have

$$\sup_{\substack{(\tilde{\boldsymbol{\zeta}},\tilde{\xi}) \in \tilde{X}_1 \\ (\tilde{\boldsymbol{\zeta}},\tilde{\xi}) \neq 0}} \frac{[B_1(\tilde{\boldsymbol{\zeta}},\tilde{\xi}),\tilde{\boldsymbol{\tau}}]}{\|(\tilde{\boldsymbol{\zeta}},\tilde{\xi})\|_{X_1}} \geq \frac{[B_1((-\tilde{\boldsymbol{\tau}},0),\tilde{\boldsymbol{\tau}}]}{\|\tilde{\boldsymbol{\tau}}\|_{[L^2(\Omega_1)]^2 \times [L^2(\Omega_2)]^2}} = \|\tilde{\boldsymbol{\tau}}\|_{[L^2(\Omega_1)]^2 \times [L^2(\Omega_2)]^2} = \|\tilde{\boldsymbol{\tau}}\|_{M_1} ,$$

which completes the proof with $\beta_1 = 1$. □

We can establish now the unique solvability and a-priori estimate for the mixed variational formulation (2.2.9) (and hence for (2.2.8)).

THEOREM 2.2.2 *There exists a unique solution $((\tilde{\boldsymbol{\theta}}, \tilde{\lambda}), \tilde{\boldsymbol{\sigma}}, (\mathbf{u}, \eta, p)) \in \tilde{X}_1 \times M_1 \times \tilde{M}$ to the two-fold saddle point system* (2.2.9). *Moreover, there exists $C > 0$, independent of the solution, such that*

$$\|((\tilde{\boldsymbol{\theta}}, \tilde{\lambda}), \tilde{\boldsymbol{\sigma}}, (\mathbf{u}, \eta, p))\|_{\tilde{X}_1 \times M_1 \times \tilde{M}}$$

$$\leq C \left\{ \|g_0\|_{H^{1/2}(\Gamma_0)} + \|g_1\|_{H^{1/2}(\Gamma_1)} + \|f_1\|_{L^2(\Omega_1)} + \|g_2\|_{H^{-1/2}(\Gamma_1)} + \|A_1(0,0)\|_{\tilde{X}_1'} \right\} .$$

PROOF. By virtue of the previous Lemmas 2.1, 2.2, and 2.3, and the boundedness of the operators and functionals $\tilde{B}$, B_1, F_1, G_1, and $\tilde{G}$, the result follows straightforward from the abstract Theorem 2.1 in [57] (see also Theorem 1 in [51], or Theorem 4.1 in [59]), and the fact that

$$\|F_1\|_{\tilde{X}_1'} + \|G_1\|_{M_1'} + \|\tilde{G}\|_{\tilde{M}'} \leq \left\{ \|g_0\|_{H^{1/2}(\Gamma_0)} + \|g_1\|_{H^{1/2}(\Gamma_1)} + \|f_1\|_{L^2(\Omega_1)} + \|g_2\|_{H^{-1/2}(\Gamma_1)} \right\} .$$

□

2.3 The Galerkin scheme

In what follows, we assume for simplicity that Γ_0 and Γ_1 are polygonal boundaries and that the interior region of Γ_1 is convex. Then, we let $\mathcal{T}_h$ be a regular triangulation of $\bar{\Omega}$ made up of straight and curved triangles T of diameter h_T such that $h :=$

$\max_{T\in\mathcal{T}_h} h_T$. We also assume that for each $T \in \mathcal{T}_h$, either $T \subseteq \bar{\Omega}_1$ or $T \subseteq \bar{\Omega}_2$. In addition, we remark that the curved triangles are those having an edge on Γ given by one of the segments $\{\,\mathbf{z}(t) : \quad t \in [t_{i-1}, t_i]\,\}$, $i \in \{1, ..., n-1\}$, where $0 = t_0 < t_1 < t_2 < \cdots < t_n = 2\pi$ is a corresponding partition of $[0, 2\pi]$, and $\mathbf{z} : [0, 2\pi] \to \Gamma$ is the usual parametrization of the circle given by $\mathbf{z}(t) := r(\cos(t), \sin(t))^{\mathbf{T}}$ for all $t \in [0, 2\pi]$.

Next, we let $\hat{T}$ be the reference triangle with vertices $\hat{P}_1 = (0,0)^{\mathbf{T}}$, $\hat{P}_2 = (1,0)^{\mathbf{T}}$, and $\hat{P}_3 = (0,1)^{\mathbf{T}}$, and consider a family of bijective mappings $F_T : \hat{T} \to T$, for all $T \in \mathcal{T}_h$. If T is a straight triangle, then F_T is the affine mapping defined by $F_T(\hat{x}) := B_T\hat{x} + b_T$, where B_T, a square matrix of order 2, and $b_T \in \mathbf{R}^2$ depend on the vertices of T. On the other hand, if T is a curved triangle with vertices P_1, P_2, and P_3, such that $P_2 = \mathbf{z}(t_{j-1}) \in \Gamma$ and $P_3 = \mathbf{z}(t_j) \in \Gamma$, then

$$F_T(\hat{x}) = B_T\hat{x} + b_T + G_T(\hat{x}) \qquad \forall\, \hat{x} := (\hat{x}_1, \hat{x}_2)^{\mathbf{T}} \in \hat{T}\,, \tag{2.3.1}$$

where

$$G_T(\hat{x}) = \frac{\hat{x}_1}{1-\hat{x}_2}\Big\{\mathbf{z}(t_{j-1} + \hat{x}_2(t_j - t_{j-1})) \,-\, [\mathbf{z}(t_{j-1}) + \hat{x}_2(\mathbf{z}(t_j) - \mathbf{z}(t_{j-1}))]\Big\}. \tag{2.3.2}$$

We now let $\mathbf{J}(F_T)$ and $D(F_T)$ be the Jacobian and the Frêchet differential, respectively, of the nonlinear mapping F_T defined by (2.3.1) and (2.3.2). Their main properties are summarized in the following lemma.

Lemma 2.3.1 *There exists $h_0 > 0$ such that for all $h \in (0, h_0)$ F_T is a diffeomorphism of class $\mathcal{C}^\infty$ that maps one-to-one $\hat{T}$ onto the curved triangle T in such a way that $F_T(\hat{P}_i) = P_i$ for all $i \in \{1,2,3\}$. In addition, $\mathbf{J}(F_T)$ does not vanish in a neighborhood of $\hat{T}$, and there exist positive constants C_i, $i \in \{1, ..., 5\}$, independent of T and h, such that for all $T \in \mathcal{T}_h$ there hold*

$$C_1\, h_T^2 \,\le\, |\mathbf{J}(F_T)| \,\le\, C_2\, h_T^2\,, \quad |\mathbf{J}(F_T)^k|_{W^{1,\infty}(\hat{T})} \,\le\, C_3\, h_T^{1+2k} \quad \forall\, k \in \{-1, 1\}\,,$$

and

$$|(DF_T)|_{W^{k,\infty}(\hat{T})} \,\le\, C_4\, h_T^{k+1}\,, \quad |(DF_T)^{-1}|_{W^{k,\infty}(\hat{T})} \,\le\, C_5\, h_T^{k-1} \quad \forall\, k \in \{0, 1\}\,.$$

Proof. See Theorem 22.4 in [98]. □

In order to introduce the Galerkin scheme, we first let

$$\mathcal{RT}_0(\hat{T}) := \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix} \right\},$$

and for each triangle $T \in \mathcal{T}_h$, we put

$$\begin{aligned} \mathcal{RT}_0(T) &:= \{ \boldsymbol{\tau} : \quad \boldsymbol{\tau} = \mathbf{J}(F_T)^{-1} (DF_T)\hat{\boldsymbol{\tau}} \circ F_T^{-1}, \quad \hat{\boldsymbol{\tau}} \in \mathcal{RT}_0(\hat{T}) \} \quad \text{and} \\ [\mathcal{P}_0(T)]^2 &:= \{ \boldsymbol{\zeta} : \quad \boldsymbol{\zeta} = \mathbf{J}(F_T)^{-1} (DF_T)\hat{\boldsymbol{\zeta}} \circ F_T^{-1}, \quad \hat{\boldsymbol{\zeta}} \in [\mathbf{P}_0(\hat{T})]^2 \}. \end{aligned} \tag{2.3.3}$$

Hereafter, given a non-negative integer k and a subset S of $\mathbf{R}$ or $\mathbf{R}^2$, we denote by $\mathbf{P}_k(S)$ the space of polynomials defined on S of degree $\leq k$.

Then, we define the following finite element subspaces

$$X_{1,h}^{\boldsymbol{\theta}_i} := \{ \boldsymbol{\zeta}_h \in [L^2(\Omega_i)]^2 : \quad \boldsymbol{\zeta}_h|_T \in [\mathcal{P}_0(T)]^2 \quad \forall\, T \in \mathcal{T}_h ,\ T \subseteq \Omega_i \} \quad i \in \{1,2\},$$

$$X_{1,h}^{\lambda} := \{ \xi_h : \Gamma \to \mathbf{R}, \quad \xi_h = \hat{\xi}_h \circ \mathbf{z}^{-1},\ \hat{\xi}_h \in X_{1,h}^{\lambda}(0,2\pi) \},$$

with

$$\begin{aligned} X_{1,h}^{\lambda}(0,2\pi) := \{ \hat{\xi}_h : [0,2\pi] \to \mathbf{R}, \quad & \hat{\xi}_h \text{ is continuous and periodic of period } 2\pi, \\ & \hat{\xi}_h|_{[t_{j-1},t_j]} \in \mathbf{P}_1(t_{j-1},t_j) \quad \forall j \in \{1,...,n\} \}, \end{aligned}$$

$$M_{1,h}^{\boldsymbol{\sigma}_i} := \{ \boldsymbol{\tau}_h \in H(\operatorname{div};\Omega_i) : \quad \boldsymbol{\tau}_h|_T \in \mathcal{RT}_0(T) \quad \forall\, T \in \mathcal{T}_h ,\ T \subseteq \Omega_i \} \quad i \in \{1,2\},$$

and

$$M_h^{u_i} := \{ v_h \in L^2(\Omega_i) : \quad v_h|_T \in \mathbf{P}_0(T), \quad \forall\, T \in \mathcal{T}_h ,\ T \subseteq \Omega_i \} \quad i \in \{1,2\}.$$

Further, we introduce an independent partition $\{\tilde{\gamma}_1, \tilde{\gamma}_2, ..., \tilde{\gamma}_m\}$ of Γ_1, denote $\tilde{h} := \max\{ |\tilde{\gamma}_j| : \ j \in \{1,...,m\} \}$, and define the finite element subspace for the unknown η as

$$M_{\tilde{h}}^{\eta} := \{ \mu_{\tilde{h}} \in H^{1/2}(\Gamma_1) : \quad \mu_{\tilde{h}}|_{\tilde{\gamma}_j} \in \mathbf{P}_1(\tilde{\gamma}_j) \quad \forall\, j \in \{1,...,m\} \}.$$

Hence, we set $X_{1,h} := X_{1,h}^{\boldsymbol{\theta}_1} \times X_{1,h}^{\boldsymbol{\theta}_2} \times X_{1,h}^{\lambda}$, $M_{1,h} := M_{1,h}^{\boldsymbol{\sigma}_1} \times M_{1,h}^{\boldsymbol{\sigma}_2}$, $M_{h,\tilde{h}} := M_h^{u_1} \times M_h^{u_2} \times M_{\tilde{h}}^{\eta}$, and state the Galerkin scheme associated with the continuous problem (2.2.8) as: *Find* $((\tilde{\boldsymbol{\theta}}_h, \lambda_h), \tilde{\boldsymbol{\sigma}}_h, (\mathbf{u}_h, \eta_{\tilde{h}})) \in X_{1,h} \times M_{1,h} \times M_{h,\tilde{h}}$ *such that*

$$\begin{aligned} &[A_1(\tilde{\boldsymbol{\theta}}_h, \lambda_h), (\tilde{\boldsymbol{\zeta}}_h, \xi_h)] + [B_1(\tilde{\boldsymbol{\zeta}}_h, \xi_h), \tilde{\boldsymbol{\sigma}}_h] &&= [F_1, (\tilde{\boldsymbol{\zeta}}_h, \xi_h)], \\ &[B_1(\tilde{\boldsymbol{\theta}}_h, \lambda_h), \tilde{\boldsymbol{\tau}}_h] + [B(\tilde{\boldsymbol{\tau}}_h), (\mathbf{u}_h, \eta_{\tilde{h}})] &&= [G_1, \tilde{\boldsymbol{\tau}}_h], \\ &[B(\tilde{\boldsymbol{\sigma}}_h), (\mathbf{v}_h, \mu_{\tilde{h}})] &&= [G, (\mathbf{v}_h, \mu_{\tilde{h}})], \end{aligned} \tag{2.3.4}$$

for all $((\tilde{\boldsymbol{\zeta}}_h, \xi_h), \tilde{\boldsymbol{\tau}}_h, (\mathbf{v}_h, \mu_{\tilde{h}})) \in X_{1,h} \times M_{1,h} \times M_{h,\tilde{h}}$.

In addition, we define $X^{\tilde{\lambda}}_{1,h} := X^{\lambda}_{1,h} \cap H^{1/2}_0(\Gamma)$, $\tilde{X}_{1,h} := X^{\boldsymbol{\theta}_1}_{1,h} \times X^{\boldsymbol{\theta}_2}_{1,h} \times X^{\tilde{\lambda}}_{1,h}$, $\tilde{M}_{h,\tilde{h}} := M^{u_1}_h \times M^{u_2}_h \times M^{\eta}_{\tilde{h}} \times \mathbf{R}$, and introduce an alternative formulation, which is the discrete analogue of (2.2.9): *Find* $((\tilde{\boldsymbol{\theta}}_h, \tilde{\lambda}_h), \tilde{\boldsymbol{\sigma}}_h, (\mathbf{u}_h, \eta_{\tilde{h}}, p_h)) \in \tilde{X}_{1,h} \times M_{1,h} \times \tilde{M}_{h,\tilde{h}}$ *such that*

$$\begin{aligned}
&[A_1(\tilde{\boldsymbol{\theta}}_h, \tilde{\lambda}_h), (\tilde{\boldsymbol{\zeta}}_h, \tilde{\xi}_h)] + [B_1(\tilde{\boldsymbol{\zeta}}_h, \tilde{\xi}_h), \tilde{\boldsymbol{\sigma}}_h] && = [F_1, (\tilde{\boldsymbol{\zeta}}_h, \tilde{\xi}_h)]\,,\\
&[B_1(\tilde{\boldsymbol{\theta}}_h, \tilde{\lambda}_h), \tilde{\boldsymbol{\tau}}_h] \qquad\qquad + \ [\tilde{B}(\tilde{\boldsymbol{\tau}}_h), (\mathbf{u}_h, \eta_{\tilde{h}}, p_h)] && = [G_1, \tilde{\boldsymbol{\tau}}_h]\,,\\
&\qquad\qquad [\tilde{B}(\tilde{\boldsymbol{\sigma}}_h), (\mathbf{v}_h, \mu_{\tilde{h}}, q_h)] && = [\tilde{G}, (\mathbf{v}_h, \mu_{\tilde{h}}, q_h)]\,,
\end{aligned} \tag{2.3.5}$$

for all $((\tilde{\boldsymbol{\zeta}}_h, \tilde{\xi}_h), \tilde{\boldsymbol{\tau}}_h, (\mathbf{v}_h, \mu_{\tilde{h}}, q_h)) \in \tilde{X}_{1,h} \times M_{1,h} \times \tilde{M}_{h,\tilde{h}}$.

Then, similarly as for the continuous case, (2.3.4) and (2.3.5) are also equivalent. More precisely, we state now the discrete analogue of Theorem 2.2.1.

THEOREM 2.3.1 *If* $((\tilde{\boldsymbol{\theta}}_h, \lambda_h), \tilde{\boldsymbol{\sigma}}_h, (\mathbf{u}_h, \eta_{\tilde{h}})) \in X_{1,h} \times M_{1,h} \times M_{h,\tilde{h}}$ *is a solution of* (2.3.4)*, where* $\lambda_h := \tilde{\lambda}_h + p_h$*, with* $\tilde{\lambda}_h \in X^{\tilde{\lambda}}_{1,h}$ *and* $p_h \in \mathbf{R}$*, then* $((\tilde{\boldsymbol{\theta}}_h, \tilde{\lambda}_h), \tilde{\boldsymbol{\sigma}}_h, (\mathbf{u}_h, \eta_{\tilde{h}}, p_h)) \in \tilde{X}_{1,h} \times M_{1,h} \times \tilde{M}_{h,\tilde{h}}$ *is a solution of* (2.3.5)*. Conversely, if* $((\tilde{\boldsymbol{\theta}}_h, \tilde{\lambda}_h), \tilde{\boldsymbol{\sigma}}_h, (\mathbf{u}_h, \eta_{\tilde{h}}, p_h)) \in \tilde{X}_{1,h} \times M_{1,h} \times \tilde{M}_{h,\tilde{h}}$ *is a solution of* (2.3.5)*, then* $((\tilde{\boldsymbol{\theta}}_h, \lambda_h), \tilde{\boldsymbol{\sigma}}_h, (\mathbf{u}_h, \eta_{\tilde{h}})) \in X_{1,h} \times M_{1,h} \times M_{h,\tilde{h}}$ *is a solution of* (2.3.4)*, with* $\lambda_h := \tilde{\lambda}_h + p_h$.

Our next goal is the stability and unique solvability of the Galerkin scheme (2.3.5). For this purpose, we consider first the equilibrium interpolation operator $\mathcal{E}_h : [H^1(\Omega)]^2 \to M^{\boldsymbol{\sigma}}_{1,h}$, where $M^{\boldsymbol{\sigma}}_{1,h} := \{\, \boldsymbol{\tau}_h \in H(\mathrm{div}\,;\Omega) : \quad \boldsymbol{\tau}_h|_T \in \mathcal{RT}_0(T) \quad \forall T \in \mathcal{T}_h \,\}$. According to the Piola transformation used in the definition of $\mathcal{RT}_0(T)$ (cf. (2.3.3)), we have

$$\mathcal{E}_h(\boldsymbol{\tau})|_T := \mathbf{J}(F_T)^{-1}\,(DF_T)\,\hat{\mathcal{E}}(\hat{\boldsymbol{\tau}}) \circ F_T^{-1} \qquad \forall\, \boldsymbol{\tau} \in [H^1(\Omega)]^2\,, \quad \forall\, T \in \mathcal{T}_h\,,$$

where $\hat{\boldsymbol{\tau}} := \mathbf{J}(F_T)\,(DF_T)^{-1}\boldsymbol{\tau} \circ F_T$ and $\hat{\mathcal{E}} : [H^1(\hat{T})]^2 \to \mathcal{RT}_0(\hat{T})$ is the local equilibrium interpolation operator on the reference triangle $\hat{T}$ (see, e.g. [25], [90]).

We remark that the approximation properties of $\mathcal{E}_h$ are well known for triangulations with just straight triangles, that is for polygonal domains (see, e.g. [90]). We follow next a similar procedure and prove an approximation property in $[L^2(\Omega)]^2$ for the present operator $\mathcal{E}_h$ involving curved triangles.

LEMMA 2.3.2 *There exists* $C > 0$*, independent of* h*, such that*

$$||\boldsymbol{\tau} - \mathcal{E}_h(\boldsymbol{\tau})||_{[L^2(\Omega)]^2} \le C\, h\, ||\boldsymbol{\tau}||_{[H^1(\Omega)]^2} \quad \forall\, \boldsymbol{\tau} \in [H^1(\Omega)]^2\,. \tag{2.3.6}$$

Proof. Let $\boldsymbol{\tau} \in [H^1(\Omega)]^2$. Using the change of variable $x = F_T(\hat{x})$, we find that

$$\begin{aligned}
\|\boldsymbol{\tau} - \mathcal{E}_h(\boldsymbol{\tau})\|^2_{[L^2(T)]^2} &= \int_T \|\boldsymbol{\tau}(x) - \mathbf{J}(F_T)^{-1}\,(DF_T)\,\hat{\mathcal{E}}(\hat{\boldsymbol{\tau}})(F_T^{-1}(x))\|_2^2\,dx \\
&= \int_{\hat{T}} |\mathbf{J}(F_T)|\,\left\|(\boldsymbol{\tau}\circ F_T)(\hat{x}) - \mathbf{J}(F_T)^{-1}\,(DF_T)\,\hat{\mathcal{E}}(\hat{\boldsymbol{\tau}})(\hat{x})\right\|_2^2\,d\hat{x} \\
&= \int_{\hat{T}} |\mathbf{J}(F_T)|\,\left\|\mathbf{J}(F_T)^{-1}\,(DF_T)\,\left[\hat{\boldsymbol{\tau}}(\hat{x}) - \hat{\mathcal{E}}(\hat{\boldsymbol{\tau}})(\hat{x})\right]\right\|_2^2\,d\hat{x} \\
&\le \int_{\hat{T}} |\mathbf{J}(F_T)|^{-1}\,\|(DF_T)\|_2^2\,\left\|\hat{\boldsymbol{\tau}}(\hat{x}) - \hat{\mathcal{E}}(\hat{\boldsymbol{\tau}})(\hat{x})\right\|_2^2\,d\hat{x}\,, \qquad (2.3.7)
\end{aligned}$$

where $\|\cdot\|_2$ is the usual euclidean norm for both vectors and matrices in $\mathbf{R}^2$ and $\mathbf{R}^{2\times 2}$, respectively.

Now, since $|\mathbf{J}(F_T)^{-1}| = O(h^{-2})$ and $\|(DF_T)\|_2 = O(h)$ (cf. Lemma 2.3.1), and because of the approximation property of $\hat{\mathcal{E}}$ (see, e.g. [25], [90]), we deduce from (2.3.7) that

$$\begin{aligned}
\|\boldsymbol{\tau} - \mathcal{E}_h(\boldsymbol{\tau})\|^2_{[L^2(T)]^2} &\le \hat{C}\,\|\hat{\boldsymbol{\tau}} - \hat{\mathcal{E}}(\hat{\boldsymbol{\tau}})\|^2_{[L^2(\hat{T})]^2} \le \hat{C}\,|\,\hat{\boldsymbol{\tau}}|^2_{[H^1(\hat{T})]^2} \\
&= \hat{C}\,|\,\mathbf{J}(F_T)\,(DF_T)^{-1}\,(\boldsymbol{\tau}\circ F_T)\,|^2_{[H^1(\hat{T})]^2} \\
&\le \hat{C}\,\Big\{\,|\mathbf{J}(F_T)|_{W^{1,\infty}(\hat{T})}\,\|(DF_T)^{-1})\|_{[W^{0,\infty}(\hat{T})]^{2\times 2}}\,\|\boldsymbol{\tau}\circ F_T\|_{[L^2(\hat{T})]^2} \\
&\quad + \|\mathbf{J}(F_T)\|_{W^{0,\infty}(\hat{T})}\,|(DF_T)^{-1}|_{[W^{1,\infty}(\hat{T})]^{2\times 2}}\,\|\boldsymbol{\tau}\circ F_T\|_{[L^2(\hat{T})]^2} \\
&\quad + \|\mathbf{J}(F_T)\|_{W^{0,\infty}(\hat{T})}\,\|(DF_T)^{-1}\|_{[W^{0,\infty}(\hat{T})]^{2\times 2}}\,|\boldsymbol{\tau}\circ F_T|_{[H^1(\hat{T})]^2}\Big\}^2\,, \qquad (2.3.8)
\end{aligned}$$

with a constant $\hat{C} > 0$, depending only on $\hat{T}$.

Then, applying the corresponding norm estimates for $\mathbf{J}(F_T)$ and $(DF_T)^{-1}$ (see again Lemma 2.3.1), changing back the variable $\hat{x}$ by $F_T^{-1}(x)$, and using chain rule in the term $|\boldsymbol{\tau}\circ F_T|_{[H^1(\hat{T})]^2}$, we conclude from (2.3.8) that

$$\|\boldsymbol{\tau} - \mathcal{E}_h(\boldsymbol{\tau})\|^2_{[L^2(T)]^2} \le \hat{C}\,h^2\,\|\boldsymbol{\tau}\|^2_{[H^1(T)]^2}\,.$$

Finally, summing up over all the triangles $T \in \mathcal{T}_h$, we get (2.3.6). □

We also need the following result.

Lemma 2.3.3 *There exists $C > 0$, independent of h, such that*

$$\|\mathrm{div}\,(\mathcal{E}_h(\boldsymbol{\tau}))\|_{L^2(\Omega)} \le C\,\|\mathrm{div}\,\boldsymbol{\tau}\|_{L^2(\Omega)} \qquad \forall\,\boldsymbol{\tau} \in [H^1(\Omega)]^2\,. \qquad (2.3.9)$$

Proof. Let $\boldsymbol{\tau} \in [H^1(\Omega)]^2$. Then, using the identity (1.49) (cf. Lemma 1.5) in chapter III of [25], Cauchy-Schwarz's inequality, and the fact that $\|\mathrm{div}\,\hat{\mathcal{E}}(\hat{\boldsymbol{\tau}})\|_{L^2(\hat{T})} \leq \|\mathrm{div}\,\hat{\boldsymbol{\tau}}\|_{L^2(\hat{T})}$, which follows from the conmuting diagram property on the reference triangle $\hat{T}$, we find that

$$\|\mathrm{div}\,\mathcal{E}_h(\boldsymbol{\tau})\|^2_{L^2(T)} := \int_T \mathrm{div}\,\mathcal{E}_h(\boldsymbol{\tau})\,\mathrm{div}\,\mathcal{E}_h(\boldsymbol{\tau})\,dx \;=\; \int_{\hat{T}} \mathrm{div}\,\widehat{\mathcal{E}_h}(\boldsymbol{\tau})\,\mathrm{div}\,\hat{\mathcal{E}}(\hat{\boldsymbol{\tau}})\,d\hat{x}$$

$$\leq\; \|\mathrm{div}\,\widehat{\mathcal{E}_h}(\boldsymbol{\tau})\|_{L^2(\hat{T})}\,\|\mathrm{div}\,\hat{\mathcal{E}}(\hat{\boldsymbol{\tau}})\|_{L^2(\hat{T})} \;\leq\; \|\mathrm{div}\,\widehat{\mathcal{E}_h}(\boldsymbol{\tau})\|_{L^2(\hat{T})}\,\|\mathrm{div}\,\hat{\boldsymbol{\tau}}\|_{L^2(\hat{T})}\,, \qquad (2.3.10)$$

where $\mathrm{div}\,\widehat{\mathcal{E}_h}(\boldsymbol{\tau})$ stands for $\mathrm{div}\,\mathcal{E}_h(\boldsymbol{\tau}) \circ F_T$.

Then, applying the inequalities (1.40) (cf. Lemma 1.4) and (1.54) (cf. Lemma 1.6) in chapter III of [25], and the estimate for $\mathbf{J}(F_T)$ given in Lemma 2.3.1, we deduce that

$$\|\mathrm{div}\,\widehat{\mathcal{E}_h}(\boldsymbol{\tau})\|_{L^2(\hat{T})} \;\leq\; C\,h_T^{-1}\,\|\mathrm{div}\,\mathcal{E}_h(\boldsymbol{\tau})\|_{L^2(T)} \quad \text{and} \quad \|\mathrm{div}\,\hat{\boldsymbol{\tau}}\|_{L^2(\hat{T})} \;\leq\; C\,h_T\,\|\mathrm{div}\,\boldsymbol{\tau}\|_{L^2(T)}\,,$$

which replaced back into (2.3.10) yields

$$\|\mathrm{div}\,(\mathcal{E}_h(\boldsymbol{\tau}))\|_{L^2(T)} \;\leq\; C\,\|\mathrm{div}\,\boldsymbol{\tau}\|_{L^2(T)} \qquad \forall\, T \in \mathcal{T}_h\,.$$

Thus, the proof of (2.3.9) is completed by summing up over all the triangles $T \in \mathcal{T}_h$. $\square$

The discrete inf-sup condition for $\tilde{B}$ is proved next through several lemmas, similarly as we did in [12].

Lemma 2.3.4 *There exists $\tilde{\beta} > 0$, independent of h and $\tilde{h}$, such that for all $(\mathbf{v}_h, \mu_{\tilde{h}}, q_h) \in \tilde{M}_{h,\tilde{h}}$ there holds*

$$\sup_{\substack{\tilde{\boldsymbol{\tau}}_h \in M_{1,h} \\ \tilde{\boldsymbol{\tau}}_h \neq 0}} \frac{[\tilde{B}(\tilde{\boldsymbol{\tau}}_h), (\mathbf{v}_h, \mu_{\tilde{h}}, q_h)]}{\|\tilde{\boldsymbol{\tau}}_h\|_{M_1}} \;\geq\; \tilde{\beta}\,\|(\mathbf{v}_h, q_h)\|_{L^2(\Omega_1)\times L^2(\Omega_2)\times \mathbf{R}}\,.$$

Proof. Let $\tilde{\Omega}$ be the interior region of the circle Γ, that is $\tilde{\Omega} := \Omega \cup \bar{\Omega}_0$. Then, given $(\mathbf{v}_h, \mu_{\tilde{h}}, q_h) \in \tilde{M}_{h,\tilde{h}}$, with $\mathbf{v}_h := (v_{1,h}, v_{2,h})$, we define

$$\tilde{v}_h := \begin{cases} v_{1,h} & \text{in} \quad \Omega_1\,, \\ v_{2,h} & \text{in} \quad \Omega_2\,, \\ -\dfrac{1}{|\Omega_0|}\left\{\displaystyle\sum_{i=1}^{2} \int_{\Omega_i} v_{i,h}\,dx \;+\; q_h|\Gamma|\right\} & \text{in} \quad \Omega_0\,. \end{cases}$$

Since $\int_{\tilde{\Omega}} \tilde{v}_h \, dx + \int_\Gamma q_h \, ds \,=\, 0$, we deduce that there exists a unique $z \in H^1(\tilde{\Omega})$ solution to the Neumann boundary value problem: $-\Delta z = \tilde{v}_h$ in $\tilde{\Omega}$, $\dfrac{\partial z}{\partial \boldsymbol{\nu}} = q_h$ on Γ, $\int_{\tilde{\Omega}} z \, dx \,=\, 0$. Moreover, since $\tilde{\Omega}$ is clearly convex, the usual regularity result (see, e.g. [65]) implies that $z \in H^2(\tilde{\Omega})$ and

$$\|z\|_{H^2(\tilde{\Omega})} \,\le\, C \left\{ \sum_{i=1}^{2} \|v_{i,h}\|_{L^2(\Omega_i)} \,+\, |q_h| \right\} .$$

Hence, we define $\bar{\boldsymbol{\tau}} := -\nabla z|_\Omega \in [H^1(\Omega)]^2$, and observe that $\operatorname{div} \bar{\boldsymbol{\tau}} = \tilde{v}_h$ in Ω, $\bar{\boldsymbol{\tau}} \cdot \boldsymbol{\nu} = -q_h$ on Γ, and

$$\|\bar{\boldsymbol{\tau}}\|_{[H^1(\Omega)]^2} \,=\, \|\nabla z\|_{[H^1(\Omega)]^2} \,\le\, \|z\|_{H^2(\tilde{\Omega})} \,\le\, C \left\{ \sum_{i=1}^{2} \|v_{i,h}\|_{L^2(\Omega_i)} \,+\, |q_h| \right\} . \tag{2.3.11}$$

Now, using the approximation property (2.3.6) (cf. Lemma 2.3.2) and the estimate (2.3.9) (cf. Lemma 2.3.3), we obtain

$$\begin{aligned} \|\mathcal{E}_h(\bar{\boldsymbol{\tau}})\|^2_{H(\operatorname{div};\Omega)} \,&=\, \|\mathcal{E}_h(\bar{\boldsymbol{\tau}})\|^2_{[L^2(\Omega)]^2} \,+\, \|\operatorname{div}(\mathcal{E}_h(\bar{\boldsymbol{\tau}}))\|^2_{L^2(\Omega)} \\ \le\, C \,&\Big\{ \|\bar{\boldsymbol{\tau}} - \mathcal{E}_h(\bar{\boldsymbol{\tau}})\|^2_{[L^2(\Omega)]^2} \,+\, \|\bar{\boldsymbol{\tau}}\|^2_{[L^2(\Omega)]^2} \,+\, \|\operatorname{div} \bar{\boldsymbol{\tau}}\|^2_{L^2(\Omega)} \Big\} \\ \le\, C \,&\Big\{ h^2 \, \|\bar{\boldsymbol{\tau}}\|^2_{[H^1(\Omega)]^2} \,+\, \|\bar{\boldsymbol{\tau}}\|^2_{[L^2(\Omega)]^2} \,+\, \|\operatorname{div} \bar{\boldsymbol{\tau}}\|^2_{L^2(\Omega)} \Big\} , \end{aligned}$$

which, according to (2.3.11), leads to

$$\|\mathcal{E}_h(\bar{\boldsymbol{\tau}})\|_{H(\operatorname{div};\Omega)} \,\le\, C \, \|\bar{\boldsymbol{\tau}}\|_{[H^1(\Omega)]^2} \,\le\, C \left\{ \sum_{i=1}^{2} \|v_{i,h}\|_{L^2(\Omega_i)} \,+\, |q_h| \right\} . \tag{2.3.12}$$

Next, we let $\mathcal{P}_h$ be the orthogonal projection from $L^2(\Omega)$ onto the finite dimensional subspace $\mathcal{M}_h := \{ w_h \in L^2(\Omega) : \quad w_h|_T \in \mathbf{P}_0(T) , \quad \forall T \in \mathcal{T}_h \}$. Because of the conmuting diagram property we know that $\mathcal{P}_h(\operatorname{div} \mathcal{E}_h(\bar{\boldsymbol{\tau}})) = \mathcal{P}_h(\operatorname{div} \bar{\boldsymbol{\tau}})$, which yields

$$\int_\Omega \tilde{v}_h \operatorname{div} \mathcal{E}_h(\bar{\boldsymbol{\tau}}) \, dx \,=\, \int_\Omega \tilde{v}_h \operatorname{div} \bar{\boldsymbol{\tau}} \, dx \,=\, \|\tilde{v}_h\|^2_{L^2(\Omega)} . \tag{2.3.13}$$

Also, since $\int_e \mathcal{E}_h(\bar{\boldsymbol{\tau}}) \cdot \boldsymbol{\nu}_e \, ds \,=\, \int_e \bar{\boldsymbol{\tau}} \cdot \boldsymbol{\nu}_e \, ds$ for all the edges e of $\mathcal{T}_h$, where $\boldsymbol{\nu}_e$ is the unit outward normal to e, and since $\bar{\boldsymbol{\tau}} \cdot \boldsymbol{\nu} = -q_h$ on Γ, we find that

$$\langle \, \mathcal{E}_h(\bar{\boldsymbol{\tau}}) \cdot \boldsymbol{\nu}, 1 \, \rangle_\Gamma \,=\, - \, q_h \, |\Gamma| . \tag{2.3.14}$$

Then we define $\bar{\boldsymbol{\tau}}_h := (\boldsymbol{\tau}_{1,h}, \boldsymbol{\tau}_{2,h}) = -(\mathcal{E}_h(\bar{\boldsymbol{\tau}})|_{\Omega_1}, \mathcal{E}_h(\bar{\boldsymbol{\tau}})|_{\Omega_2}) \in M_{1,h}$ and deduce, by virtue of (2.3.12), (2.3.13) and (2.3.14), that

$$\|\bar{\boldsymbol{\tau}}_h\|_{M_1} \leq \|\mathcal{E}_h(\bar{\boldsymbol{\tau}})\|_{H(\mathrm{div}\,;\Omega)} \leq C \left\{ \|\tilde{v}_h\|^2_{L^2(\Omega)} + q_h^2 \right\}^{1/2} \tag{2.3.15}$$

and

$$[\tilde{B}(\bar{\boldsymbol{\tau}}_h), (\mathbf{v}_h, \mu_{\tilde{h}}, q_h)] := -\sum_{i=1}^{2} \int_{\Omega_i} v_{i,h} \,\mathrm{div}\, \boldsymbol{\tau}_{i,h}\, dx - \langle \boldsymbol{\tau}_{1,h}\cdot\mathbf{n} - \boldsymbol{\tau}_{2,h}\cdot\mathbf{n}, \mu_{\tilde{h}} \rangle_{\Gamma_1} + q_h \langle \boldsymbol{\tau}_{2,h}\cdot\boldsymbol{\nu}, 1\rangle_{\Gamma}$$

$$= \int_{\Omega} \tilde{v}_h \,\mathrm{div}\, \mathcal{E}_h(\bar{\boldsymbol{\tau}})\, dx - q_h \langle \mathcal{E}_h(\bar{\boldsymbol{\tau}}) \cdot \boldsymbol{\nu}, 1 \rangle_{\Gamma} = \|\tilde{v}_h\|^2_{L^2(\Omega)} + q_h^2\, |\Gamma| \,. \tag{2.3.16}$$

Finally, we conclude from (2.3.15) and (2.3.16) that

$$\sup_{\substack{\tilde{\boldsymbol{\tau}}_h \in M_{1,h} \\ \tilde{\boldsymbol{\tau}}_h \neq 0}} \frac{[\tilde{B}(\tilde{\boldsymbol{\tau}}_h), (\mathbf{v}_h, \mu_{\tilde{h}}, q_h)]}{\|\tilde{\boldsymbol{\tau}}_h\|_{M_1}} \geq \frac{[\tilde{B}(\bar{\boldsymbol{\tau}}_h), (\mathbf{v}_h, \mu_{\tilde{h}}, q_h)]}{\|\bar{\boldsymbol{\tau}}_h\|_{M_1}} \geq \tilde{\beta}\, \|(\mathbf{v}_h, q_h)\|_{L^2(\Omega_1)\times L^2(\Omega_2)\times \mathbf{R}} \,,$$

which completes the proof. □

We now let $\tilde{\Omega}_1$ be the interior region of Γ_1, that is $\tilde{\Omega}_1 := \Omega_1 \cup \bar{\Omega}_0$, and define the finite element subspace

$$\tilde{M}^{\boldsymbol{\sigma}_1}_{1,h} := \{ \boldsymbol{\sigma}_h \in H(\mathrm{div}\,; \tilde{\Omega}_1) : \quad \boldsymbol{\sigma}_h|_T \in \mathcal{RT}_0(T) \quad \forall T \in \mathcal{T}_h\,,\ T \subseteq \Omega_1\,, \quad \forall T \in \mathcal{T}_{h,0} \} \,,$$

where $\mathcal{T}_{h,0}$ is a triangulation of $\bar{\Omega}_0$ whose nodes on the polygonal curve Γ_0 are those induced by $\mathcal{T}_h$. Then we recall that the corresponding equilibrium interpolation operator $\tilde{\mathcal{E}}^1_h : [H^1(\tilde{\Omega}_1)]^2 \to \tilde{M}^{\boldsymbol{\sigma}_1}_{1,h}$ can also be defined from the larger space $[H^\delta(\tilde{\Omega}_1)]^2 \cap H(\mathrm{div}\,; \tilde{\Omega}_1)$ onto $\tilde{M}^{\boldsymbol{\sigma}_1}_{1,h}$ for all $\delta \in (0,1)$ (see Theorem 3.1 in [2]). Moreover, as established by Theorem 3.4 in [2], there holds the analogue of the approximation property (2.3.6), that is, there exists $C > 0$, independent of h, such that

$$||\boldsymbol{\tau} - \tilde{\mathcal{E}}^1_h(\boldsymbol{\tau})||_{[L^2(\tilde{\Omega}_1)]^2} \leq C\, h^\delta\, ||\boldsymbol{\tau}||_{[H^\delta(\tilde{\Omega}_1)]^2} \quad \forall\, \boldsymbol{\tau} \in [H^\delta(\tilde{\Omega}_1)]^2 \cap H(\mathrm{div}\,; \tilde{\Omega}_1)\,. \tag{2.3.17}$$

Further, let $\{\gamma_1, \gamma_2, ..., \gamma_n\}$ be the partition on Γ_1 induced by the triangulation $\mathcal{T}_h$, and define

$$H_h^{-1/2} := \{ \chi_h \in L^2(\Gamma_1) : \quad \chi_h|_{\gamma_j} \in \mathbf{P}_0(\gamma_j) \quad \forall\, j \in \{1, ..., n\} \} \,.$$

We assume that $\{\mathcal{T}_h\}_{h>0}$ is uniformly regular near Γ_1, which means that there exists $C > 0$, independent of h, such that $|\gamma_j| \geq Ch$ for all $j \in \{1, ..., n\}$, for all $h > 0$. It is well known that this condition yields the inverse inequality for the space $H_h^{-1/2}$, that is, for any real numbers s and t with $-1/2 \leq s \leq t \leq 0$, there exists $C > 0$ such that

$$\|\chi_h\|_{H^t(\Gamma_1)} \;\leq\; C\, h^{s-t}\, \|\chi_h\|_{H^s(\Gamma_1)} \qquad \forall\, \chi_h \in H_h^{-1/2}\,. \tag{2.3.18}$$

Then, we have the following result.

Lemma 2.3.5 *There exists $\bar{\beta} > 0$, independent of h and $\tilde{h}$, such that for all $(\mathbf{v}_h, \mu_{\tilde{h}}, q_h) \in \tilde{M}_{h,\tilde{h}}$ there holds*

$$\sup_{\substack{\tilde{\boldsymbol{\tau}}_h \in M_{1,h}\\ \tilde{\boldsymbol{\tau}}_h \neq 0}} \frac{[\tilde{B}(\tilde{\boldsymbol{\tau}}_h), (\mathbf{v}_h, \mu_{\tilde{h}}, q_h)]}{\|\tilde{\boldsymbol{\tau}}_h\|_{M_1}} \;\geq\; \bar{\beta} \sup_{\substack{\chi_h \in H_h^{-1/2}\\ \chi_h \neq 0}} \frac{\langle \chi_h, \mu_{\tilde{h}}\rangle_{\Gamma_1}}{\|\chi_h\|_{H^{-1/2}(\Gamma_1)}}\,.$$

Proof. Given $\chi_h \in H_h^{-1/2}$, $\chi_h \neq 0$, we set $\tilde{v}_h := \begin{cases} 0 & \text{in} \quad \Omega_1 \\ -\frac{1}{|\Omega_0|}\,\langle \chi_h, 1\rangle_{\Gamma_1} & \text{in} \quad \Omega_0 \end{cases}$, and let $z \in H^1(\tilde{\Omega}_1)$ be the unique solution of the Neumann boundary value problem: $-\Delta z = \tilde{v}_h$ in $\tilde{\Omega}_1$, $\dfrac{\partial z}{\partial \mathbf{n}} = \chi_h$ on Γ_1, $\displaystyle\int_{\tilde{\Omega}_1} z\, dx = 0$. Since $\tilde{\Omega}_1$ is convex, $\tilde{v}_h \in L^2(\tilde{\Omega}_1)$, and $H_h^{-1/2} \subseteq H^{-1/2+\delta}(\Gamma_1)$ for any $\delta \in [0,1)$, we deduce (see [65]) that $z \in H^{1+\delta}(\tilde{\Omega}_1)$ and $\|z\|_{H^{1+\delta}(\tilde{\Omega}_1)} \;\leq\; C\, \|\chi_h\|_{H^{-1/2+\delta}(\Gamma_1)}$. In what follows, we choose a fixed $\delta \in (0, 1/2)$. Then we define $\bar{\boldsymbol{\tau}} := -\nabla z \in [H^\delta(\tilde{\Omega}_1)]^2$, and observe that $\operatorname{div} \bar{\boldsymbol{\tau}} = \tilde{v}_h$ in $\tilde{\Omega}_1$, $\bar{\boldsymbol{\tau}} \cdot \mathbf{n} = -\chi_h$ on Γ_1, and

$$\|\bar{\boldsymbol{\tau}}\|_{[H^\delta(\tilde{\Omega}_1)]^2} \;\leq\; \|z\|_{H^{1+\delta}(\tilde{\Omega}_1)} \;\leq\; C\, \|\chi_h\|_{H^{-1/2+\delta}(\Gamma_1)}\,. \tag{2.3.19}$$

Also, noting that $\operatorname{div} \bar{\boldsymbol{\tau}} = \tilde{v}_h$ in $\tilde{\Omega}_1$, and applying the continuous dependence result for the above Neumann problem, we find that

$$\|\bar{\boldsymbol{\tau}}\|_{H(\operatorname{div};\tilde{\Omega}_1)} \;\leq\; C\, \Big\{ \|z\|_{H^1(\tilde{\Omega}_1)} \;+\; \|\chi_h\|_{H^{-1/2}(\Gamma_1)} \Big\} \;\leq\; C\, \|\chi_h\|_{H^{-1/2}(\Gamma_1)}\,. \tag{2.3.20}$$

Now, according to the conmuting diagram property of $\tilde{\mathcal{E}}_h^1$, we deduce that $\operatorname{div} \tilde{\mathcal{E}}_h^1(\bar{\boldsymbol{\tau}}) = \operatorname{div} \bar{\boldsymbol{\tau}} = \tilde{v}_h$ in $\tilde{\Omega}_1$ and $\tilde{\mathcal{E}}_h^1(\bar{\boldsymbol{\tau}}) \cdot \mathbf{n} = \bar{\boldsymbol{\tau}} \cdot \mathbf{n} = -\chi_h$ on Γ_1. Thus, using the approximation property (2.3.17), the estimates (2.3.19) and (2.3.20), and the inverse inequality (2.3.18), we get

$$\|\tilde{\mathcal{E}}_h^1(\bar{\boldsymbol{\tau}})\|_{H(\operatorname{div};\Omega_1)} \;=\; \|\tilde{\mathcal{E}}_h^1(\bar{\boldsymbol{\tau}})\|_{[L^2(\Omega_1)]^2} \;\leq\; \|\tilde{\mathcal{E}}_h^1(\bar{\boldsymbol{\tau}})\|_{[L^2(\tilde{\Omega}_1)]^2}$$

$$\leq\; \|\bar{\boldsymbol{\tau}} - \tilde{\mathcal{E}}_h^1(\bar{\boldsymbol{\tau}})\|_{[L^2(\tilde{\Omega}_1)]^2} \;+\; \|\bar{\boldsymbol{\tau}}\|_{[L^2(\tilde{\Omega}_1)]^2} \;\leq\; C\, h^\delta\, \|\bar{\boldsymbol{\tau}}\|_{[H^\delta(\tilde{\Omega}_1)]^2} \;+\; \|\bar{\boldsymbol{\tau}}\|_{H(\operatorname{div};\tilde{\Omega}_1)}$$

$$\leq C\,h^{\delta}\,\|\chi_h\|_{H^{-1/2+\delta}(\Gamma_1)} + C\,\|\chi_h\|_{H^{-1/2}(\Gamma_1)} \leq C\,\|\chi_h\|_{H^{-1/2}(\Gamma_1)}\,,$$

and hence

$$\|\tilde{\mathcal{E}}_h^1(\bar{\boldsymbol{\tau}})\|_{H(\mathrm{div}\,;\Omega_1)} \leq C\,\|\chi_h\|_{H^{-1/2}(\Gamma_1)}\,. \tag{2.3.21}$$

Then we define $\bar{\boldsymbol{\tau}}_h := (\boldsymbol{\tau}_{1,h}, \boldsymbol{\tau}_{2,h}) = (\tilde{\mathcal{E}}_h^1(\bar{\boldsymbol{\tau}})|_{\Omega_1}, 0) \in M_{1,h}$ and deduce, according to (2.3.21), the above stated properties of $\tilde{\mathcal{E}}_h^1$, and the definition of $\tilde{B}$, that

$$\|\bar{\boldsymbol{\tau}}_h\|_{M_1} \leq C\,\|\chi_h\|_{H^{-1/2}(\Gamma_1)} \tag{2.3.22}$$

and

$$[\tilde{B}(\bar{\boldsymbol{\tau}}_h), (\mathbf{v}_h, \mu_{\tilde{h}}, q_h)] = \langle \chi_h, \mu_{\tilde{h}} \rangle_{\Gamma_1} \qquad \forall\,(\mathbf{v}_h, \mu_{\tilde{h}}, q_h) \in \tilde{M}_{h,\tilde{h}}\,. \tag{2.3.23}$$

Therefore, (2.3.22) and (2.3.23) yield

$$\sup_{\substack{\tilde{\boldsymbol{\tau}}_h \in M_{1,h}\\ \tilde{\boldsymbol{\tau}}_h \neq 0}} \frac{[\tilde{B}(\tilde{\boldsymbol{\tau}}_h), (\mathbf{v}_h, \mu_{\tilde{h}}, q_h)]}{\|\tilde{\boldsymbol{\tau}}_h\|_{M_1}} \geq \frac{[\tilde{B}(\bar{\boldsymbol{\tau}}_h), (\mathbf{v}_h, \mu_{\tilde{h}}, q_h)]}{\|\bar{\boldsymbol{\tau}}_h\|_{M_1}} \geq C\,\frac{\langle \chi_h, \mu_{\tilde{h}} \rangle_{\Gamma_1}}{\|\chi_h\|_{H^{-1/2}(\Gamma_1)}}\,,$$

which finishes the proof. □

We now state additional properties of the spaces $M_{\tilde{h}}^{\eta}$ and $H_h^{-1/2}$. To this end, we assume that the partition $\{\tilde{\gamma}_1, ..., \tilde{\gamma}_m\}$ of Γ_1 is uniformly regular, that is there exists $C > 0$ such that $|\tilde{\gamma}_j| \geq C\tilde{h}$ for all $j \in \{1, ..., m\}$. Thus, since $M_{\tilde{h}}^{\eta} \subseteq H^1(\Gamma_1)$, there holds the corresponding inverse inequality, that is for any real numbers s and t with $1/2 \leq s \leq t \leq 1$, there exists $C > 0$ such that

$$\|\mu_{\tilde{h}}\|_{H^t(\Gamma_1)} \leq C\,\tilde{h}^{s-t}\,\|\mu_{\tilde{h}}\|_{H^s(\Gamma_1)} \qquad \forall\,\mu_{\tilde{h}} \in M_{\tilde{h}}^{\eta}\,. \tag{2.3.24}$$

Also, we recall that the subspace $H_h^{-1/2}$ satisfies the usual approximation property, that is there exists $C > 0$ such that for all $s \in (-1/2, 1/2]$ and for all $\chi \in H^s(\Gamma_1)$, there exists $\hat{\chi}_h \in H_h^{-1/2}$ such that

$$\|\chi - \hat{\chi}_h\|_{H^{-1/2}(\Gamma_1)} \leq C\,h^{s+1/2}\,\|\chi\|_{H^s(\Gamma_1)}\,. \tag{2.3.25}$$

Then we have the following result completing the lower estimate provided in Lemma 2.3.5.

Lemma 2.3.6 *There exist C_0, $\hat{\beta} > 0$, independent of h and $\tilde{h}$, such that for all $h \leq C_0\tilde{h}$ and for all $\mu_{\tilde{h}} \in M_{\tilde{h}}^{\eta}$ there holds*

$$\sup_{\substack{\chi_h \in H_h^{-1/2}\\ \chi_h \neq 0}} \frac{\langle \chi_h, \mu_{\tilde{h}} \rangle_{\Gamma_1}}{\|\chi_h\|_{H^{-1/2}(\Gamma_1)}} \geq \hat{\beta}\,\|\mu_{\tilde{h}}\|_{H^{1/2}(\Gamma_1)}\,.$$

PROOF. Given $\mu_{\tilde{h}} \in M^{\eta}_{\tilde{h}}$ we let $z \in H^1(\tilde{\Omega}_1)$ be the unique solution of the Dirichlet problem: $-\Delta z + z = 0$ in $\tilde{\Omega}_1$, $z = \mu_{\tilde{h}}$ on Γ_1. Since $\tilde{\Omega}_1$ is convex and $M^{\eta}_{\tilde{h}} \subseteq H^1(\Gamma_1)$, the usual regularity result implies that $z \in H^{1+\delta}(\tilde{\Omega}_1)$ and $\|z\|_{H^{1+\delta}(\tilde{\Omega}_1)} \le C\, \|\mu_{\tilde{h}}\|_{H^{1/2+\delta}(\Gamma_1)}$ for all $\delta \in [0, 1/2]$. We also note that the normal derivative of z on Γ_1 is well defined for all $\delta \in [0, 1/2)$, and satisfies $\dfrac{\partial z}{\partial \mathbf{n}} \in H^{-1/2+\delta}(\Gamma_1)$ with $\left\| \dfrac{\partial z}{\partial \mathbf{n}} \right\|_{H^{-1/2+\delta}(\Gamma_1)} \le C\, \|z\|_{H^{1+\delta}(\tilde{\Omega}_1)}$ (see Theorem 2.4 in [2]). In particular, we have $\dfrac{\partial z}{\partial \mathbf{n}} \in H^{-1/2}(\Gamma_1)$, and applying the trace theorem we find that

$$\langle \frac{\partial z}{\partial \mathbf{n}}, \mu_{\tilde{h}} \rangle_{\Gamma_1} \;=\; \langle \frac{\partial z}{\partial \mathbf{n}}, z \rangle_{\Gamma_1} \;=\; \|z\|^2_{H^1(\tilde{\Omega}_1)} \;\ge\; C\, \|\mu_{\tilde{h}}\|^2_{H^{1/2}(\Gamma_1)} \,. \tag{2.3.26}$$

We now choose a fixed $\delta \in (0, 1/2)$. Thus, approximation property (2.3.25) yields the existence of $\hat{\chi}_h \in H_h^{-1/2}$ such that

$$\left\| \frac{\partial z}{\partial \mathbf{n}} - \hat{\chi}_h \right\|_{H^{-1/2}(\Gamma_1)} \;\le\; C\, h^{\delta} \left\| \frac{\partial z}{\partial \mathbf{n}} \right\|_{H^{-1/2+\delta}(\Gamma_1)} \;\le\; C\, h^{\delta}\; \|\mu_{\tilde{h}}\|_{H^{1/2+\delta}(\Gamma_1)} \,,$$

which, according to the inverse inequality (2.3.24), gives

$$\left\| \frac{\partial z}{\partial \mathbf{n}} - \hat{\chi}_h \right\|_{H^{-1/2}(\Gamma_1)} \;\le\; \bar{C} \left(\frac{h}{\tilde{h}} \right)^{\delta} \|\mu_{\tilde{h}}\|_{H^{1/2}(\Gamma_1)} \,, \tag{2.3.27}$$

and hence

$$\|\hat{\chi}_h\|_{H^{-1/2}(\Gamma_1)} \;\le\; C \left(\frac{h}{\tilde{h}} \right)^{\delta} \|\mu_{\tilde{h}}\|_{H^{1/2}(\Gamma_1)} \,+\, C\, \|z\|_{H^1(\tilde{\Omega}_1)} \;\le\; \tilde{C}\, \|\mu_{\tilde{h}}\|_{H^{1/2}(\Gamma_1)} \quad \forall\, h \le \tilde{h} \,.$$

It follows that

$$\sup_{\substack{\chi_h \in H_h^{-1/2} \\ \chi_h \ne 0}} \frac{\langle \chi_h, \mu_{\tilde{h}} \rangle_{\Gamma_1}}{\|\chi_h\|_{H^{-1/2}(\Gamma_1)}} \;\ge\; \frac{\langle \hat{\chi}_h, \mu_{\tilde{h}} \rangle_{\Gamma_1}}{\|\hat{\chi}_h\|_{H^{-1/2}(\Gamma_1)}} \;\ge\; \frac{1}{\tilde{C}} \frac{\langle \hat{\chi}_h, \mu_{\tilde{h}} \rangle_{\Gamma_1}}{\|\mu_{\tilde{h}}\|_{H^{1/2}(\Gamma_1)}}$$

$$=\; \frac{1}{\tilde{C}\, \|\mu_{\tilde{h}}\|_{H^{1/2}(\Gamma_1)}} \left\{ \langle \frac{\partial z}{\partial \mathbf{n}}, \mu_{\tilde{h}} \rangle_{\Gamma_1} \,-\, \langle \frac{\partial z}{\partial \mathbf{n}} - \hat{\chi}_h, \mu_{\tilde{h}} \rangle_{\Gamma_1} \right\} \,,$$

which, by virtue of (2.3.26) and (2.3.27), implies

$$\sup_{\substack{\chi_h \in H_h^{-1/2} \\ \chi_h \ne 0}} \frac{\langle \chi_h, \mu_{\tilde{h}} \rangle_{\Gamma_1}}{\|\chi_h\|_{H^{-1/2}(\Gamma_1)}} \;\ge\; \frac{1}{\tilde{C}} \left\{ C \,-\, \bar{C} \left(\frac{h}{\tilde{h}} \right)^{\delta} \right\} \|\mu_{\tilde{h}}\|_{H^{1/2}(\Gamma_1)} \,.$$

This inequality shows the existence of C_0, $\hat{\beta} > 0$, independent of h and $\tilde{h}$, such that for all $h \le C_0 \tilde{h}$ and for all $\mu_{\tilde{h}} \in M_{\tilde{h}}^{\eta}$, there holds $\displaystyle\sup_{\substack{\chi_h \in H_h^{-1/2} \\ \chi_h \neq 0}} \frac{\langle \chi_h, \mu_{\tilde{h}} \rangle_{\Gamma_1}}{\| \chi_h \|_{H^{-1/2}(\Gamma_1)}} \ge$ $\hat{\beta}\, \| \mu_{\tilde{h}} \|_{H^{1/2}(\Gamma_1)}$, which completes the proof. $\square$

As a consequence of the previous analysis, we conclude that $\tilde{B}$ satisfies the discrete inf-sup condition uniformly. More precisely, we have the following result.

LEMMA 2.3.7 *There exist C_0, $\beta^* > 0$, independent of h and $\tilde{h}$, such that for all $h \le C_0 \tilde{h}$ and for all $(\mathbf{v}_h, \mu_{\tilde{h}}, q_h) \in \tilde{M}_{h,\tilde{h}}$ there holds*

$$\sup_{\substack{\tilde{\boldsymbol{\tau}}_h \in M_{1,h} \\ \tilde{\boldsymbol{\tau}}_h \neq 0}} \frac{[\tilde{B}(\tilde{\boldsymbol{\tau}}_h), (\mathbf{v}_h, \mu_{\tilde{h}}, q_h)]}{\| \tilde{\boldsymbol{\tau}}_h \|_{M_1}} \;\ge\; \beta^* \, \| (\mathbf{v}_h, \mu_{\tilde{h}}, q_h) \|_{\tilde{M}} \,.$$

PROOF. It follows straightforward from Lemmas 2.3.4, 2.3.5, and 2.3.6. $\square$

We now introduce the discrete kernel of $\tilde{B}$, that is

$$\tilde{V}_h := \{ \, \tilde{\boldsymbol{\tau}}_h \in M_{1,h} : \quad [\tilde{B}(\tilde{\boldsymbol{\tau}}_h), (\mathbf{v}_h, \mu_{\tilde{h}}, q_h)] \;=\; 0 \quad \forall \, (\mathbf{v}_h, \mu_{\tilde{h}}, q_h) \in \tilde{M}_{h,\tilde{h}} \, \} \,,$$

which yields (see Lemma 5.7 in [59] for details)

$$\tilde{V}_h := \Big\{ \, \tilde{\boldsymbol{\tau}}_h := (\boldsymbol{\tau}_{1,h}, \boldsymbol{\tau}_{2,h}) \in M_{1,h} : \quad \operatorname{div} \boldsymbol{\tau}_{i,h} = 0 \quad \text{in} \quad \Omega_i \quad \forall \, i \in \{1, 2\} \,,$$

$$\langle \boldsymbol{\tau}_{1,h} \cdot \mathbf{n} - \boldsymbol{\tau}_{2,h} \cdot \mathbf{n}, \mu_{\tilde{h}} \rangle_{\Gamma_1} = 0 \quad \forall \, \mu_{\tilde{h}} \in M_{\tilde{h}}^{\eta} \,, \quad \text{and} \quad \langle \boldsymbol{\tau}_{2,h} \cdot \boldsymbol{\nu}, 1 \rangle_{\Gamma} \;=\; 0 \Big\} \,.$$

Hence, the discrete inf-sup condition for the operator B_1 follows easily.

LEMMA 2.3.8 *For all $\tilde{\boldsymbol{\tau}}_h \in \tilde{V}_h$ there holds*

$$\sup_{\substack{(\tilde{\boldsymbol{\zeta}}_h, \tilde{\xi}_h) \in \tilde{X}_{1,h} \\ (\tilde{\boldsymbol{\zeta}}_h, \tilde{\xi}_h) \neq 0}} \frac{[B_1(\tilde{\boldsymbol{\zeta}}_h, \tilde{\xi}_h), \tilde{\boldsymbol{\tau}}_h]}{\| (\tilde{\boldsymbol{\zeta}}_h, \tilde{\xi}_h) \|_{X_1}} \;\ge\; \| \tilde{\boldsymbol{\tau}}_h \|_{M_1} \,.$$

PROOF. Similarly as for the proof of Lemma 2.2.3, it suffices to observe that for all $\tilde{\boldsymbol{\tau}}_h := (\boldsymbol{\tau}_{1,h}, \boldsymbol{\tau}_{2,h}) \in \tilde{V}_h$ there holds $(-\tilde{\boldsymbol{\tau}}_h, 0) \in \tilde{X}_{1,h}$ and $\| \boldsymbol{\tau}_{i,h} \|_{[L^2(\Omega_i)]^2} \;=\; \| \boldsymbol{\tau}_{i,h} \|_{H(\operatorname{div};\Omega_i)}$ for all $i \in \{1, 2\}$. We omit further details. $\square$

We are now in a position to establish the unique solvability, stability, and convergence of the mixed finite element scheme (2.3.5).

Theorem 2.3.2 *Let $C_0 > 0$ be the constant from Lemma 2.3.7. Then, for all $h \leq C_0\tilde{h}$ there exists a unique $((\tilde{\boldsymbol{\theta}}_h, \tilde{\lambda}_h), \tilde{\boldsymbol{\sigma}}_h, (\mathbf{u}_h, \eta_{\tilde{h}}, p_h)) \in \tilde{X}_{1,h} \times M_{1,h} \times \tilde{M}_{h,\tilde{h}}$ solution of* (2.3.5)*. Moreover, there exist c, $C > 0$, independent of h and $\tilde{h}$, such that*

$$\|((\tilde{\boldsymbol{\theta}}_h, \tilde{\lambda}_h), \tilde{\boldsymbol{\sigma}}_h, (\mathbf{u}_h, \eta_{\tilde{h}}, p_h))\|_{\tilde{X}_1 \times M_1 \times \tilde{M}}$$

$$\leq c \left\{ \|g_0\|_{H^{1/2}(\Gamma_0)} + \|g_1\|_{H^{1/2}(\Gamma_1)} + \|f_1\|_{L^2(\Omega_1)} + \|g_2\|_{H^{-1/2}(\Gamma_1)} + \|A_1(0,0)\|_{\tilde{X}_1'} \right\},$$

and

$$\|((\tilde{\boldsymbol{\theta}}, \tilde{\lambda}), \tilde{\boldsymbol{\sigma}}, (\mathbf{u}, \eta, p)) - ((\tilde{\boldsymbol{\theta}}_h, \tilde{\lambda}_h), \tilde{\boldsymbol{\sigma}}_h, (\mathbf{u}_h, \eta_{\tilde{h}}, p_h))\|_{\tilde{X}_1 \times M_1 \times \tilde{M}}$$

$$\leq C \inf_{\substack{((\tilde{\boldsymbol{\zeta}}_h, \tilde{\xi}_h), \tilde{\boldsymbol{\tau}}_h, (\mathbf{v}_h, \mu_{\tilde{h}})) \\ \in \tilde{X}_{1,h} \times M_{1,h} \times \hat{M}_{h,\tilde{h}}}} \|((\tilde{\boldsymbol{\theta}}, \tilde{\lambda}), \tilde{\boldsymbol{\sigma}}, (\mathbf{u}, \eta)) - ((\tilde{\boldsymbol{\zeta}}_h, \tilde{\xi}_h), \tilde{\boldsymbol{\tau}}_h, (\mathbf{v}_h, \mu_{\tilde{h}}))\|_{\tilde{X}_1 \times M_1 \times \hat{M}},$$

where $\hat{M} := L^2(\Omega_1) \times L^2(\Omega_2) \times H^{1/2}(\Gamma_1)$ and $\hat{M}_{h,\tilde{h}} := M_h^{u_1} \times M_h^{u_2} \times M_{\tilde{h}}^{\eta}$.

Proof. According to Lemmas 2.2.1, 2.3.7, and 2.3.8, the proof follows straightforward from the abstract Theorems 3.1 and 3.3 in [57] (see also Theorems 3 and 5 in [51], or Theorems 4.2 and 4.3 in [59]). □

Because of the condition $h \leq C_0\tilde{h}$, we assume hereafter that each edge in $\{\gamma_1, ..., \gamma_n\}$, the partition on Γ_1 induced by $\mathcal{T}_h$, is contained in an edge of the independent partition $\{\tilde{\gamma}_1, ..., \tilde{\gamma}_m\}$ of Γ_1. This requires implicitly that the end points of each $\tilde{\gamma}_j$ be vertices of $\mathcal{T}_h$, which is also assumed in what follows.

We now introduce the approximation properties of the finite element subspaces $\tilde{X}_{1,h}$, $M_{1,h}$, and $\hat{M}_{h,\tilde{h}}$. More precisely, according to Lemma 2.3.2 and the corresponding results in [10], [25], and [90], we have:

($\mathrm{AP}_{1,h}^{\boldsymbol{\theta}_i}$) For all $\boldsymbol{\zeta} \in [H^1(\Omega_i)]^2$ there exists $\boldsymbol{\zeta}_h \in X_{1,h}^{\boldsymbol{\theta}_i}$ such that

$$\|\boldsymbol{\zeta} - \boldsymbol{\zeta}_h\|_{[L^2(\Omega_i)]^2} \leq C\, h \, \|\boldsymbol{\zeta}\|_{[H^1(\Omega_i)]^2}.$$

($\mathrm{AP}_{1,h}^{\tilde{\lambda}}$) For all $\tilde{\xi} \in H^{3/2}(\Gamma) \cap H_0^{1/2}(\Gamma)$ there exists $\tilde{\xi}_h \in X_{1,h}^{\tilde{\lambda}}$ such that

$$\|\tilde{\xi} - \tilde{\xi}_h\|_{H^{1/2}(\Gamma)} \leq C\, h \, \|\tilde{\xi}\|_{H^{3/2}(\Gamma)}.$$

($\mathrm{AP}_{1,h}^{\boldsymbol{\sigma}_i}$) For all $\boldsymbol{\tau} \in [H^1(\Omega_i)]^2$ with $\operatorname{div} \boldsymbol{\tau} \in H^1(\Omega_i)$ there exists $\boldsymbol{\tau}_h \in M_{1,h}^{\boldsymbol{\sigma}_i}$ such that

$$\|\boldsymbol{\tau} - \boldsymbol{\tau}_h\|_{H(\operatorname{div};\Omega_i)} \leq C\, h \left\{ \|\boldsymbol{\tau}\|_{[H^1(\Omega_i)]^2} + \|\operatorname{div} \boldsymbol{\tau}\|_{H^1(\Omega_i)} \right\}.$$

($\mathrm{AP}_h^{u_i}$) For all $v \in H^1(\Omega_i)$ there exists $v_h \in M_h^{u_i}$ such that

$$\|v - v_h\|_{L^2(\Omega_i)} \leq C\, h\, \|v\|_{H^1(\Omega_i)}\,.$$

($\mathrm{AP}_{\tilde h}^{\eta}$) For all $\mu \in H^{3/2}(\Gamma_1)$ there exists $\mu_{\tilde h} \in M_{\tilde h}^{\eta}$ such that

$$\|\mu - \mu_{\tilde h}\|_{H^{1/2}(\Gamma_1)} \leq C\, \tilde h\, \|\mu\|_{H^{3/2}(\Gamma_1)}\,.$$

Then, the rate of convergence of the mixed finite element solution is established as follows.

THEOREM 2.3.3 *Let* $((\tilde{\boldsymbol\theta}, \tilde\lambda), \tilde{\boldsymbol\sigma}, (\mathbf{u}, \eta, p))$ *and* $((\tilde{\boldsymbol\theta}_h, \tilde\lambda_h), \tilde{\boldsymbol\sigma}_h, (\mathbf{u}_h, \eta_{\tilde h}, p_h))$ *be the unique solutions of the continuous and discrete mixed formulations* (2.2.9) *and* (2.3.5)*, respectively. Assume that* $\tilde{\boldsymbol\theta} := (\boldsymbol\theta_1, \boldsymbol\theta_2) \in H^1(\Omega_1) \times H^1(\Omega_2)$*,* $\tilde\lambda \in H^{3/2}(\Gamma)$*,* $\tilde{\boldsymbol\sigma} := (\boldsymbol\sigma_1, \boldsymbol\sigma_2) \in [H^1(\Omega_1)]^2 \times [H^1(\Omega_2)]^2$, $\operatorname{div} \boldsymbol\sigma_i \in H^1(\Omega_i)$*,* $\mathbf{u} := (u_1, u_2) \in H^1(\Omega_1) \times H^1(\Omega_2)$*, and* $\eta \in H^{3/2}(\Gamma_1)$*. Then there exists* $C > 0$*, independent of* h *and* $\tilde h$*, such that*

$$\|((\tilde{\boldsymbol\theta}, \tilde\lambda), \tilde{\boldsymbol\sigma}, (\mathbf{u}, \eta, p)) - ((\tilde{\boldsymbol\theta}_h, \tilde\lambda_h), \tilde{\boldsymbol\sigma}_h, (\mathbf{u}_h, \eta_{\tilde h}, p_h))\|_{\tilde X_1 \times M_1 \times \tilde M}$$

$$\leq C\, h \sum_{i=1}^{2} \left\{ \|\boldsymbol\theta_i\|_{[H^1(\Omega_i)]^2} + \|\boldsymbol\sigma_i\|_{[H^1(\Omega_i)]^2} + \|\operatorname{div} \boldsymbol\sigma_i\|_{H^1(\Omega_i)} + \|u_i\|_{H^1(\Omega_i)} \right\}$$

$$+ C\, h\, \|\tilde\lambda\|_{H^{3/2}(\Gamma)} + C\, \tilde h\, \|\eta\|_{H^{3/2}(\Gamma_1)}\,.$$

PROOF. It follows straightforward from the Céa estimate in Theorem 2.3.2 and the above approximation properties. □

2.4 A-posteriori error analysis

We first introduce some notations. Given $T \in \mathcal{T}_h$, $\langle \cdot, \cdot \rangle_{H(\mathrm{div}\,;T)}$ denotes the inner product of $H(\mathrm{div}\,;T)$, and $\boldsymbol\nu_T$ stands for the unit outward normal to ∂T. Also, we let $E(T)$ be the set of edges of T, and let E_h be the set of all edges of the triangulation $\mathcal{T}_h$. Then, we put $E_h(\Gamma_0) := \{e \in E_h : \quad e \subseteq \Gamma_0\}$, $E_h(\Gamma_1) := \{e \in E_h : \quad e \subseteq \Gamma_1\}$, and $E_h(\Gamma) := \{e \in E_h : \quad e \subseteq \Gamma\}$. Further, h_e denotes the diameter of $e \in E_h$, and for each $e \in E_h(\Gamma_1)$ we set $\tilde h_e := |\tilde\gamma_j|$, where $\tilde\gamma_j$ is the segment containing edge e.

On the other hand, given a polygonal domain $\mathcal{S} \subset R^2$ and $s \in (1, \infty)$, the Sobolev space $W^{1,s}(\mathcal{S})$ is the space of functions $v \in L^s(\mathcal{S})$ such that the first order distributional derivatives of v belong to $L^s(\mathcal{S})$ (see [76]). It is well known that

$W^{1,s}(\mathcal{S})$ provided with the norm $\|v\|_{W^{1,s}(\mathcal{S})} := \left(\|v\|^s_{L^s(\mathcal{S})} + \|\nabla v\|^s_{[L^s(\mathcal{S})]^2}\right)^{1/s}$ is a Banach space. The trace Theorem ensures that there exists a linear continuous map $\gamma :\, W^{1,s}(\mathcal{S}) \mapsto L^s(\partial\mathcal{S})$ such that $\gamma v = v|_{\partial\mathcal{S}}$ for each $v \in W^{1,s}(\mathcal{S}) \cap C(\bar{\mathcal{S}})$. Hence, it is usual to denote $W^{1-1/s,s}(\partial\mathcal{S}) := \gamma(W^{1,s}(\mathcal{S}))$ which is a strict subspace of $L^s(\partial\mathcal{S})$ (see [76]). Furthermore, by virtue of a Sobolev imbedding theorem, we find that $W^{1,s}(\mathcal{S}) \subset C(\bar{\mathcal{S}})$ if $s > 2$.

We now consider in particular $\mathcal{S} := T \in \mathcal{T}_h$. Then for $s = 2$ we use the standard notation and write $H^{1/2}(\partial T)$ instead of $W^{1/2,2}(\partial T)$. Equivalently, the Sobolev space $H^{1/2}(\partial T)$ is defined as the completion of $C^\infty(\partial T)$ with respect to the norm:

$$\|v\|_{H^{1/2}(\partial T)} = \Big(\|v\|^2_{L^2(\partial T)} + |v|^2_{H^{1/2}(\partial T)} \Big)^{1/2},$$

where

$$|v|^2_{H^{1/2}(\partial T)} := \int_{\partial T} \int_{\partial T} \frac{|v(x) - v(y)|^2}{|x-y|^2}\, ds_x ds_y.$$

The dual space of $H^{1/2}(\partial T)$ is $H^{-1/2}(\partial T)$, and we denote by $\langle \cdot, \cdot \rangle_{\partial T}$ the corresponding duality pairing with respect to the $L^2(\partial T)$-inner product.

Let us now take an edge $e \in E(T)$. We know that $H^1_0(e)$ stands for the closure in $H^1(e)$ of the space $C^\infty_0(e)$. Also, we recall that the interpolation space with index $1/2$ between $H^1_0(e)$ and $L^2(e)$ is $H^{1/2}_{00}(e)$ (cf. [76]), and its norm is given by

$$\|v\|_{H^{1/2}_{00}(e)} = \left(|v|^2_{H^{1/2}(e)} + \int_e \frac{v^2(x)}{|x-a_e|}\, ds_x + \int_e \frac{v^2(x)}{|x-b_e|}\, ds_x \right)^{1/2},$$

where a_e and b_e are the end points of the edge e. Alternatively, the space $H^{1/2}_{00}(e)$ may be defined as the subspace of functions in $H^{1/2}(e)$ whose extensions by zero to the rest of ∂T belong to $H^{1/2}(\partial T)$.

It is important to retain here that the restriction of an element in $H^{-1/2}(\partial T)$ over edge e does not belong in general to $H^{-1/2}(e)$, but to the dual of $H^{1/2}_{00}(e)$, denoted by $H^{-1/2}_{00}(e)$, and which is larger than $H^{-1/2}(e)$. In what follows we denote by $\langle \cdot, \cdot \rangle_e$ the duality pairing between $H^{-1/2}_{00}(e)$ and $H^{1/2}_{00}(e)$ with respect to the $L^2(e)$-inner product.

The starting point of our a-posteriori error analysis is the introduction of a Ritz projection of the error. To this end, we let $\tilde{X} := \tilde{X}_1 \times M_1$, where $\tilde{X}_1 := [L^2(\Omega_1)]^2 \times [L^2(\Omega_2)]^2 \times H^{1/2}_0(\Gamma)$, $M_1 := H(\mathrm{div}\,;\Omega_1) \times H(\mathrm{div}\,;\Omega_2)$, and define the nonlinear saddle point operator $A : \tilde{X} \to \tilde{X}'$ as the one resulting from the first two rows and two columns of (2.2.9), that is

$$[A((\tilde{\boldsymbol{\theta}}, \tilde{\lambda}), \tilde{\boldsymbol{\sigma}}), ((\tilde{\boldsymbol{\zeta}}, \tilde{\xi}), \tilde{\boldsymbol{\tau}})] := [A_1(\tilde{\boldsymbol{\theta}}, \tilde{\lambda}), (\tilde{\boldsymbol{\zeta}}, \tilde{\xi})] + [B_1(\tilde{\boldsymbol{\zeta}}, \tilde{\xi}), \tilde{\boldsymbol{\sigma}}] + [B_1(\tilde{\boldsymbol{\theta}}, \tilde{\lambda}), \tilde{\boldsymbol{\tau}}] \quad (2.4.1)$$

for all $((\tilde{\boldsymbol{\theta}}, \tilde{\lambda}), \tilde{\boldsymbol{\sigma}})$, $((\tilde{\boldsymbol{\zeta}}, \tilde{\xi}), \tilde{\boldsymbol{\tau}}) \in \tilde{X}$.

Further, let $\mathcal{A} : \tilde{X} \to \tilde{X}'$ be the linear operator defined by

$$[\mathcal{A}((\tilde{\boldsymbol{\theta}}, \tilde{\lambda}), \tilde{\boldsymbol{\sigma}}), ((\tilde{\boldsymbol{\zeta}}, \tilde{\xi}), \tilde{\boldsymbol{\tau}})] := \sum_{i=1}^{2} \int_{\Omega_i} \boldsymbol{\theta}_i \cdot \boldsymbol{\zeta}_i \, dx + \langle \mathbf{W}(\tilde{\lambda}), \tilde{\xi} \rangle_\Gamma + \sum_{i=1}^{2} \langle \boldsymbol{\sigma}_i, \boldsymbol{\tau}_i \rangle_{H(\mathrm{div}\,;\Omega_i)} \quad (2.4.2)$$

for all $((\tilde{\boldsymbol{\theta}}, \tilde{\lambda}), \tilde{\boldsymbol{\sigma}})$, $((\tilde{\boldsymbol{\zeta}}, \tilde{\xi}), \tilde{\boldsymbol{\tau}}) \in \tilde{X}$, with $\tilde{\boldsymbol{\theta}} := (\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)$, $\tilde{\boldsymbol{\zeta}} := (\boldsymbol{\zeta}_1, \boldsymbol{\zeta}_2)$, $\tilde{\boldsymbol{\sigma}} := (\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2)$ and $\tilde{\boldsymbol{\tau}} := (\boldsymbol{\tau}_1, \boldsymbol{\tau}_2)$. We remark, according to the properties of the boundary integral operator $\mathbf{W}$, that $\mathcal{A}$ induces an inner product on $\tilde{X}$.

In what follows, $((\tilde{\boldsymbol{\theta}}, \tilde{\lambda}), \tilde{\boldsymbol{\sigma}}, (\mathbf{u}, \eta, p)) \in \tilde{X}_1 \times M_1 \times \tilde{M}$ and $((\tilde{\boldsymbol{\theta}}_h, \tilde{\lambda}_h), \tilde{\boldsymbol{\sigma}}_h, (\mathbf{u}_h, \eta_{\tilde{h}}, p_h))$ $\in \tilde{X}_{1,h} \times M_{1,h} \times \tilde{M}_{h,\tilde{h}}$ stand for the unique solutions of the continuous and discrete formulations (2.2.9) and (2.3.5), respectively. Then we define the $\mathcal{A}$-Ritz projection of the associated Galerkin error as the unique $((\bar{\boldsymbol{\theta}}, \bar{\lambda}), \bar{\boldsymbol{\sigma}}) \in \tilde{X}$ such that

$$\begin{aligned} [\mathcal{A}((\bar{\boldsymbol{\theta}}, \bar{\lambda}), \bar{\boldsymbol{\sigma}}), ((\tilde{\boldsymbol{\zeta}}, \tilde{\xi}), \tilde{\boldsymbol{\tau}})] \;&=\; [A((\tilde{\boldsymbol{\theta}}, \tilde{\lambda}), \tilde{\boldsymbol{\sigma}}), ((\tilde{\boldsymbol{\zeta}}, \tilde{\xi}), \tilde{\boldsymbol{\tau}})] \\ -\; [A((\tilde{\boldsymbol{\theta}}_h, \tilde{\lambda}_h), \tilde{\boldsymbol{\sigma}}_h), ((\tilde{\boldsymbol{\zeta}}, \tilde{\xi}), \tilde{\boldsymbol{\tau}})] \;&+\; [\tilde{B}(\tilde{\boldsymbol{\tau}}), (\mathbf{u}, \eta, p) - (\mathbf{u}_h, \eta_{\tilde{h}}, p_h)] \end{aligned} \quad (2.4.3)$$

for all $((\tilde{\boldsymbol{\zeta}}, \tilde{\xi}), \tilde{\boldsymbol{\tau}}) \in \tilde{X}$.

The following theorem provides an a-priori estimate for $((\bar{\boldsymbol{\theta}}, \bar{\lambda}), \bar{\boldsymbol{\sigma}})$.

THEOREM 2.4.1 *Let* $\tilde{\boldsymbol{\theta}}_h := (\boldsymbol{\theta}_{1,h}, \boldsymbol{\theta}_{2,h})$, $\tilde{\boldsymbol{\sigma}}_h := (\boldsymbol{\sigma}_{1,h}, \boldsymbol{\sigma}_{2,h})$, *and* $\mathbf{u}_h := (u_{1,h}, u_{2,h})$. *Assume there exists* $s > 2$ *such that* $g_0 \in H^{1/2}(\Gamma_0) \cap W^{1-1/s,s}(\Gamma_0)$, $g_1 \in H^{1/2}(\Gamma_1) \cap W^{1-1/s,s}(\Gamma_1)$, *and let* $(\varphi_{1,h}, \varphi_{2,h})$ *be a pair of auxiliary functions satisfying the following conditions:*

a) $\varphi_{i,h} \in H^1(\Omega_i) \cap W^{1,s}(\Omega_i)$ *for each* $i \in \{1, 2\}$,

b) $\varphi_{1,h}(\bar{\mathbf{x}}) = g_0(\bar{\mathbf{x}})$ *and* $\varphi_{1,h}(\bar{\mathbf{x}}) = g_1(\bar{\mathbf{x}}) - \eta_{\tilde{h}}(\bar{\mathbf{x}})$ *for each vertex* $\bar{\mathbf{x}}$ *of* $\mathcal{T}_h$ *lying, respectively, on* Γ_0 *and* Γ_1,

c) $\varphi_{2,h}(\bar{\mathbf{x}}) = -\eta_{\tilde{h}}(\bar{\mathbf{x}})$ *and* $\varphi_{2,h}(\bar{\mathbf{x}}) = \tilde{\lambda}_h(\bar{\mathbf{x}}) + p_h$ *for each vertex* $\bar{\mathbf{x}}$ *of* $\mathcal{T}_h$ *lying, respectively, on* Γ_1 *and* Γ.

Also, for each $T \in \mathcal{T}_h$ *define* $\hat{\boldsymbol{\theta}}_T := \begin{cases} \boldsymbol{\sigma}_{1,h} - \mathbf{a}(\cdot, \boldsymbol{\theta}_{1,h}) & \text{if} \quad T \subseteq \Omega_1 \\ \boldsymbol{\sigma}_{2,h} - \boldsymbol{\theta}_{2,h} & \text{if} \quad T \subseteq \Omega_2 \end{cases}$, *and let* $\hat{\boldsymbol{\sigma}}_T \in H(\mathrm{div}\,;T)$ *be the unique solution of the local problem*

$$\langle \hat{\boldsymbol{\sigma}}_T, \boldsymbol{\tau} \rangle_{H(\mathrm{div}\,;T)} = \mathcal{F}_{h,T}(\boldsymbol{\tau}) \qquad \forall\, \boldsymbol{\tau} \in H(\mathrm{div}\,;T)\,, \tag{2.4.4}$$

where, for $T \subseteq \Omega_1$,

$$\begin{aligned} \mathcal{F}_{h,T}(\boldsymbol{\tau}) &:= \int_T \boldsymbol{\theta}_{1,h} \cdot \boldsymbol{\tau}\, dx + \int_T u_{1,h} \,\mathrm{div}\, \boldsymbol{\tau}\, dx - \langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}_T, \varphi_{1,h} \rangle_{\partial T} \\ &+ \sum_{e \in E(T) \cap E_h(\Gamma_0)} \langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}_T, \varphi_{1,h} - g_0 \rangle_e + \sum_{e \in E(T) \cap E_h(\Gamma_1)} \langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}_T, \varphi_{1,h} - g_1 + \eta_{\tilde{h}} \rangle_e\,, \end{aligned} \tag{2.4.5}$$

and for $T \subseteq \Omega_2$,

$$\begin{aligned} \mathcal{F}_{h,T}(\boldsymbol{\tau}) &:= \int_T \boldsymbol{\theta}_{2,h} \cdot \boldsymbol{\tau}\, dx + \int_T u_{2,h} \,\mathrm{div}\, \boldsymbol{\tau}\, dx - \langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}_T, \varphi_{2,h} \rangle_{\partial T} \\ &+ \sum_{e \in E(T) \cap E_h(\Gamma_1)} \langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}_T, \varphi_{2,h} + \eta_{\tilde{h}} \rangle_e + \sum_{e \in E(T) \cap E_h(\Gamma)} \langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}_T, \varphi_{2,h} - \tilde{\lambda}_h - p_h \rangle_e\,. \end{aligned} \tag{2.4.6}$$

Then there exists $C > 0$, *independent of* h *and* $\tilde{h}$, *such that*

$$\begin{aligned} \|((\bar{\boldsymbol{\theta}}, \bar{\lambda}), \bar{\boldsymbol{\sigma}})\|^2_{\tilde{X}} &\leq \sum_{T \in \mathcal{T}_h} \left\{ \|\hat{\boldsymbol{\theta}}_T\|^2_{[L^2(T)]^2} + \|\hat{\boldsymbol{\sigma}}_T\|^2_{H(\mathrm{div}\,;T)} \right\} \\ &+ C\, \|2\mathbf{W}(\tilde{\lambda}_h) + \boldsymbol{\sigma}_{2,h} \cdot \boldsymbol{\nu}\|^2_{H^{-1/2}(\Gamma)}\,. \end{aligned} \tag{2.4.7}$$

Proof. From the first two equations of (2.2.9) we have

$$[A((\tilde{\boldsymbol{\theta}}, \tilde{\lambda}), \tilde{\boldsymbol{\sigma}}), ((\tilde{\boldsymbol{\zeta}}, \tilde{\xi}), \tilde{\boldsymbol{\tau}})] + [\tilde{B}(\tilde{\boldsymbol{\tau}}), (\mathbf{u}, \eta, p)] = -\langle \boldsymbol{\tau}_1 \cdot \mathbf{n}, g_0 \rangle_{\Gamma_0} - \langle \boldsymbol{\tau}_1 \cdot \mathbf{n}, g_1 \rangle_{\Gamma_1}\,,$$

and hence (2.4.3) becomes

$$\begin{aligned} [\mathcal{A}((\bar{\boldsymbol{\theta}}, \bar{\lambda}), \bar{\boldsymbol{\sigma}}), ((\tilde{\boldsymbol{\zeta}}, \tilde{\xi}), \tilde{\boldsymbol{\tau}})] &= -\langle \boldsymbol{\tau}_1 \cdot \mathbf{n}, g_0 \rangle_{\Gamma_0} - \langle \boldsymbol{\tau}_1 \cdot \mathbf{n}, g_1 \rangle_{\Gamma_1} \\ &- [A((\tilde{\boldsymbol{\theta}}_h, \tilde{\lambda}_h), \tilde{\boldsymbol{\sigma}}_h), ((\tilde{\boldsymbol{\zeta}}, \tilde{\xi}), \tilde{\boldsymbol{\tau}})] - [\tilde{B}(\tilde{\boldsymbol{\tau}}), (\mathbf{u}_h, \eta_{\tilde{h}}, p_h)] \end{aligned} \tag{2.4.8}$$

for all $((\tilde{\boldsymbol{\zeta}}, \tilde{\xi}), \tilde{\boldsymbol{\tau}}) \in \tilde{X}$.

Equivalently, denoting $\bar{\boldsymbol{\theta}} := (\bar{\boldsymbol{\theta}}_1, \bar{\boldsymbol{\theta}}_2)$ and $\bar{\boldsymbol{\sigma}} := (\bar{\boldsymbol{\sigma}}_1, \bar{\boldsymbol{\sigma}}_2)$, and using the definitions of the operators A and $\tilde{B}$, we find that (2.4.8) can be split as follows:

$$\int_{\Omega_1} \bar{\boldsymbol{\theta}}_1 \cdot \boldsymbol{\zeta}_1\, dx = \int_{\Omega_1} \boldsymbol{\sigma}_{1,h} \cdot \boldsymbol{\zeta}_1\, dx - \int_{\Omega_1} \mathbf{a}(\cdot, \boldsymbol{\theta}_{1,h}) \cdot \boldsymbol{\zeta}_1\, dx \qquad \forall\, \boldsymbol{\zeta}_1 \in [L^2(\Omega_1)]^2\,, \tag{2.4.9}$$

$$\int_{\Omega_2} \bar{\boldsymbol{\theta}}_2 \cdot \boldsymbol{\zeta}_2\, dx = \int_{\Omega_2} \boldsymbol{\sigma}_{2,h} \cdot \boldsymbol{\zeta}_2\, dx - \int_{\Omega_2} \boldsymbol{\theta}_{2,h} \cdot \boldsymbol{\zeta}_2\, dx \qquad \forall\, \boldsymbol{\zeta}_2 \in [L^2(\Omega_2)]^2\,, \tag{2.4.10}$$

$$\langle \mathbf{W}(\bar{\lambda}), \tilde{\xi} \rangle_\Gamma \;=\; - \langle 2\,\mathbf{W}(\tilde{\lambda}_h) + \boldsymbol{\sigma}_{2,h} \cdot \boldsymbol{\nu}, \tilde{\xi} \rangle_\Gamma \qquad \forall\, \tilde{\xi} \in H_0^{1/2}(\Gamma)\,, \tag{2.4.11}$$

$$\begin{aligned} \langle \bar{\boldsymbol{\sigma}}_1, \boldsymbol{\tau}_1 \rangle_{H(\mathrm{div}\,;\Omega_1)} \;&=\; \int_{\Omega_1} \boldsymbol{\theta}_{1,h} \cdot \boldsymbol{\tau}_1 \, dx + \int_{\Omega_1} u_{1,h} \, \mathrm{div}\, \boldsymbol{\tau}_1 \, dx + \langle \boldsymbol{\tau}_1 \cdot \mathbf{n}, \eta_{\tilde{h}} \rangle_{\Gamma_1} \\ &- \langle \boldsymbol{\tau}_1 \cdot \mathbf{n}, g_0 \rangle_{\Gamma_0} - \langle \boldsymbol{\tau}_1 \cdot \mathbf{n}, g_1 \rangle_{\Gamma_1} \qquad \forall\, \boldsymbol{\tau}_1 \in H(\mathrm{div}\,;\Omega_1)\,, \end{aligned} \tag{2.4.12}$$

and

$$\begin{aligned} \langle \bar{\boldsymbol{\sigma}}_2, \boldsymbol{\tau}_1 \rangle_{H(\mathrm{div}\,;\Omega_2)} \;&=\; \int_{\Omega_2} \boldsymbol{\theta}_{2,h} \cdot \boldsymbol{\tau}_2 \, dx + \int_{\Omega_2} u_{2,h} \, \mathrm{div}\, \boldsymbol{\tau}_2 \, dx \\ - \langle \boldsymbol{\tau}_2 \cdot \mathbf{n}, \eta_{\tilde{h}} \rangle_{\Gamma_1} &- \langle \boldsymbol{\tau}_2 \cdot \boldsymbol{\nu}, \tilde{\lambda}_h + p_h \rangle_\Gamma \qquad \forall\, \boldsymbol{\tau}_2 \in H(\mathrm{div}\,;\Omega_2)\,. \end{aligned} \tag{2.4.13}$$

It follows from (2.4.9) and (2.4.10) that

$$\bar{\boldsymbol{\theta}}_1 \;:=\; \boldsymbol{\sigma}_{1,h} \;-\; \mathbf{a}(\cdot, \boldsymbol{\theta}_{1,h}) \quad \text{and} \quad \bar{\boldsymbol{\theta}}_2 \;:=\; \boldsymbol{\sigma}_{2,h} \;-\; \boldsymbol{\theta}_{2,h}\,. \tag{2.4.14}$$

Further, we deduce from (2.4.11) and the strong coerciveness of $\mathbf{W}$ (cf. (2.2.1)) that there exists $C > 0$, independent of h, such that

$$\|\bar{\lambda}\|^2_{H_0^{1/2}(\Gamma)} \;\leq\; C\; \|2\mathbf{W}(\tilde{\lambda}_h) + \boldsymbol{\sigma}_{2,h} \cdot \boldsymbol{\nu}\|^2_{H^{-1/2}(\Gamma)}\,. \tag{2.4.15}$$

Now, it is easy to see that

$$-\frac{1}{2}\,\|\bar{\boldsymbol{\sigma}}_1\|^2_{H(\mathrm{div}\,;\Omega_1)} \;=\; \min_{\boldsymbol{\tau}_1 \in H(\mathrm{div}\,;\Omega_1)} \left\{ \frac{1}{2}\,\|\boldsymbol{\tau}_1\|^2_{H(\mathrm{div}\,;\Omega_1)} \;-\; \langle \bar{\boldsymbol{\sigma}}_1, \boldsymbol{\tau}_1 \rangle_{H(\mathrm{div}\,;\Omega_1)} \right\}\,,$$

that is

$$-\frac{1}{2}\,\|\bar{\boldsymbol{\sigma}}_1\|^2_{H(\mathrm{div}\,;\Omega_1)} \;=\; \min_{\boldsymbol{\tau}_1 \in H(\mathrm{div}\,;\Omega_1)} \Phi_1(\boldsymbol{\tau}_1)\,, \tag{2.4.16}$$

where

$$\begin{aligned} \Phi_1(\boldsymbol{\tau}_1) \;:=\; &\frac{1}{2}\,\|\boldsymbol{\tau}_1\|^2_{H(\mathrm{div}\,;\Omega_1)} \;-\; \int_{\Omega_1} \boldsymbol{\theta}_{1,h} \cdot \boldsymbol{\tau}_1 \, dx \;-\; \int_{\Omega_1} u_{1,h} \, \mathrm{div}\, \boldsymbol{\tau}_1 \, dx \\ &- \langle \boldsymbol{\tau}_1 \cdot \mathbf{n}, \eta_{\tilde{h}} \rangle_{\Gamma_1} \;+\; \langle \boldsymbol{\tau}_1 \cdot \mathbf{n}, g_0 \rangle_{\Gamma_0} \;+\; \langle \boldsymbol{\tau}_1 \cdot \mathbf{n}, g_1 \rangle_{\Gamma_1}\,. \end{aligned} \tag{2.4.17}$$

Next, using Gauss's formula on each triangle and then on Ω_1, we deduce that

$$\sum_{\substack{T \in \mathcal{T}_h \\ T \subseteq \Omega_1}} \langle \boldsymbol{\tau}_1 \cdot \boldsymbol{\nu}_T, \varphi_{1,h} \rangle_{\partial T} \;=\; \sum_{\substack{T \in \mathcal{T}_h \\ T \subseteq \Omega_1}} \left\{ \int_T \nabla \varphi_{1,h} \cdot \boldsymbol{\tau}_1 \, dx + \int_T \varphi_{1,h} \, \mathrm{div}\, \boldsymbol{\tau}_1 \, dx \right\}$$

$$= \int_{\Omega_1} \nabla \varphi_{1,h} \cdot \boldsymbol{\tau}_1 \, dx + \int_{\Omega_1} \varphi_{1,h} \, \mathrm{div}\, \boldsymbol{\tau}_1 \, dx \;=\; \langle \boldsymbol{\tau}_1 \cdot \mathbf{n}, \varphi_{1,h} \rangle_{\Gamma_0} \;+\; \langle \boldsymbol{\tau}_1 \cdot \mathbf{n}, \varphi_{1,h} \rangle_{\Gamma_1}\,,$$

that is

$$\sum_{\substack{T \in \mathcal{T}_h \\ T \subseteq \Omega_1}} \langle \boldsymbol{\tau}_1 \cdot \boldsymbol{\nu}_T, \varphi_{1,h} \rangle_{\partial T} \;-\; \langle \boldsymbol{\tau}_1 \cdot \mathbf{n}, \varphi_{1,h} \rangle_{\Gamma_0} \;-\; \langle \boldsymbol{\tau}_1 \cdot \mathbf{n}, \varphi_{1,h} \rangle_{\Gamma_1} \;=\; 0\,. \tag{2.4.18}$$

At this point we remark that the hypotheses on g_0, g_1, $\varphi_{1,h}$, and $\varphi_{2,h}$ guarantee, according to the Sobolev imbedding theorems, that these functions are continuous and that the traces $(\varphi_{1,h} - g_0)|_e$, $(\varphi_{1,h} - g_1 + \eta_{\tilde{h}})|_e$, $(\varphi_{2,h} + \eta_{\tilde{h}})|_e$, and $(\varphi_{2,h} - \tilde{\lambda}_h - p_h)|_e$ belong to $H_{00}^{1/2}(e)$ for each edge e in $E_h(\Gamma_0)$, $E_h(\Gamma_1)$, $E_h(\Gamma_1)$, and $E_h(\Gamma)$, respectively.

Thus, we can *decompose* the duality pairings on Γ_0 and Γ_1, and write

$$\langle \boldsymbol{\tau}_1 \cdot \mathbf{n}, \varphi_{1,h} - g_0 \rangle_{\Gamma_0} \;=\; \sum_{e \in E_h(\Gamma_0)} \langle \boldsymbol{\tau}_1 \cdot \mathbf{n}, \varphi_{1,h} - g_0 \rangle_e \,, \tag{2.4.19}$$

and

$$\langle \boldsymbol{\tau}_1 \cdot \mathbf{n}, \varphi_{1,h} - g_1 + \eta_{\tilde{h}} \rangle_{\Gamma_1} \;=\; \sum_{e \in E_h(\Gamma_1)} \langle \boldsymbol{\tau}_1 \cdot \mathbf{n}, \varphi_{1,h} - g_1 + \eta_{\tilde{h}} \rangle_e \,. \tag{2.4.20}$$

Then, including (2.4.18) into the definition of Φ_1, and using (2.4.19) and (2.4.20), problem (2.4.16) becomes

$$-\frac{1}{2}\, \|\bar{\boldsymbol{\sigma}}_1\|^2_{H(\mathrm{div}\,;\Omega_1)} \;=\; \min_{\boldsymbol{\tau}_1 \in H(\mathrm{div}\,;\Omega_1)} \left\{ \sum_{\substack{T \in \mathcal{T}_h \\ T \subseteq \Omega_1}} \Phi_{1,T}(\boldsymbol{\tau}_{1,T}) \right\}, \tag{2.4.21}$$

where $\boldsymbol{\tau}_{1,T}$ is the restriction of $\boldsymbol{\tau}_1$ to the triangle T, and

$$\Phi_{1,T}(\boldsymbol{\tau}_{1,T}) \;:=\; \frac{1}{2}\, \|\boldsymbol{\tau}_{1,T}\|^2_{H(\mathrm{div}\,;T)} \;-\; \mathcal{F}_{h,T}(\boldsymbol{\tau}_{1,T})\,,$$

with $\mathcal{F}_{h,T}$ given by (2.4.5).

In this way, we conclude from (2.4.21) that

$$-\frac{1}{2}\, \|\bar{\boldsymbol{\sigma}}_1\|^2_{H(\mathrm{div}\,;\Omega_1)} \;\geq\; \sum_{\substack{T \in \mathcal{T}_h \\ T \subseteq \Omega_1}} \left\{ \min_{\boldsymbol{\tau}_{1,T} \in H(\mathrm{div}\,;T)} \Phi_{1,T}(\boldsymbol{\tau}_{1,T}) \right\} \;=\; -\,\frac{1}{2} \sum_{\substack{T \in \mathcal{T}_h \\ T \subseteq \Omega_1}} \|\hat{\boldsymbol{\sigma}}_T\|^2_{H(\mathrm{div}\,;T)}\,,$$

that is

$$\|\bar{\boldsymbol{\sigma}}_1\|^2_{H(\mathrm{div}\,;\Omega_1)} \;\leq\; \sum_{\substack{T \in \mathcal{T}_h \\ T \subseteq \Omega_1}} \|\hat{\boldsymbol{\sigma}}_T\|^2_{H(\mathrm{div}\,;T)}\,. \tag{2.4.22}$$

A similar analysis, which is omitted here, can be applied to prove that

$$\|\bar{\boldsymbol{\sigma}}_2\|^2_{H(\mathrm{div}\,;\Omega_2)} \;\leq\; \sum_{\substack{T \in \mathcal{T}_h \\ T \subseteq \Omega_2}} \|\hat{\boldsymbol{\sigma}}_T\|^2_{H(\mathrm{div}\,;T)}\,. \tag{2.4.23}$$

Finally, (2.4.14), (2.4.15), (2.4.22), and (2.4.23) yield (2.4.7) and complete the proof. □

We now utilize the $\mathcal{A}$-Ritz projection $((\bar{\boldsymbol{\theta}}, \bar{\lambda}), \bar{\boldsymbol{\sigma}})$ and the corresponding a-priori estimate provided by Theorem 2.4.1 to derive a first reliable a-posteriori error estimate for the Galerkin scheme (2.3.5).

THEOREM 2.4.2 *Assume the same hypotheses of Theorem* 2.4.1, *and for each* $T \in \mathcal{T}_h$ *define* $\hat{\mathbf{f}}_T := \begin{cases} f_1 + \operatorname{div} \boldsymbol{\sigma}_{1,h} & \text{if} \quad T \subseteq \Omega_1 \\ \operatorname{div} \boldsymbol{\sigma}_{2,h} & \text{if} \quad T \subseteq \Omega_2 \end{cases}$, *and* $\vartheta_T^2 := \|\hat{\boldsymbol{\theta}}_T\|^2_{[L^2(T)]^2} + \|\hat{\boldsymbol{\sigma}}_T\|^2_{H(\operatorname{div};T)} + \|\hat{\mathbf{f}}_T\|^2_{L^2(T)}$. *Then there exists* $C > 0$, *independent of* h *and* $\tilde{h}$, *such that*

$$\|((\tilde{\boldsymbol{\theta}}, \tilde{\lambda}), \tilde{\boldsymbol{\sigma}}, (\mathbf{u}, \eta, p)) \, - \, ((\tilde{\boldsymbol{\theta}}_h, \tilde{\lambda}_h), \tilde{\boldsymbol{\sigma}}_h, (\mathbf{u}_h, \eta_{\tilde{h}}, p_h))\|_{\tilde{X}_1 \times M_1 \times \tilde{M}} \, \leq \, C \, \boldsymbol{\vartheta} \,, \qquad (2.4.24)$$

where

$$\boldsymbol{\vartheta} := \left\{ \sum_{T \in \mathcal{T}_h} \vartheta_T^2 \, + \, \|2\, \mathbf{W}(\tilde{\lambda}_h) + \boldsymbol{\sigma}_{2,h} \cdot \boldsymbol{\nu}\|^2_{H^{-1/2}(\Gamma)} \, + \, \|g_2 - (\boldsymbol{\sigma}_{1,h} - \boldsymbol{\sigma}_{2,h}) \cdot \mathbf{n}\|^2_{H^{-1/2}(\Gamma_1)} \right\}^{1/2} .$$

PROOF. We first observe, according to the Lipschitz continuity and strong monotonicity of the operator A_1 (cf. Lemma 2.2.1), that for all $(\hat{\boldsymbol{\zeta}}, \hat{\xi}) \in \tilde{X}_1$ the Gâteaux derivative $\mathcal{D}A_1(\hat{\boldsymbol{\zeta}}, \hat{\xi})$ defines a uniformly bounded and uniformly elliptic bilinear form on $\tilde{X}_1 \times \tilde{X}_1$ (see Lemma 3.1 in [57] for details).

Therefore, since the operators B_1 and $\tilde{B}$ satisfy the continuous inf-sup conditions (cf. Lemmas 2.2.2 and 2.2.3), we apply the abstract Theorem 2.2 in [57] and find that the linear operator arising from the left hand side of (2.2.9) after replacing A_1 by $\mathcal{D}A_1(\hat{\boldsymbol{\zeta}}, \hat{\xi})$ at any $(\hat{\boldsymbol{\zeta}}, \hat{\xi}) \in \tilde{X}_1$, satisfies the global inf-sup condition with a constant $\hat{C} > 0$ that is independent of $(\hat{\boldsymbol{\zeta}}, \hat{\xi})$. Hence, we consider in particular $(\hat{\boldsymbol{\zeta}}, \hat{\xi}) \in \tilde{X}_1$ such that

$$\mathcal{D}A_1(\hat{\boldsymbol{\zeta}}, \hat{\xi}) \Big((\tilde{\boldsymbol{\theta}}, \tilde{\lambda}) - (\tilde{\boldsymbol{\theta}}_h, \tilde{\lambda}_h), (\tilde{\boldsymbol{\zeta}}, \tilde{\xi}) \Big) \, = \, [A_1(\tilde{\boldsymbol{\theta}}, \tilde{\lambda}), (\tilde{\boldsymbol{\zeta}}, \tilde{\xi})] \, - \, [A_1(\tilde{\boldsymbol{\theta}}_h, \tilde{\lambda}_h), (\tilde{\boldsymbol{\zeta}}, \tilde{\xi})]$$

for all $(\tilde{\boldsymbol{\zeta}}, \tilde{\xi}) \in \tilde{X}_1$, use the definition of the operator A in (2.4.1), and deduce that

$$\frac{1}{\hat{C}} \, \|((\tilde{\boldsymbol{\theta}}, \tilde{\lambda}), \tilde{\boldsymbol{\sigma}}, (\mathbf{u}, \eta, p)) \, - \, ((\tilde{\boldsymbol{\theta}}_h, \tilde{\lambda}_h), \tilde{\boldsymbol{\sigma}}_h, (\mathbf{u}_h, \eta_{\tilde{h}}, p_h))\|_{\tilde{X}_1 \times M_1 \times \tilde{M}}$$

$$\leq \sup_{\|((\tilde{\boldsymbol{\zeta}}, \tilde{\xi}), \tilde{\boldsymbol{\tau}}, (\mathbf{v}, \mu, q))\| \leq 1} \left\{ \begin{array}{c} [A((\tilde{\boldsymbol{\theta}}, \tilde{\lambda}), \tilde{\boldsymbol{\sigma}}), ((\tilde{\boldsymbol{\zeta}}, \tilde{\xi}), \tilde{\boldsymbol{\tau}})] \, - \, [A((\tilde{\boldsymbol{\theta}}_h, \tilde{\lambda}_h), \tilde{\boldsymbol{\sigma}}_h), ((\tilde{\boldsymbol{\zeta}}, \tilde{\xi}), \tilde{\boldsymbol{\tau}})] \\ + [\tilde{B}(\tilde{\boldsymbol{\tau}}), (\mathbf{u}, \eta, p) - (\mathbf{u}_h, \eta_{\tilde{h}}, p_h)] \, + \, [\tilde{B}(\tilde{\boldsymbol{\sigma}} - \tilde{\boldsymbol{\sigma}}_h), (\mathbf{v}, \mu, q)] \end{array} \right\}$$

Next, using the definition of the $\mathcal{A}$-Ritz projection $((\bar{\boldsymbol{\theta}}, \bar{\lambda}), \bar{\boldsymbol{\sigma}}) \in \tilde{X}$ (cf. (2.4.3)), taking into account the third equation of the continuous formulation (2.2.9), and noting from the third equation of the discrete scheme (2.3.5) that $\langle \boldsymbol{\sigma}_{2,h} \cdot \boldsymbol{\nu}, 1 \rangle_\Gamma = 0$, the above estimate becomes

$$\frac{1}{\hat{C}}\,\|((\tilde{\boldsymbol{\theta}},\tilde{\lambda}),\tilde{\boldsymbol{\sigma}},(\mathbf{u},\eta,p))\;-\;((\tilde{\boldsymbol{\theta}}_h,\tilde{\lambda}_h),\tilde{\boldsymbol{\sigma}}_h,(\mathbf{u}_h,\eta_{\tilde{h}},p_h))\|_{\tilde{X}_1\times M_1\times\tilde{M}}$$

$$\leq \sup_{\|((\tilde{\boldsymbol{\zeta}},\tilde{\xi}),\tilde{\boldsymbol{\tau}},(\mathbf{v},\mu,q))\|\leq 1}\left\{\begin{array}{c}[\mathcal{A}((\bar{\boldsymbol{\theta}},\bar{\lambda}),\bar{\boldsymbol{\sigma}}),((\tilde{\boldsymbol{\zeta}},\tilde{\xi}),\tilde{\boldsymbol{\tau}})]-\langle g_2-(\boldsymbol{\sigma}_{1,h}-\boldsymbol{\sigma}_{2,h})\cdot\mathbf{n},\mu\rangle_{\Gamma_1}\\ +\displaystyle\int_{\Omega_1} v_1\,(f_1+\operatorname{div}\boldsymbol{\sigma}_{1,h})\,dx\;+\;\int_{\Omega_2} v_2\operatorname{div}\boldsymbol{\sigma}_{2,h}\,dx\end{array}\right\}. \tag{2.4.25}$$

In this way, (2.4.25), the boundedness of $\mathcal{A}$, Theorem 2.4.1, and Cauchy-Schwarz's inequality yield (2.4.24) and complete the proof. □

The following two lemmas give a-priori estimates for the solutions of the local problems (2.4.4). We will use them next to show the *quasi-efficiency* of $\boldsymbol{\vartheta}$, and then to derive a fully explicit reliable a-posteriori error estimate that can be expressed in terms of local indicators (see Theorems 2.4.3 and 2.4.4 below).

Lemma 2.4.1 *Assume the same hypotheses of Theorem 2.4.1. Then there exists $C>0$, independent of h and $\tilde{h}$, such that for each $T\in\mathcal{T}_h$, $T\subseteq\Omega_1$, there holds*

$$\|\hat{\boldsymbol{\sigma}}_T\|^2_{H(\operatorname{div};T)}\;\leq\;C\Bigg\{\|\boldsymbol{\theta}_{1,h}-\nabla\varphi_{1,h}\|^2_{[L^2(T)]^2}\;+\;\|u_{1,h}-\varphi_{1,h}\|^2_{L^2(T)}$$
$$+\sum_{e\in E(T)\cap E_h(\Gamma_0)}\|\varphi_{1,h}-g_0\|^2_{H^{1/2}_{00}(e)}\;+\sum_{e\in E(T)\cap E_h(\Gamma_1)}\|\varphi_{1,h}-g_1+\eta_{\tilde{h}}\|^2_{H^{1/2}_{00}(e)}\Bigg\}. \tag{2.4.26}$$

Moreover, for any $z_1\in H^1(\Omega_1)\cap W^{1,s}(\Omega_1)$, with $s>2$, such that $z_1=g_0$ on Γ_0, we get

$$\|\hat{\boldsymbol{\sigma}}_T\|^2_{H(\operatorname{div};T)}\;\leq\;C\left\{\|\boldsymbol{\theta}_{1,h}-\nabla z_1\|^2_{[L^2(T)]^2}+\|u_{1,h}-z_1\|^2_{L^2(T)}+\|\mathcal{J}_{1,T}\|^2_{H^{1/2}(\partial T)}\right\}, \tag{2.4.27}$$

where $\mathcal{J}_{1,T}:=\left\{\begin{array}{ll}0 \quad \text{on} \quad e\in E(T)\cap E_h(\Gamma_0)\\ z_1-g_1+\eta_{\tilde{h}} \quad \text{on} \quad e\in E(T)\cap E_h(\Gamma_1)\\ z_1-\varphi_{1,h} \quad \text{otherwise}\end{array}\right.$.

Proof. We first recall from (2.4.4) and (2.4.5) that $\|\hat{\boldsymbol{\sigma}}\|_{H(\operatorname{div};T)}=\|\mathcal{F}_{h,T}\|_{H(\operatorname{div};T)'}$, where

$$\mathcal{F}_{h,T}(\boldsymbol{\tau}):=\int_T\boldsymbol{\theta}_{1,h}\cdot\boldsymbol{\tau}\,dx\;+\;\int_T u_{1,h}\operatorname{div}\boldsymbol{\tau}\,dx\;-\;\langle\boldsymbol{\tau}\cdot\boldsymbol{\nu}_T,\varphi_{1,h}\rangle_{\partial T}$$
$$+\sum_{e\in E(T)\cap E_h(\Gamma_0)}\langle\boldsymbol{\tau}\cdot\boldsymbol{\nu}_T,\varphi_{1,h}-g_0\rangle_e+\sum_{e\in E(T)\cap E_h(\Gamma_1)}\langle\boldsymbol{\tau}\cdot\boldsymbol{\nu}_T,\varphi_{1,h}-g_1+\eta_{\tilde{h}}\rangle_e$$

for all $\boldsymbol{\tau} \in H(\mathrm{div}\,;T)$. Hence, using that $\langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}_T, \varphi_{1,h} \rangle_{\partial T} = \int_T \nabla \varphi_{1,h} \cdot \boldsymbol{\tau}\, dx + \int_T \varphi_{1,h} \,\mathrm{div}\, \boldsymbol{\tau}\, dx$, replacing this back into the above equation, and applying Cauchy-Schwarz's inequality and the corresponding duality pairings, we conclude (2.4.26).

The derivation of (2.4.27) is similar. It suffices to observe that

$$\begin{aligned} & - \langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}_T, z_1 \rangle_{\partial T} + \langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}_T, z_1 - \varphi_{1,h} \rangle_{\partial T} \\ + & \sum_{e \in E(T) \cap E_h(\Gamma_0)} \langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}_T, \varphi_{1,h} - g_0 \rangle_e + \sum_{e \in E(T) \cap E_h(\Gamma_1)} \langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}_T, \varphi_{1,h} - g_1 + \eta_{\tilde{h}} \rangle_e \\ & = - \langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}_T, z_1 \rangle_{\partial T} + \langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}_T, \mathcal{J}_{1,T} \rangle_{\partial T}, \end{aligned}$$

which follows from the fact that $z_1 = g_0$ on Γ_0, and that the extensions of $(\varphi_{1,h} - g_0)|_e$ and $(\varphi_{1,h} - g_1 + \eta_{\tilde{h}})|_e$ by zero to the rest of ∂T belong to the corresponding space $H^{1/2}(\partial T)$. Then, we apply Gauss's formula to the term $\langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}_T, z_1 \rangle_{\partial T}$ and conclude (2.4.27). $\square$

LEMMA 2.4.2 *Assume the same hypotheses of Theorem* 2.4.1. *Then there exists* $C > 0$, *independent of* h *and* $\tilde{h}$, *such that for each* $T \in \mathcal{T}_h$, $T \subseteq \Omega_2$, *there holds*

$$\begin{aligned} \|\hat{\boldsymbol{\sigma}}_T\|^2_{H(\mathrm{div}\,;T)} \leq C \Bigg\{ & \|\boldsymbol{\theta}_{2,h} - \nabla \varphi_{2,h}\|^2_{[L^2(T)]^2} + \|u_{2,h} - \varphi_{2,h}\|^2_{L^2(T)} \\ + \sum_{e \in E(T) \cap E_h(\Gamma_1)} & \|\varphi_{2,h} + \eta_{\tilde{h}}\|^2_{H^{1/2}_{00}(e)} + \sum_{e \in E(T) \cap E_h(\Gamma)} \|\varphi_{2,h} - \tilde{\lambda}_h - p_h\|^2_{H^{1/2}_{00}(e)} \Bigg\}. \end{aligned} \tag{2.4.28}$$

Moreover, for any $z_2 \in H^1(\Omega_2)$, *we get*

$$\|\hat{\boldsymbol{\sigma}}_T\|^2_{H(\mathrm{div}\,;T)} \leq C \left\{ \|\boldsymbol{\theta}_{2,h} - \nabla z_2\|^2_{[L^2(T)]^2} + \|u_{2,h} - z_2\|^2_{L^2(T)} + \|\mathcal{J}_{2,T}\|^2_{H^{1/2}(\partial T)} \right\}, \tag{2.4.29}$$

where $\mathcal{J}_{2,T} := \begin{cases} z_2 + \eta_{\tilde{h}} & \text{on} \quad e \in E(T) \cap E_h(\Gamma_1) \\ z_2 - \tilde{\lambda}_h - p_h & \text{on} \quad e \in E(T) \cap E_h(\Gamma) \\ z_2 - \varphi_{2,h} & \text{otherwise} \end{cases}$.

PROOF. It is similar to the proof of Lemma 2.4.1. Here we use (2.4.6) instead of (2.4.5), and the fact that the extensions of $(\varphi_{2,h} + \eta_{\tilde{h}})|_e$ and $(\varphi_{2,h} - \tilde{\lambda}_h - p_h)|_e$ by zero to the rest of ∂T also belong to the corresponding space $H^{1/2}(\partial T)$. We omit further details. $\square$

In the following theorem we establish the *quasi-efficiency* of $\boldsymbol{\vartheta}$ (see also [4], [16], [61]), which means that this a-posteriori error estimate is efficient up to the traces of

$(u_1 - \varphi_{1,h})$ and $(u_2 - \varphi_{2,h})$ on the edges of $\mathcal{T}_h$. We remark that this property certainly restricts the possible choices of the auxiliary functions $\varphi_{1,h}$ and $\varphi_{2,h}$. At the end of this section we suggest an heuristic procedure to choose them.

THEOREM 2.4.3 *In addition to the hypotheses of Theorem* 2.4.1*, assume that* $u_1 \in W^{1,s}(\Omega_1)$ *and* $u_2 \in W^{1,s}(\Omega_2)$*, with* $s > 2$*. Then there exists* $C > 0$*, independent of* h *and* $\tilde{h}$*, such that*

$$\begin{aligned}\boldsymbol{\vartheta}^2 \le C \Big\{ & \|((\tilde{\boldsymbol{\theta}}, \tilde{\lambda}), \tilde{\boldsymbol{\sigma}}, (\mathbf{u}, \eta, p)) - ((\tilde{\boldsymbol{\theta}}_h, \tilde{\lambda}_h), \tilde{\boldsymbol{\sigma}}_h, (\mathbf{u}_h, \eta_{\tilde{h}}, p_h))\|^2_{\tilde{X}_1 \times M_1 \times \tilde{M}} \\ & + \sum_{\substack{T \in \mathcal{T}_h \\ T \subseteq \Omega_1}} \|\mathcal{J}_{1,T}\|^2_{H^{1/2}(\partial T)} + \sum_{\substack{T \in \mathcal{T}_h \\ T \subseteq \Omega_2}} \|\mathcal{J}_{2,T}\|^2_{H^{1/2}(\partial T)} \Big\},\end{aligned} \tag{2.4.30}$$

where

$$\mathcal{J}_{1,T} := \begin{cases} 0 \quad \text{on} \quad e \in E(T) \cap E_h(\Gamma_0) \\ \eta_{\tilde{h}} - \eta \quad \text{on} \quad e \in E(T) \cap E_h(\Gamma_1) \\ u_1 - \varphi_{1,h} \quad \text{otherwise} \end{cases}, \tag{2.4.31}$$

and

$$\mathcal{J}_{2,T} := \begin{cases} \eta_{\tilde{h}} - \eta \quad \text{on} \quad e \in E(T) \cap E_h(\Gamma_1) \\ \tilde{\lambda} + p - (\tilde{\lambda}_h + p_h) \quad \text{on} \quad e \in E(T) \cap E_h(\Gamma) \\ u_2 - \varphi_{2,h} \quad \text{otherwise} \end{cases}. \tag{2.4.32}$$

PROOF. Taking a sufficiently smooth test function $\tilde{\boldsymbol{\tau}}$ in the second equation of (2.2.9), and then integrating backward by parts, we deduce that $\boldsymbol{\theta}_1 = \nabla u_1$ in Ω_1 and $\boldsymbol{\theta}_2 = \nabla u_2$ in Ω_2, whence $u_1 \in H^1(\Omega_1)$ and $u_2 \in H^1(\Omega_2)$. Moreover, the same equation implies that $u_1 = g_0$ on Γ_0, $u_1 - u_2 = g_1$ on Γ_1, $u_2 = -\eta$ on Γ_1, and $u_2 = \tilde{\lambda} + p$ on Γ. Then, applying (2.4.27) (cf. Lemma 2.4.1) and (2.4.29) (cf. Lemma 2.4.2) with $z_1 = u_1$ and $z_2 = u_2$, respectively, and summing up over all triangles $T \in \mathcal{T}_h$, we deduce that there exists $C > 0$, independent of h and $\tilde{h}$, such that

$$\begin{aligned} & \sum_{T \in \mathcal{T}_h} \|\hat{\boldsymbol{\sigma}}_T\|^2_{H(\mathrm{div}\,;T)} \\ \le C \Big\{ & \|\boldsymbol{\theta}_{1,h} - \boldsymbol{\theta}_1\|^2_{[L^2(\Omega_1)]^2} + \|u_{1,h} - u_1\|^2_{L^2(\Omega_1)} + \sum_{\substack{T \in \mathcal{T}_h \\ T \subseteq \Omega_1}} \|\mathcal{J}_{1,T}\|^2_{H^{1/2}(\partial T)} \\ & + \|\boldsymbol{\theta}_{2,h} - \boldsymbol{\theta}_2\|^2_{[L^2(\Omega_2)]^2} + \|u_{2,h} - u_2\|^2_{L^2(\Omega_2)} + \sum_{\substack{T \in \mathcal{T}_h \\ T \subseteq \Omega_2}} \|\mathcal{J}_{2,T}\|^2_{H^{1/2}(\partial T)} \Big\}, \end{aligned} \tag{2.4.33}$$

where $\mathcal{J}_{1,T}$ and $\mathcal{J}_{2,T}$ are given now by (2.4.31) and (2.4.32).

From the first equation of (2.2.9) we get $\boldsymbol{\sigma}_1 = \mathbf{a}(\cdot, \boldsymbol{\theta}_1)$ in Ω_1, $\boldsymbol{\sigma}_2 = \boldsymbol{\theta}_2$ in Ω_2, and also $2\mathbf{W}(\tilde{\lambda}) + \boldsymbol{\sigma}_2 \cdot \boldsymbol{\nu} = 0$ on Γ. Further, the third equation of (2.2.9) yields $\operatorname{div} \boldsymbol{\sigma}_1 = -f_1$ in Ω_1, $\operatorname{div} \boldsymbol{\sigma}_2 = 0$ in Ω_2, $\boldsymbol{\sigma}_1 \cdot \mathbf{n} - \boldsymbol{\sigma}_2 \cdot \mathbf{n} = g_2$ on Γ_1, and $\langle \boldsymbol{\sigma}_2 \cdot \boldsymbol{\nu}, 1 \rangle_\Gamma = 0$. Hence, using triangle inequality and the Lipschitz continuity of the nonlinear part of the operator A_1 (restricted to each triangle $T \subseteq \Omega_1$), and then summing up over all triangles $T \in \mathcal{T}_h$, we find

$$\begin{aligned}
&\sum_{T \in \mathcal{T}_h} \left\{ \|\hat{\boldsymbol{\theta}}_T\|^2_{[L^2(T)]^2} + \|\hat{\mathbf{f}}_T\|^2_{L^2(T)} \right\} \\
&\leq C \Big\{ \|\boldsymbol{\sigma}_{1,h} - \boldsymbol{\sigma}_1\|^2_{[L^2(\Omega_1)]^2} + \|\boldsymbol{\theta}_{1,h} - \boldsymbol{\theta}_1\|^2_{[L^2(\Omega_1)]^2} + \|\operatorname{div}(\boldsymbol{\sigma}_{1,h} - \boldsymbol{\sigma}_1)\|^2_{L^2(\Omega_1)} \\
&\quad + \|\boldsymbol{\sigma}_{2,h} - \boldsymbol{\sigma}_2\|^2_{[L^2(\Omega_2)]^2} + \|\boldsymbol{\theta}_{2,h} - \boldsymbol{\theta}_2\|^2_{[L^2(\Omega_2)]^2} + \|\operatorname{div}(\boldsymbol{\sigma}_{2,h} - \boldsymbol{\sigma}_2)\|^2_{L^2(\Omega_2)} \Big\}.
\end{aligned} \tag{2.4.34}$$

Now, applying the boundedness of $\mathbf{W}$, and the trace theorem in $H(\operatorname{div}; \Omega_2)$ and $H(\operatorname{div}; \Omega_1)$, we easily prove that

$$\|2\,\mathbf{W}(\tilde{\lambda}_h) + \boldsymbol{\sigma}_{2,h} \cdot \boldsymbol{\nu}\|^2_{H^{-1/2}(\Gamma)} \leq C \left\{ \|\tilde{\lambda}_h - \tilde{\lambda}\|^2_{H^{1/2}(\Gamma)} + \|\boldsymbol{\sigma}_{2,h} - \boldsymbol{\sigma}_2\|^2_{H(\operatorname{div};\Omega_2)} \right\}, \tag{2.4.35}$$

and

$$\|g_2 - (\boldsymbol{\sigma}_{1,h} - \boldsymbol{\sigma}_{2,h}) \cdot \mathbf{n}\|^2_{H^{-1/2}(\Gamma_1)} \leq C \left\{ \|\boldsymbol{\sigma}_{1,h} - \boldsymbol{\sigma}_1\|^2_{H(\operatorname{div};\Omega_1)} + \|\boldsymbol{\sigma}_{2,h} - \boldsymbol{\sigma}_2\|^2_{H(\operatorname{div};\Omega_2)} \right\}. \tag{2.4.36}$$

In this way, (2.4.33), (2.4.34), (2.4.35), and (2.4.36) yield (2.4.30) and finish the proof. □

On the other hand, the boundary terms appearing in the definition of $\boldsymbol{\vartheta}$ (cf. Theorem 2.4.2) can be bounded by local quantities defined on the edges of Γ and Γ_1. More precisely, we have the following lemma.

LEMMA 2.4.3 *Assume that $g_2 \in L^2(\Gamma_1)$, and define the constants*

$$C_h(\Gamma) := \max \left\{ \frac{h_e}{h_{e'}} : \quad e \text{ and } e' \text{ are neighbour edges in } \Gamma \right\},$$

and

$$C_{\tilde{h}}(\Gamma_1) := \max \left\{ \frac{|\tilde{\gamma}_i|}{|\tilde{\gamma}_j|} : \quad |i-j| = 1 \,\text{or}\; m-1\,, \quad i,j \in \{1, ..., m\} \right\}.$$

Then there exists $C > 0$, independent of h, such that

$$\begin{aligned} &\|2\,\mathbf{W}(\tilde{\lambda}_h) + \boldsymbol{\sigma}_{2,h}\cdot\boldsymbol{\nu}\|^2_{H^{-1/2}(\Gamma)} \\ &\le C\,\log\left[1 + C_h(\Gamma)\right] \sum_{e\in E_h(\Gamma)} h_e\,\|2\,\mathbf{W}(\tilde{\lambda}_h) + \boldsymbol{\sigma}_{2,h}\cdot\boldsymbol{\nu}\|^2_{L^2(e)}\,, \end{aligned} \tag{2.4.37}$$

and

$$\begin{aligned} &\|g_2 - (\boldsymbol{\sigma}_{1,h} - \boldsymbol{\sigma}_{2,h})\cdot\mathbf{n}\|^2_{H^{-1/2}(\Gamma_1)} \\ &\le C\,\log\left[1 + C_{\tilde{h}}(\Gamma_1)\right] \sum_{e\in E_h(\Gamma_1)} \tilde{h}_e\,\|g_2 - (\boldsymbol{\sigma}_{1,h} - \boldsymbol{\sigma}_{2,h})\cdot\mathbf{n}\|^2_{L^2(e)}\,. \end{aligned} \tag{2.4.38}$$

Proof. We first notice from the definitions of the subspaces $X^{\tilde{\lambda}}_{1,h}$ and $M^{\boldsymbol{\sigma}_2}_{1,h}$ (see Section 2.3) that $\tilde{\lambda}_h \in H^1(\Gamma)$ and $(\boldsymbol{\sigma}_{2,h}\cdot\boldsymbol{\nu})|_\Gamma \in L^2(\Gamma)$, and hence a well known mapping property of $\mathbf{W}$ implies that $\left(2\,\mathbf{W}(\tilde{\lambda}_h) + \boldsymbol{\sigma}_{2,h}\cdot\boldsymbol{\nu}\right) \in L^2(\Gamma)$.

Now, taking $\tilde{\boldsymbol{\zeta}}_h = \mathbf{0}$ in the first equation of (2.3.5) and $(\mathbf{v}_h, \mu_{\tilde{h}}, q_h) = (\mathbf{0}, 0, 1)$ in the third one, we find, respectively, that $\langle 2\,\mathbf{W}(\tilde{\lambda}_h) + \boldsymbol{\sigma}_{2,h}\cdot\boldsymbol{\nu}, \tilde{\xi}_h\rangle_\Gamma = 0$ for all $\tilde{\xi}_h \in X^{\tilde{\lambda}}_{1,h}$, and $\langle \boldsymbol{\sigma}_{2,h}\cdot\boldsymbol{\nu}, 1\rangle_\Gamma = 0$. Thus, since $X^{\tilde{\lambda}}_{1,h} = X^{\lambda}_{1,h} \oplus \mathbf{R}$, $\mathbf{W}$ is symmetric, and $\mathbf{W}(1) = 0$, we find that $\left(2\,\mathbf{W}(\tilde{\lambda}_h) + \boldsymbol{\sigma}_{2,h}\cdot\boldsymbol{\nu}\right)$ is $L^2(\Gamma)$-orthogonal to $X^{\lambda}_{1,h}$. Therefore, (2.4.37) follows from a direct application of Theorem 2 in [34].

Next, according to the assumption on g_2 and the definitions of the subspaces $M^{\boldsymbol{\sigma}_1}_{1,h}$ and $M^{\boldsymbol{\sigma}_2}_{1,h}$, it is clear that $(g_2 - (\boldsymbol{\sigma}_{1,h} - \boldsymbol{\sigma}_{2,h})\cdot\mathbf{n}) \in L^2(\Gamma_1)$. Then, taking $(\mathbf{v}_h, q_h) = (\mathbf{0}, 0)$ in the third equation of (2.3.5), we get $\langle g_2 - (\boldsymbol{\sigma}_{1,h} - \boldsymbol{\sigma}_{2,h})\cdot\mathbf{n}, \mu_{\tilde{h}}\rangle_{\Gamma_1} = 0$ for all $\mu_{\tilde{h}} \in M^{\eta}_{\tilde{h}}$, and hence Theorem 2 in [34] gives

$$\begin{aligned} &\|g_2 - (\boldsymbol{\sigma}_{1,h} - \boldsymbol{\sigma}_{2,h})\cdot\mathbf{n}\|^2_{H^{-1/2}(\Gamma_1)} \\ &\le C\,\log\left[1 + C_{\tilde{h}}(\Gamma_1)\right] \sum_{j=1}^{m} |\tilde{\gamma}_j|\,\|g_2 - (\boldsymbol{\sigma}_{1,h} - \boldsymbol{\sigma}_{2,h})\cdot\mathbf{n}\|^2_{L^2(\tilde{\gamma}_j)}\,. \end{aligned}$$

But, since each $e \in E_h(\Gamma_1)$ is contained in some segment $\tilde{\gamma}_j$, we easily see that

$$\sum_{j=1}^{m} |\tilde{\gamma}_j|\,\|g_2 - (\boldsymbol{\sigma}_{1,h} - \boldsymbol{\sigma}_{2,h})\cdot\mathbf{n}\|^2_{L^2(\tilde{\gamma}_j)} \;=\; \sum_{e\in E_h(\Gamma_1)} \tilde{h}_e\,\|g_2 - (\boldsymbol{\sigma}_{1,h} - \boldsymbol{\sigma}_{2,h})\cdot\mathbf{n}\|^2_{L^2(e)}\,,$$

which yields (2.4.38) and completes the proof. □

We are now in a position to provide a fully explicit and reliable a-posteriori error estimate for the Galerkin scheme (2.3.5). Indeed, as a consequence of the previous results, we have the following theorem.

THEOREM 2.4.4 *In addition to the hypotheses of Theorem* 2.4.1, *assume that* $g_2 \in L^2(\Gamma_1)$. *Then there exists* $C > 0$, *independent of* h *and* $\tilde{h}$, *such that*

$$\|((\tilde{\boldsymbol{\theta}}, \tilde{\lambda}), \tilde{\boldsymbol{\sigma}}, (\mathbf{u}, \eta, p)) \,-\, ((\tilde{\boldsymbol{\theta}}_h, \tilde{\lambda}_h), \tilde{\boldsymbol{\sigma}}_h, (\mathbf{u}_h, \eta_{\tilde{h}}, p_h))\|_{\tilde{X}_1 \times M_1 \times \tilde{M}} \,\leq\, C\, \tilde{\boldsymbol{\vartheta}}\,, \tag{2.4.39}$$

where $\tilde{\boldsymbol{\vartheta}} := \left\{ \sum_{T \in \mathcal{T}_h} \tilde{\vartheta}_T^2 \right\}^{1/2}$, *and* $\tilde{\vartheta}_T^2$ *is given, respectively, for* $T \subseteq \Omega_1$ *and* $T \subseteq \Omega_2$, *by*

$$\begin{aligned}
\tilde{\vartheta}_T^2 \,&:=\, \|\boldsymbol{\theta}_{1,h} - \nabla \varphi_{1,h}\|^2_{[L^2(T)]^2} \,+\, \|u_{1,h} - \varphi_{1,h}\|^2_{L^2(T)} \\
&+ \sum_{e \in E(T) \cap E_h(\Gamma_0)} \|\varphi_{1,h} - g_0\|^2_{H_{00}^{1/2}(e)} \,+\, \sum_{e \in E(T) \cap E_h(\Gamma_1)} \|\varphi_{1,h} - g_1 + \eta_{\tilde{h}}\|^2_{H_{00}^{1/2}(e)} \\
&+ \|\boldsymbol{\sigma}_{1,h} - \mathbf{a}(\cdot, \boldsymbol{\theta}_{1,h})\|^2_{[L^2(T)]^2} \,+\, \|f_1 + \operatorname{div} \boldsymbol{\sigma}_{1,h}\|^2_{L^2(T)} \\
&+\, \log\,[1 + C_{\tilde{h}}(\Gamma_1)] \sum_{e \in E(T) \cap E_h(\Gamma_1)} \tilde{h}_e\, \|g_2 - (\boldsymbol{\sigma}_{1,h} - \boldsymbol{\sigma}_{2,h}) \cdot \mathbf{n}\|^2_{L^2(e)}\,,
\end{aligned} \tag{2.4.40}$$

and

$$\begin{aligned}
\tilde{\vartheta}_T^2 \,&:=\, \|\boldsymbol{\theta}_{2,h} - \nabla \varphi_{2,h}\|^2_{[L^2(T)]^2} \,+\, \|u_{2,h} - \varphi_{2,h}\|^2_{L^2(T)} \\
&+ \sum_{e \in E(T) \cap E_h(\Gamma_1)} \|\varphi_{2,h} + \eta_{\tilde{h}}\|^2_{H_{00}^{1/2}(e)} \,+\, \sum_{e \in E(T) \cap E_h(\Gamma)} \|\varphi_{2,h} - \tilde{\lambda}_h - p_h\|^2_{H_{00}^{1/2}(e)} \\
&+ \|\boldsymbol{\sigma}_{2,h} - \boldsymbol{\theta}_{2,h}\|^2_{[L^2(T)]^2} \,+\, \|\operatorname{div} \boldsymbol{\sigma}_{2,h}\|^2_{L^2(T)} \\
&+\, \log\,[1 + C_{\tilde{h}}(\Gamma_1)] \sum_{e \in E(T) \cap E_h(\Gamma_1)} \tilde{h}_e\, \|g_2 - (\boldsymbol{\sigma}_{1,h} - \boldsymbol{\sigma}_{2,h}) \cdot \mathbf{n}\|^2_{L^2(e)} \\
&+ \log\,[1 + C_h(\Gamma)] \sum_{e \in E(T) \cap E_h(\Gamma)} h_e\, \|2\, \mathbf{W}(\tilde{\lambda}_h) + \boldsymbol{\sigma}_{2,h} \cdot \boldsymbol{\nu}\|^2_{L^2(e)}\,.
\end{aligned} \tag{2.4.41}$$

PROOF. It follows from Theorem 2.4.2, and Lemmas 2.4.1, 2.4.2, and 2.4.3. □

We notice that the $H_{00}^{1/2}(e)$ norms needed for the computation of $\tilde{\vartheta}_T^2$ can be bounded by using the corresponding interpolation theorem, that is $\|\cdot\|^2_{H_{00}^{1/2}(e)} \leq \|\cdot\|_{L^2(e)}\,\|\cdot\|_{H_0^1(e)}$.

At this point we compare the a-posteriori error estimates $\boldsymbol{\vartheta}$ and $\tilde{\boldsymbol{\vartheta}}$. On one hand, the utilization of $\boldsymbol{\vartheta}$ requires the exact solution of the local problem (2.4.4), which, however, lives in the infinite dimensional space $H(\operatorname{div}; T)$. This implies that in practice (2.4.4) must be solved approximately by using, for instance, h or $h - p$ versions of the finite element method, which naturally yields approximations of ϑ_T

(and hence of $\boldsymbol{\vartheta}$). Nevertheless, the advantage of $\boldsymbol{\vartheta}$, at least from a theoretical point of view (as proved by Theorems 2.4.2 and 2.4.3), is that it constitutes a reliable and *quasi-efficient* a-posteriori error estimate. Alternatively, the advantage of $\tilde{\boldsymbol{\vartheta}}$, whose eventual *efficiency* or *quasi-efficiency* is an open question, lies on the fact that it does not require neither the exact nor any approximate solution of the local problem (2.4.4), and hence, it constitutes a fully explicit reliable a-posteriori error estimate. In any case, we also need to specify the auxiliary functions $\varphi_{1,h}$ and $\varphi_{2,h}$. To this respect, we observe from Theorem 2.4.3 that $\varphi_{1,h}$ and $\varphi_{2,h}$ should be as close as possible to the exact solutions u_1 and u_2, respectively. Naturally, since these solutions are not known, this criterion must be understood in an empirical sense. According to this, and motivated also by the fact that $\boldsymbol{\theta}_{i,h}$ becomes an approximation of ∇u_i, we propose next a suitable choice for $\varphi_{1,h}$ and $\varphi_{2,h}$.

We first compute local functions $\tilde{\varphi}_{i,T}$, for each $i \in \{1,2\}$ and for each $T \subseteq \Omega_i$, satisfying the following conditions: $\tilde{\varphi}_{i,T} \in \mathbf{P}_1(T)$; $\nabla \tilde{\varphi}_{i,T} = \boldsymbol{\theta}_{i,h}|_T$; and $\tilde{\varphi}_{i,T}(\bar{\mathbf{x}}_T) = u_{i,h}|_T$, where $\bar{\mathbf{x}}_T$ is the barycenter of the triangle T. We remark that each $\tilde{\varphi}_{i,T}$ is uniquely determined by these constraints, and that its computation is quite straightforward. Then, we define $\varphi_{i,h}$ as the $\mathbf{P}_1$-*continuous average* of the functions $\tilde{\varphi}_{i,T}$, for all $T \subseteq \Omega_i$. In other words, $\varphi_{i,h}$ is the unique function in $C(\bar{\Omega}_i)$ such that:

1. $\varphi_{i,h}|_T \in \mathbf{P}_1(T)$ for all $T \subseteq \Omega_i$.

2. $\varphi_{i,h}$ satisfies the assumptions b) and c) stated in Theorem 2.4.1.

3. for each vertex $\bar{\mathbf{x}}$ of $\mathcal{T}_h$ not lying on the boundary $\partial\Omega_i$, $\varphi_{i,h}(\bar{\mathbf{x}})$ is the average of the values of $\tilde{\varphi}_{i,T}(\bar{\mathbf{x}})$ on all the triangles $T \subseteq \Omega_i$ to which $\bar{\mathbf{x}}$ belongs.

In the case of a curved triangle T, the above definitions of $\tilde{\varphi}_{2,T}$ and $\varphi_{2,h}$ are modified accordingly by considering the functions $\tilde{\varphi}_{2,T} \circ F_T$ and $\varphi_{2,h} \circ F_T$ on the reference triangle $\hat{T}$, where $F_T : \hat{T} \to T$ is the bijective mapping defined by (2.3.1). Now, it is not difficult to see that the functions $\varphi_{1,h}$ and $\varphi_{2,h}$ suggested here also satisfy the assumption a) of Theorem 2.4.1. Further, we observe in this case that $\varphi_{2,h}$ coincide with $-\eta_{\tilde{h}}$ on the whole Γ_1, and hence the term involving $(\varphi_{2,h} + \eta_{\tilde{h}})$ dissapears from the definition (2.4.41) of $\tilde{\vartheta}_T^2$. The numerical examples presented below in Section 2.5 consider the explicit a-posteriori error estimate $\tilde{\boldsymbol{\vartheta}}$ with the above choices of $\varphi_{1,h}$ and $\varphi_{2,h}$.

2.5 Numerical results

We provide here some numerical results illustrating the performance of the present mixed finite element method and the explicit a-posteriori error estimate $\tilde{\boldsymbol{\vartheta}}$. We emphasize, in virtue of Theorem 2.3.1, that our computations rely on the original Galerkin scheme (2.3.4) instead of the equivalent one (2.3.5). The notation to be used in this section is described next. N denotes the number of degrees of freedom defining the subspaces $X_{1,h}$, $M_{1,h}$, and $M_{h,\tilde{h}}$, that is $N := 3$(number of triangles of $\mathcal{T}_h$) + (number of edges of $\mathcal{T}_h$) + (number of edges on Γ_1) + (number of nodes of $\mathcal{T}_h$ on Γ) + (number of independent nodes on Γ_1). Due to the requirement on the mesh sizes h and $\tilde{h}$ established in Theorem 2.3.2, we set a vertex of the independent partition $\{\tilde{\gamma}_1, ..., \tilde{\gamma}_m\}$ every two vertices of $\mathcal{T}_h$ on Γ_1.

Given $((\tilde{\boldsymbol{\theta}}, \lambda), \tilde{\boldsymbol{\sigma}}, (\mathbf{u}, \eta)) \in X_1 \times M_1 \times M$ and $((\tilde{\boldsymbol{\theta}}_h, \lambda_h), \tilde{\boldsymbol{\sigma}}_h, (\mathbf{u}_h, \eta_{\tilde{h}})) \in X_{1,h} \times M_{1,h} \times M_{h,\tilde{h}}$, the unique solutions of (2.2.8) and (2.3.4), we use the following individual and total errors:

$$\mathbf{e}(\tilde{\boldsymbol{\theta}}) := \left\{ \|\boldsymbol{\theta}_1 - \boldsymbol{\theta}_{1,h}\|^2_{[L^2(\Omega_1)]^2} + \|\boldsymbol{\theta}_2 - \boldsymbol{\theta}_{2,h}\|^2_{[L^2(\Omega_2)]^2} \right\}^{1/2},$$

$$\mathbf{e}(\lambda) := \|\lambda - \lambda_h\|_{H^{1/2}(\Gamma)}, \quad \mathbf{e}(\eta) := \|\eta - \eta_{\tilde{h}}\|_{H^{1/2}(\Gamma_1)},$$

$$\mathbf{e}(\tilde{\boldsymbol{\sigma}}) := \left\{ \|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_{1,h}\|^2_{H(\mathrm{div}\,;\Omega_1)} + \|\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_{2,h}\|^2_{H(\mathrm{div}\,;\Omega_2)} \right\}^{1/2},$$

$$\mathbf{e}(\mathbf{u}) := \left\{ \|u_1 - u_{1,h}\|^2_{L^2(\Omega_1)} + \|u_2 - u_{2,h}\|^2_{L^2(\Omega_2)} \right\}^{1/2},$$

and

$$\mathbf{e} := \left\{ [\mathbf{e}(\tilde{\boldsymbol{\theta}})]^2 + [\mathbf{e}(\lambda)]^2 + [\mathbf{e}(\tilde{\boldsymbol{\sigma}})]^2 + [\mathbf{e}(\mathbf{u})]^2 + [\mathbf{e}(\eta)]^2 \right\}^{1/2}.$$

We recall here, according to Theorems 2.2.1 and 2.3.1, that the connection between the solutions of (2.2.8) and (2.2.9) (resp. (2.3.4) and (2.3.5)) is given by the relation $\lambda = \tilde{\lambda} + p$ (resp. $\lambda_h = \tilde{\lambda}_h + p_h$). Thus, since $\mathbf{W}(1) = 0$ we note that $\mathbf{W}(\tilde{\lambda}_h) = \mathbf{W}(\lambda_h)$.

Also, given two consecutive triangulations with degrees of freedom N and N', and corresponding total errors $\mathbf{e}$ and $\mathbf{e}'$, the experimental rate of convergence is defined by $\boldsymbol{\gamma} := -2\,\dfrac{\log(\mathbf{e}/\mathbf{e}')}{\log(N/N')}$. Now, the adaptive algorithm to be used in the computation of the solutions of (2.3.4) reads as follows (see [96]):

1. Start with a coarse mesh $\mathcal{T}_h$.

2. Solve the Galerkin scheme (2.3.4) for the actual mesh $\mathcal{T}_h$.

3. Compute $\tilde{\vartheta}_T$ for each triangle $T \in \mathcal{T}_h$, according to (2.4.40) and (2.4.41).

4. Consider stopping criterion and decide to finish or go to next step.

5. Use *blue-green* procedure to refine each $T' \in \mathcal{T}_h$ whose local indicator $\tilde{\vartheta}_{T'}$ satisfies $\tilde{\vartheta}_{T'} \geq \dfrac{1}{2} \max\{ \tilde{\vartheta}_T : \quad T \in \mathcal{T}_h\}$.

6. Define resulting mesh as the new $\mathcal{T}_h$ and go to step 2.

The numerical results presented in this section were obtained in a *Compaq Alpha ES40 Parallel Computer* using a Matlab code. The discrete scheme (2.3.4), which becomes a nonlinear algebraic system with N unknowns, is solved by Newton's method with the initial guess $(1, ..., 1)^{\mathbf{T}} \in \mathbf{R}^N$, and a tolerance of 10^{-11} for the corresponding residual. The linear systems arising in the Newton steps are solved by a suitable preconditioned algorithm from the Matlab library.

We consider three examples. Example 1 considers a linear version of the boundary value problem (2.1.1), which means that $\mathbf{a}(x, \boldsymbol{\zeta}) = \boldsymbol{\zeta}$ for all $(x, \boldsymbol{\zeta}) \in \Omega_1 \times \mathbf{R}^2$. Then we take $\Omega_0 := (-0.5, 0.5)^2$, $\Omega_1 := (-2, 2)^2 \backslash \bar{\Omega}_0$, and choose the data g_0, f_1, g_1, and g_2 such that the exact solution is $u_1(x_1, x_2) = \dfrac{x_1 x_2}{(x_1 - 0.45)^2 + x_2^2}$ and $u_2(x_1, x_2) = \dfrac{x_1 - x_2}{x_1^2 + x_2^2}$. We notice that u_1 blows up at $(0.45, 0)$, and hence u_1 has a singular behaviour around the point $(0.5, 0)$.

The second example considers a pure nonlinear boundary value problem (2.1.1) with $\mathbf{a}(x, \boldsymbol{\zeta}) := \left(2 + \dfrac{1}{1 + \|\boldsymbol{\zeta}\|}\right) \boldsymbol{\zeta}$ for all $(x, \boldsymbol{\zeta}) \in \Omega_1 \times \mathbf{R}^2$. It is easy to check that this mapping satisfies the hypotheses (H.1), (H.2), (H.3), and (H.4) (see Section 2.2). We take Ω_0 as the L-shaped domain $(-1, 1)^2 \backslash [0, 1]^2$, set $\Omega_1 := (-2, 2)^2 \backslash \bar{\Omega}_0$, and choose the data g_0, f_1, g_1, and g_2 such that the exact solution is $u_1(x_1, x_2) := \dfrac{x_1 x_2}{(x_1 + 0.05)^2 + x_2^2}$ and $u_2(x_1, x_2) := \dfrac{x_1 - x_2}{x_1^2 + x_2^2}$. We observe here that u_1 blows up at $(-0.05, 0)$, and hence this function shows a singular behaviour in a neighborhood of $(0, 0)$.

Finally, our third example uses again the geometry of Example 1, considers the same nonlinearity of Example 2, and take the data g_0, f_1, g_1, and g_2 such that the exact solution is $u_1(x_1, x_2) := \dfrac{x_1 x_2}{(x_1 - 0.45)^2 + x_2^2}$ and $u_2(x_1, x_2) := \dfrac{2(x_1 + 1.8) + 3x_2}{(x_1 + 1.8)^2 + x_2^2}$.

Thus, we notice that u_1 and u_2 blow up at $(0.45, 0)$ and $(-1.8, 0)$, and hence they present singular behaviours around $(0.5, 0)$ and $(-2, 0)$, respectively.

We always take Γ as the circle with center at the origin and radius 4.

In Tables 2.5.1, 2.5.2, and 2.5.3, we give the errors for each unknown, the error estimate $\tilde{\boldsymbol{\vartheta}}$, the effectivity index $\mathbf{e}/\tilde{\boldsymbol{\vartheta}}$, and the experimental rate of convergence $\boldsymbol{\gamma}$. All the errors are computed on each triangle using a 7 points Gaussian quadrature rule. We notice here that the effectivity indexes, which are bounded above and below, confirm the reliability of $\tilde{\boldsymbol{\vartheta}}$ and provide numerical evidences for the eventual efficiency of it. Then, Figures 2.5.1, 2.5.3, and 2.5.5 show that the total error $\mathbf{e}$ of the adaptive algorithm decreases much faster than the one obtained with the uniform refinement. This is also supported by the values of $\boldsymbol{\gamma}$ displayed in the tables, which show that the adaptive procedure yields the quasi-optimal linear rate of convergence $O(h)$ guaranteed by Theorem 2.3.3. Finally, Figures 2.5.2, 2.5.4, and 2.5.6 show some intermediate meshes obtained with the algorithm. We notice, as expected, that the adaptivity is able to recognize the singular regions of each example. In other words, the adapted meshes are highly refined around the points $(0.5, 0)$, $(0, 0)$, and $(0.5, 0)$ together with $(-2, 0)$, for Examples 1, 2, and 3, respectively. Furthermore, although the refinement around $(-2, 0)$ certainly violates the assumptions of uniformity of $\mathcal{T}_h$ around Γ_1 (needed in Section 2.3), the numerical results provided by Example 3 support the conjecture that the present algorithm may behave quite well even in a case not fully covered by the theory.

Consequently, these examples confirm that the scheme (2.3.4) can be solved much more efficiently by using the adaptive method instead of the uniform discretization procedure.

Table 2.5.1: individual errors, error estimate $\tilde{\boldsymbol{\vartheta}}$, effectivity index, and rate of convergence for the uniform and adaptive refinements (Example 1).

N	$\mathbf{e}(\tilde{\boldsymbol{\theta}})$	$\mathbf{e}(\lambda)$	$\mathbf{e}(\tilde{\boldsymbol{\sigma}})$	$\mathbf{e}(\mathbf{u})$	$\mathbf{e}(\eta)$	$\tilde{\boldsymbol{\vartheta}}$	$\mathbf{e}/\tilde{\boldsymbol{\vartheta}}$	$\boldsymbol{\gamma}$
2357	6.2368	0.0230	18.4233	0.3034	0.0464	19.9547	0.9748	——
9196	5.4487	0.0053	12.0095	0.1819	0.0096	13.8410	0.9529	0.5709
36320	4.0722	0.0012	8.3926	0.1123	0.0023	9.9591	0.9367	0.5041
144352	2.6019	0.0002	5.6223	0.0597	0.0006	6.4944	0.9540	0.5932
2357	6.2368	0.0230	18.4233	0.3034	0.0464	19.9547	0.9748	——
2439	5.5359	0.0122	12.1930	0.2760	0.0469	14.0431	0.9538	21.8255
2512	4.3545	0.0150	9.1118	0.2659	0.0479	10.9267	0.9246	19.1252
2666	3.2551	0.0111	6.9189	0.2445	0.0460	8.1138	0.9429	9.3452
2812	2.5638	0.0113	5.4180	0.2421	0.0460	6.3965	0.9379	9.1213
3735	1.6385	0.0064	3.4797	0.2098	0.0457	4.1088	0.9375	3.1211
5219	1.2136	0.0037	2.4864	0.2012	0.0444	2.9833	0.9300	1.9619
8655	0.8787	0.0044	1.7905	0.1740	0.0412	2.1818	0.9178	1.2891
16623	0.6264	0.0032	1.2646	0.1373	0.0250	1.5532	0.9130	1.0575
27069	0.4911	0.0030	0.9818	0.1169	0.0125	1.2172	0.9070	1.0270
55840	0.3481	0.0018	0.6966	0.0803	0.0074	0.8582	0.9122	0.9494
77488	0.2959	0.0014	0.5875	0.0685	0.0045	0.7246	0.9127	1.0292
95293	0.2674	0.0012	0.5309	0.0637	0.0036	0.6557	0.9117	0.9774
156559	0.2117	0.0007	0.4265	0.0488	0.0023	0.5221	0.9167	0.8958

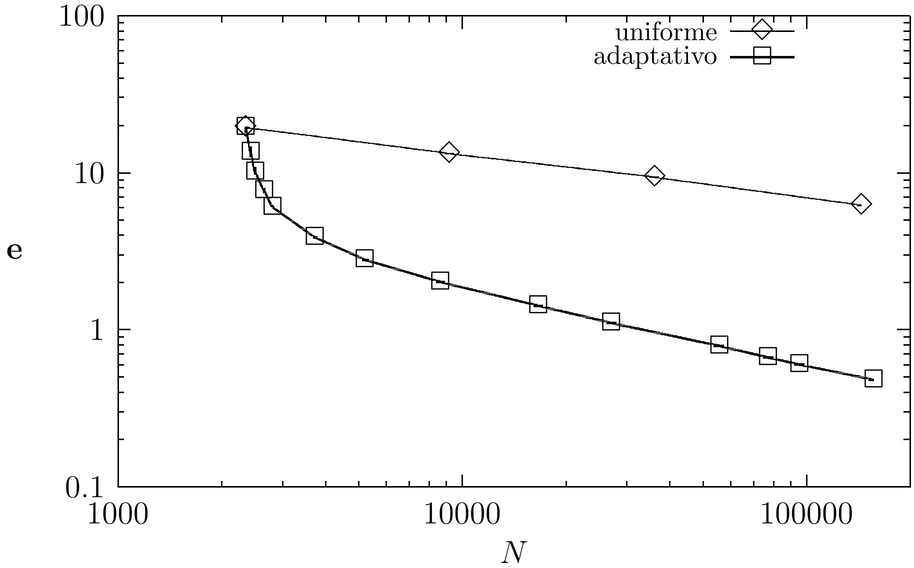

Figure 2.5.1: total error **e** for the uniform and adaptive refinements (Example 1).

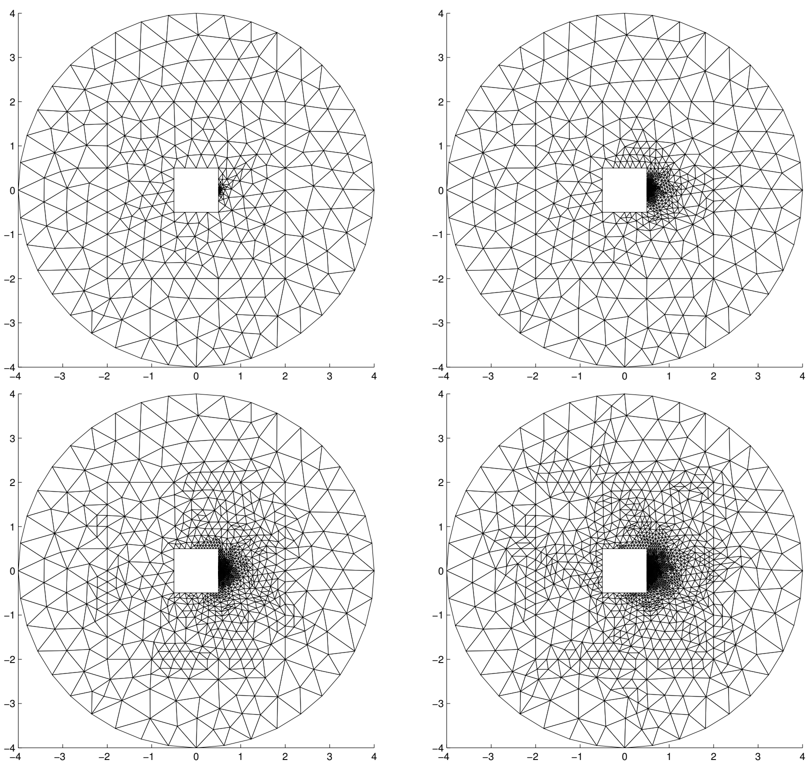

Figure 2.5.2: adapted intermediate meshes with 2812, 8655, 16623 and 27069 degrees of freedom, respectively, for Example 1.

Table 2.5.2: individual errors, error estimate $\tilde{\boldsymbol{\vartheta}}$, effectivity index, and rate of convergence for the uniform and adaptive refinements (Example 2).

N	$\mathbf{e}(\tilde{\boldsymbol{\theta}})$	$\mathbf{e}(\lambda)$	$\mathbf{e}(\tilde{\boldsymbol{\sigma}})$	$\mathbf{e}(\mathbf{u})$	$\mathbf{e}(\eta)$	$\tilde{\boldsymbol{\vartheta}}$	$\mathbf{e}/\tilde{\boldsymbol{\vartheta}}$	$\boldsymbol{\gamma}$
1613	0.9465	0.1457	24.8538	0.3321	0.2108	24.8414	1.0014	——
6232	0.7134	0.1722	26.6078	0.2660	0.1930	26.5987	1.0008	——
24488	0.5207	0.1672	17.0046	0.2330	0.1778	16.9929	1.0014	0.6540
97072	0.4045	0.1555	10.5563	0.2097	0.1622	10.5389	1.0028	0.6916
1613	0.9465	0.1457	24.8538	0.3321	0.2108	24.8404	1.0014	——
1659	0.8059	0.1752	26.6523	0.3312	0.2109	26.6479	1.0007	——
1714	0.6818	0.1814	17.1935	0.3335	0.2127	17.1891	1.0014	26.8481
1769	0.6248	0.1822	11.2691	0.3338	0.2130	11.2615	1.0030	26.6771
1870	0.6041	0.1823	8.7195	0.3338	0.2130	8.7102	1.0047	9.1906
1975	0.5991	0.1823	6.7824	0.3338	0.2130	6.7710	1.0076	9.1137
2567	0.4220	0.0160	4.6006	0.2001	0.0430	4.6576	0.9929	2.9667
3377	0.4165	0.0160	3.4549	0.2000	0.0430	3.5306	0.9874	2.0610
5520	0.3219	0.0136	2.4811	0.1715	0.0418	2.5771	0.9733	1.3398
10630	0.2552	0.0099	1.7426	0.1332	0.0232	1.8110	0.9754	1.0701
15805	0.2079	0.0094	1.4009	0.1192	0.0159	1.4623	0.9720	1.0958
30918	0.1601	0.0077	1.0036	0.0909	0.0115	1.0554	0.9668	0.9879
59038	0.1152	0.0064	0.7340	0.0651	0.0087	0.7669	0.9726	0.9688
78847	0.1003	0.0067	0.6350	0.0579	0.0088	0.6634	0.9731	0.9986
158642	0.0750	0.0064	0.4509	0.0427	0.0085	0.4711	0.9749	0.9741

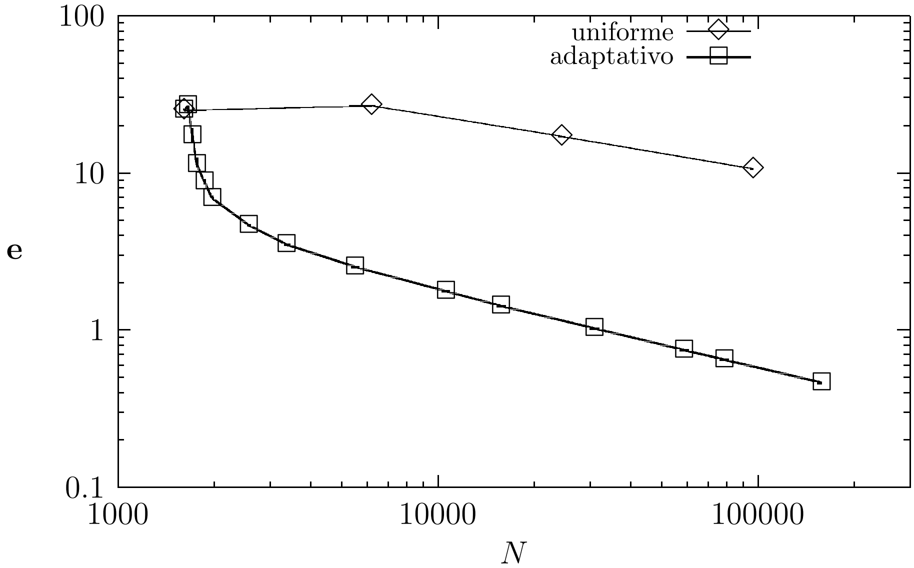

Figure 2.5.3: total error **e** for the uniform and adaptive refinements (Example 2).

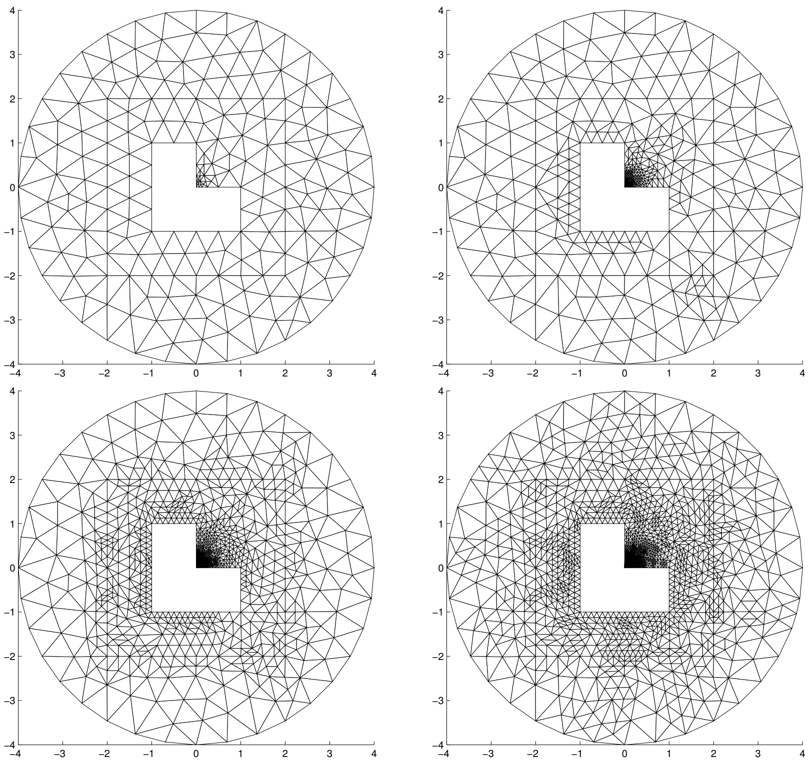

Figure 2.5.4: adapted intermediate meshes with 1769, 5520, 15805 and 30918 degrees of freedom, respectively, for Example 2.

Table 2.5.3: individual errors, error estimate $\tilde{\boldsymbol{\vartheta}}$, effectivity index, and rate of convergence for the uniform and adaptive refinements (Example 3).

N	$\mathbf{e}(\tilde{\boldsymbol{\theta}})$	$\mathbf{e}(\lambda)$	$\mathbf{e}(\tilde{\boldsymbol{\sigma}})$	$\mathbf{e}(\mathbf{u})$	$\mathbf{e}(\eta)$	$\tilde{\boldsymbol{\vartheta}}$	$\mathbf{e}/\tilde{\boldsymbol{\vartheta}}$	γ
2357	17.4739	0.4125	44.0447	2.7299	5.8914	65.3193	0.7322	——
9196	13.9420	0.4128	30.3579	1.4239	3.5459	45.9814	0.7313	0.5176
36320	8.0055	0.3805	19.6237	0.7126	1.3731	25.7026	0.8269	0.6680
144352	3.7795	0.3503	11.8582	0.5126	0.4150	12.7147	0.9806	0.7730
2357	17.4739	0.4135	44.0447	2.7299	5.8914	65.3193	0.7322	——
2577	12.6557	0.4596	29.1018	1.5505	3.3097	46.9136	0.6810	9.0442
2799	8.5183	0.4703	20.5283	1.1527	1.1268	29.1943	0.7635	8.7131
3122	6.7504	0.4573	16.9764	1.0253	0.9357	22.2095	0.8252	3.5835
3357	5.4420	0.4596	14.8794	1.0005	0.5678	17.0991	0.9294	3.9301
4388	4.2300	0.0218	10.4067	0.6567	0.2466	11.8574	0.9492	2.5759
6363	3.1313	0.0273	7.0145	0.5463	0.1994	8.4505	0.9116	2.0405
7877	2.6708	0.0260	6.4294	0.4901	0.0848	7.2596	0.9615	0.9246
12310	2.0215	0.0210	4.6988	0.4079	0.0607	5.3701	0.9556	1.3778
20883	1.5398	0.0169	3.5107	0.3200	0.0387	4.0907	0.9404	1.0904
25819	1.4171	0.0156	3.1918	0.2964	0.0346	3.6083	0.9714	0.8778
40311	1.1120	0.0115	2.4817	0.2382	0.0204	2.9045	0.9399	1.1219
62708	0.8843	0.0098	1.9801	0.1939	0.0155	2.3408	0.9302	1.0238
79465	0.8038	0.0094	1.7777	0.1768	0.0142	2.1033	0.9314	0.8923

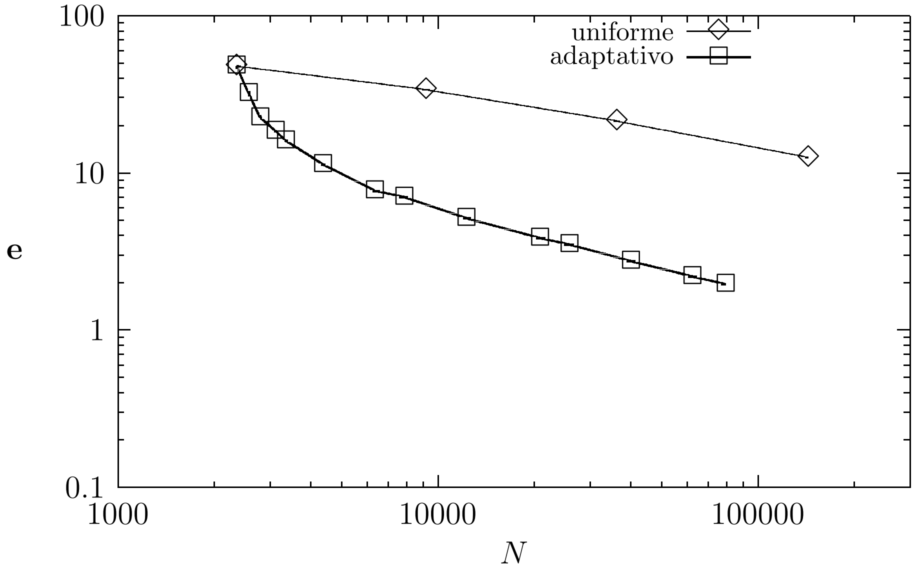

Figure 2.5.5: total error **e** for the uniform and adaptive refinements (Example 3).

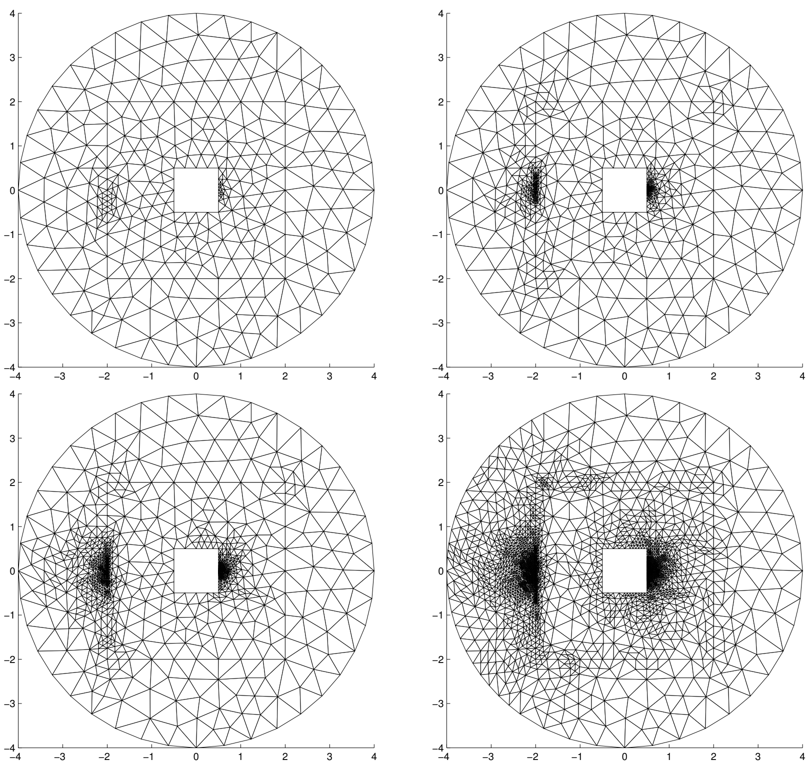

Figure 2.5.6: adapted intermediate meshes with 2799, 6363, 12310 and 40311 degrees of freedom, respectively, for Example 3.

Part II

The LDG method for nonlinear problems

Chapter 3

A local discontinuous Galerkin method for nonlinear diffusion problems with mixed boundary conditions

In this chapter we present and analyze a local discontinuous Galerkin method for a class of nonlinear diffusion problems in polygonal regions of $\mathbf{R}^2$. Our analysis follows known approaches previously applied to linear problems and considers convex and non-convex domains. We provide solvability and stability of the discrete scheme for a wide range of polynomial approximations, and derive a-priori error estimates in the energy and L^2 norms. Numerical experiments illustrating these results are also provided.

3.1 Introduction

The discontinuous Galerkin (DG) methods have been frequently applied over the last years to solve non-linear hyperbolic, convection dominated, and more recently diffusion dominated and purely elliptic problems. The main advantages of these methods, as compared to the continuous finite element schemes, are the high order of approximation provided by them, their high degree of parallelism, and their suitability for h, p, and hp refinements (because of the use of arbitrary polynomial degrees on different finite elements). We refer to [44] for a general overview, to [7] and [8] for a unified analysis providing stability and optimal error estimates, and to [37] for the numerical and computational performance, of most of the existing DG methods. These include, among others, the Babuška-Zlámal penalty scheme, the interior penalty approach, and the local discontinuous Galerkin (LDG) method.

We are particularly interested here in the LDG (see, e.g. [38], [40], [45], [83] and [84]). More precisely, the purpose of this work is to extend the applicability of the local discontinuous Galerkin method to a class of nonlinear elliptic boundary value problems arising in heat conduction. In other words, we generalize the results provided in [83] to the case of nonlinear diffusion problems with mixed boundary conditions. Some parts of our analysis follow the general framework introduced in [8]. In order to describe the model of interest, we let Ω be a bounded and simply connected domain in $\mathbf{R}^2$ with polygonal boundary Γ, and let Γ_D and Γ_N be parts of Γ such that $|\Gamma_D| \neq 0$, $\Gamma = \bar{\Gamma}_D \cup \bar{\Gamma}_N$, and $\Gamma_D \cap \Gamma_N = \phi$. Also, we let $a_i : \Omega \times \mathbf{R}^2 \to \mathbf{R}$, $i = 1, 2$, be nonlinear mappings satisfying certain conditions (specified below) and denote by $\mathbf{a} : \Omega \times \mathbf{R}^2 \to \mathbf{R}^2$ the vector function defined by $\mathbf{a}(x, \boldsymbol{\zeta}) := (a_1(x, \boldsymbol{\zeta}), a_2(x, \boldsymbol{\zeta}))^{\mathbf{T}}$ for all $(x, \boldsymbol{\zeta}) \in \Omega \times \mathbf{R}^2$. Then, given $f \in L^2(\Omega)$, $g_D \in H^{1/2}(\Gamma_D)$, and $g_N \in L^2(\Gamma_N)$, we look for $u \in H^1(\Omega)$ such that

$$\begin{gathered} -\operatorname{div} \mathbf{a}(\cdot, \nabla u(\cdot)) = f \quad \text{in} \quad \Omega\,, \\ u = g_D \quad \text{on} \quad \Gamma_D \quad \text{and} \quad \mathbf{a}(\cdot, \nabla u(\cdot)) \cdot \boldsymbol{\nu} = g_N \quad \text{on} \quad \Gamma_N\,, \end{gathered} \tag{3.1.1}$$

where div is the usual divergence operator and $\boldsymbol{\nu}$ denotes the unit outward normal to $\partial\Omega$.

The rest of the chapter is organized as follows. In Section 3.2 we introduce the local discontinuous Galerkin scheme, including the definition of the corresponding numerical fluxes, for the case in which Ω is convex. The associated primal formulation is provided in Section 3.3. Then, the unique solvability and stability of it is established in Section 3.4. In Section 3.5 we derive the a-priori error estimates in energy and L^2 norms. Next, in Section 3.6 we consider the non-convex case and present the corresponding a-priori error estimates in energy and L^2 norms. Finally, several numerical experiments are reported in Section 3.7.

3.2 The LDG formulation

From now up to the end of Section 3.5 we assume that Ω is convex. We follow [40] (see also [56], [59], [83]) and introduce the gradient $\boldsymbol{\theta} := \nabla u$ in Ω and the flux $\boldsymbol{\sigma} := \mathbf{a}(\cdot, \boldsymbol{\theta})$ in Ω as additional unknowns. In this way, (3.1.1) can be reformulated as the following problem in $\bar{\Omega}$: *Find* $(\boldsymbol{\theta}, \boldsymbol{\sigma}, u)$ *in appropriate spaces such that, in the distributional sense,*

$$\begin{gathered}\boldsymbol{\theta} = \nabla u \quad \text{in} \quad \Omega\,, \qquad \boldsymbol{\sigma} = \mathbf{a}(\cdot, \boldsymbol{\theta}) \quad \text{in} \quad \Omega\,, \qquad -\mathrm{div}\,\boldsymbol{\sigma} = f \quad \text{in} \quad \Omega\,, \\ u = g_D \quad \text{on} \quad \Gamma_D\,, \quad \text{and} \quad \boldsymbol{\sigma}\cdot\boldsymbol{\nu} = g_N \quad \text{on} \quad \Gamma_N.\end{gathered} \tag{3.2.1}$$

Hereafter, the mappings a_i defining $\mathbf{a}$ are supposed to satisfy the following hypotheses:

(H.1) *Carathéodory condition.* The function $a_i(\cdot, \boldsymbol{\zeta})$, $i = 1, 2$, is measurable in Ω for all $\boldsymbol{\zeta} \in \mathbf{R}^2$, and $a_i(x, \cdot)$ is continuous in $\mathbf{R}^2$ for almost all $x \in \Omega$.

(H.2) *Growth condition.* There exist $C > 0$ and $\phi_i \in L^2(\Omega)$, $i = 1, 2$, such that

$$|a_i(x, \boldsymbol{\zeta})| \;\leq\; C\,\{1 + |\boldsymbol{\zeta}|\} + |\phi_i(x)|\,,$$

for all $\boldsymbol{\zeta} \in \mathbf{R}^2$ and for almost all $x \in \Omega$.

(H.3) The function $a_i(x, \cdot)$, $i = 1, 2$, has continuous first order partial derivatives in $\mathbf{R}^2$ for almost all $x \in \Omega$. In addition, there exists $C > 0$ such that

$$\sum_{i,j=1}^{2} \frac{\partial}{\partial \zeta_j}\, a_i(x, \boldsymbol{\zeta})\, \tilde{\zeta}_i\, \tilde{\zeta}_j \;\geq\; C \sum_{i=1}^{2} \tilde{\zeta}_i^2\,,$$

for all $\boldsymbol{\zeta} := (\zeta_1, \zeta_2)^{\mathbf{T}}$, $\tilde{\boldsymbol{\zeta}} := (\tilde{\zeta}_1, \tilde{\zeta}_2)^{\mathbf{T}} \in \mathbf{R}^2$ and for almost all $x \in \Omega$.

(H.4) The function $a_i(x, \cdot)$, $i = 1, 2$, has continuous first order partial derivatives in $\mathbf{R}^2$ for almost all $x \in \Omega$. In addition, there exists $C > 0$ such that for each $i, j \in \{1, 2\}$, $\frac{\partial}{\partial \zeta_j}\, a_i(x, \boldsymbol{\zeta})$ satisfies the Carathéodory condition (H.1), and $\left|\frac{\partial}{\partial \zeta_j}\, a_i(x, \boldsymbol{\zeta})\right| \leq C$, for all $\boldsymbol{\zeta} \in \mathbf{R}^2$ and for almost all $x \in \Omega$.

We now let $\mathcal{T}_h$ be a shape-regular triangulation of $\bar{\Omega}$ (with possible hanging nodes) made up of straight triangles T with diameter h_T and unit outward normal to ∂T given by $\boldsymbol{\nu}_T$. As usual, the index h also denotes $h := \max\limits_{T \in \mathcal{T}_h} h_T$. In addition, we define the edges of $\mathcal{T}_h$ as follows. An *interior edge* of $\mathcal{T}_h$ is the (non-empty) interior of $\partial T \cap \partial T'$, where T and T' are two adjacent elements of $\mathcal{T}_h$, not necessarily matching.

Similarly, a *boundary edge* of $\mathcal{T}_h$ is the (non-empty) interior of $\partial T \cap \partial\Omega$, where T is a boundary element of $\mathcal{T}_h$. We denote by $\mathcal{E}_I$ the union of all interior edges of $\mathcal{T}_h$, by $\mathcal{E}_D$ and $\mathcal{E}_N$ the union of all Dirichlet and Neumann boundary edges, respectively, and set $\mathcal{E} := \mathcal{E}_I \cup \mathcal{E}_D \cup \mathcal{E}_N$ the union of all edges of $\mathcal{T}_h$. Further, for each $e \subseteq \mathcal{E}$, h_e represents its diameter. Also, in what follows we assume that $\mathcal{T}_h$ is of *bounded variation*, that is there exists a constant $l > 1$, independent of the meshsize h, such that

$$l^{-1} \;\leq\; \frac{h_T}{h_{T'}} \;\leq\; l \tag{3.2.2}$$

for each pair T, $T' \in \mathcal{T}_h$ sharing an interior edge.

The LDG variational formulation is described next. We first multiply the first three equations of (3.2.1) by smooth test functions $\boldsymbol{\zeta}$, $\boldsymbol{\tau}$ and v, respectively, integrate by parts over each $T \in \mathcal{T}_h$, and obtain

$$\begin{aligned}
&\int_T \mathbf{a}(\cdot,\boldsymbol{\theta}) \cdot \boldsymbol{\zeta}\, dx - \int_T \boldsymbol{\sigma} \cdot \boldsymbol{\zeta}\, dx = 0\,, \\
&\int_T \boldsymbol{\theta} \cdot \boldsymbol{\tau}\, dx + \int_T u \operatorname{div} \boldsymbol{\tau}\, dx - \int_{\partial T} u\, \boldsymbol{\tau} \cdot \boldsymbol{\nu}_T\, ds = 0\,, \\
&\int_T \boldsymbol{\sigma} \cdot \nabla v\, dx - \int_{\partial T} v\, \boldsymbol{\sigma} \cdot \boldsymbol{\nu}_T\, ds = \int_T f\, v\, dx\,.
\end{aligned} \tag{3.2.3}$$

Then, we want to approximate the exact solution $(\boldsymbol{\theta}, \boldsymbol{\sigma}, u)$ by discrete functions $(\boldsymbol{\theta}_h, \boldsymbol{\sigma}_h, u_h)$ in the finite element space $\boldsymbol{\Sigma}_h \times \boldsymbol{\Sigma}_h \times V_h$, where

$$\boldsymbol{\Sigma}_h := \Big\{ \boldsymbol{\theta}_h \in [L^2(\Omega)]^2 : \quad \boldsymbol{\theta}_h|_T \in [\mathbf{P}_r(T)]^2 \quad \forall\, T \in \mathcal{T}_h \Big\},$$

$$V_h := \Big\{ v_h \in L^2(\Omega) : \quad v_h|_T \in \mathbf{P}_k(T) \quad \forall\, T \in \mathcal{T}_h \Big\}\,,$$

with $k \geq 1$ and $r = k$ or $r = k - 1$. Hereafter, given an integer $\kappa \geq 0$ we denote by $\mathbf{P}_\kappa(T)$ the space of polynomials of degree at most κ on T.

From now on, the space $\boldsymbol{\Sigma}_h$ is provided with the usual product norm of $[L^2(\Omega)]^2$, which is denoted by $\|\cdot\|_{[L^2(\Omega)]^2}$. The norm for V_h will be defined later on in Section 3.3.

The idea of the LDG method is to enforce the conservation laws given in (3.2.3) with the traces of $\boldsymbol{\sigma}$ and $\mathbf{u}$ on the boundary of each $T \in \mathcal{T}_h$ being replaced by suitable numerical approximations of them. In other words, following [45] and [83], we consider the following formulation: *Find* $(\boldsymbol{\theta}_h, \boldsymbol{\sigma}_h, u_h) \in \boldsymbol{\Sigma}_h \times \boldsymbol{\Sigma}_h \times V_h$ *such that* *for all $T \in \mathcal{T}_h$ we have*

$$
\begin{aligned}
&\int_T \mathbf{a}(\cdot,\boldsymbol{\theta}_h)\cdot\boldsymbol{\zeta}_h\,dx - \int_T \boldsymbol{\sigma}_h\cdot\boldsymbol{\zeta}_h\,dx = 0 && \forall\,\boldsymbol{\zeta}_h \in \boldsymbol{\Sigma}_h\,,\\
&\int_T \boldsymbol{\theta}_h\cdot\boldsymbol{\tau}_h\,dx + \int_T u_h \,\mathrm{div}\,\boldsymbol{\tau}_h\,dx - \int_{\partial T} \widehat{u}\,\boldsymbol{\tau}_h\cdot\boldsymbol{\nu}_T\,ds = 0 && \forall\,\boldsymbol{\tau}_h \in \boldsymbol{\Sigma}_h\,,\\
&\int_T \boldsymbol{\sigma}_h\cdot\nabla v_h\,dx - \int_{\partial T} v_h\,\widehat{\boldsymbol{\sigma}}\cdot\boldsymbol{\nu}_T\,ds = \int_T f v_h\,dx && \forall\,v_h \in V_h\,,
\end{aligned}
\tag{3.2.4}
$$

where the *numerical fluxes* $\widehat{u}$ and $\widehat{\boldsymbol{\sigma}}$, which usually depend on u_h, $\boldsymbol{\sigma}_h$, and the boundary conditions, are chosen so that some compatibility conditions are satisfied.

In order to specify these conditions, we let $H^1(\mathcal{T}_h)$ (resp. $H(\mathrm{div}\,;\mathcal{T}_h)$) be the space of functions on Ω whose restriction to each $T \in \mathcal{T}_h$ belongs to $H^1(T)$ (resp. $H(\mathrm{div}\,;T)$), and consider the scalar numerical flux $\widehat{u} = (\widehat{u}_T)_{T\in\mathcal{T}_h}$ and the vector numerical flux $\widehat{\boldsymbol{\sigma}} = (\widehat{\boldsymbol{\sigma}}_T)_{T\in\mathcal{T}_h}$ as linear operators

$$
\begin{array}{rccl}
\widehat{u}: & H^1(\mathcal{T}_h) & \rightarrow & \prod\limits_{T\in\mathcal{T}_h} L^2(\partial T)\\
& v & \rightarrow & \widehat{u}(v) := (\widehat{u}_T(v))_{T\in\mathcal{T}_h}\,,
\end{array}
\tag{3.2.5}
$$

and

$$
\begin{array}{rccl}
\widehat{\boldsymbol{\sigma}}: & H^1(\mathcal{T}_h)\times H(\mathrm{div}\,;\mathcal{T}_h) & \rightarrow & \prod\limits_{T\in\mathcal{T}_h} [L^2(\partial T)]^2\\
& (v,\boldsymbol{\tau}) & \rightarrow & \widehat{\boldsymbol{\sigma}}(v,\boldsymbol{\tau}) := (\widehat{\boldsymbol{\sigma}}_T(v,\boldsymbol{\tau}))_{T\in\mathcal{T}_h}\,.
\end{array}
\tag{3.2.6}
$$

Then, we say that these numerical fluxes are *consistent* if

$$
\widehat{u}_T(v) = v|_{\partial T} \quad\text{and}\quad \widehat{\boldsymbol{\sigma}}_T(v,\mathbf{a}(\cdot,\nabla v(\cdot))) = \mathbf{a}(\cdot,\nabla v(\cdot))|_{\partial T} \qquad \forall\,T\in\mathcal{T}_h\,,
$$

whenever v is a smooth function satisfying the boundary conditions. In addition, we say that $\widehat{u}$ and $\widehat{\boldsymbol{\sigma}}$ are *conservative* if for each pair of smooth functions $(v,\boldsymbol{\tau})$ and for each $e \subseteq \mathcal{E}_I$, there hold

$$
\widehat{u}_T(v) = \widehat{u}_{T'}(v) \quad\text{and}\quad \widehat{\boldsymbol{\sigma}}_T(v,\boldsymbol{\tau}) = \widehat{\boldsymbol{\sigma}}_{T'}(v,\boldsymbol{\tau}) \quad\text{on}\quad e = \partial T\cap\partial T'\,.
$$

In order to define the specific numerical fluxes that we will use in our formulation, we need some additional notations. In fact, given $w := (w_T)_{T\in\mathcal{T}_h} \in \prod\limits_{T\in\mathcal{T}_h} L^2(\partial T)$ and $\boldsymbol{\zeta} := (\boldsymbol{\zeta}_T)_{T\in\mathcal{T}_h} \in \prod\limits_{T\in\mathcal{T}_h} [L^2(\partial T)]^2$, we denote $w_{T,e} := w_T|_e$ and $\boldsymbol{\zeta}_{T,e} := \boldsymbol{\zeta}_T|_e$ for each $(T,e) \in \mathcal{T}_h\times\mathcal{E}$. Further, let us consider an interior edge e of $\mathcal{T}_h$, shared by two adjacent elements T and T' of $\mathcal{T}_h$. Then, we define the *average* and the *jump* of w across e by

$$\{w\} := \frac{1}{2}(w_{T,e} + w_{T',e}) \quad \text{and} \quad [\![w]\!] := w_{T,e}\,\boldsymbol{\nu}_T + w_{T',e}\,\boldsymbol{\nu}_{T'}\,. \tag{3.2.7}$$

Analogously, the corresponding *average* and *jump* of $\boldsymbol{\zeta}$ across $e \subseteq \mathcal{E}_I$ are defined by

$$\{\boldsymbol{\zeta}\} := \frac{1}{2}(\boldsymbol{\zeta}_{T,e} + \boldsymbol{\zeta}_{T',e}) \quad \text{and} \quad [\![\boldsymbol{\zeta}]\!] := \boldsymbol{\zeta}_{T,e} \cdot \boldsymbol{\nu}_T + \boldsymbol{\zeta}_{T',e} \cdot \boldsymbol{\nu}_{T'}\,. \tag{3.2.8}$$

We notice, that for any $e \subseteq \mathcal{E}^{\partial} := \mathcal{E}_D \cup \mathcal{E}_N$, the traces of every scalar and vector functions $w \in \prod_{T\in\mathcal{T}_h} L^2(\partial T)$ and $\boldsymbol{\zeta} \in \prod_{T\in\mathcal{T}_h} [L^2(\partial T)]^2$ on e are uniquely defined, and hence we set

$$[\![w]\!] := w_{T,e}\,\boldsymbol{\nu}_T \quad \text{and} \quad \{\boldsymbol{\zeta}\} := \boldsymbol{\zeta}_{T,e}\,.$$

We are now ready to complete the LDG formulation (3.2.4). Indeed, using the approach from [83] and [38], we define the numerical fluxes $\widehat{u}$ and $\widehat{\boldsymbol{\sigma}}$ for each $T \in \mathcal{T}_h$, as follows:

$$\widehat{u}_{T,e} := \begin{cases} \{u_h\} + \boldsymbol{\beta} \cdot [\![u_h]\!] & \text{if } e \subseteq \mathcal{E}_I\,, \\ g_D & \text{if } e \subseteq \mathcal{E}_D\,, \\ u_h & \text{if } e \subseteq \mathcal{E}_N\,, \end{cases} \tag{3.2.9}$$

and

$$\widehat{\boldsymbol{\sigma}}_{T,e} := \begin{cases} \{\boldsymbol{\sigma}_h\} - \boldsymbol{\beta}[\![\boldsymbol{\sigma}_h]\!] - \alpha[\![u_h]\!] & \text{if } e \subseteq \mathcal{E}_I\,, \\ \boldsymbol{\sigma}_h - \alpha(u_h - g_D)\boldsymbol{\nu} & \text{if } e \subseteq \mathcal{E}_D\,, \\ \widehat{\boldsymbol{\sigma}}_{T,e} \cdot \boldsymbol{\nu} = g_N & \text{if } e \subseteq \mathcal{E}_N\,, \end{cases} \tag{3.2.10}$$

where the auxiliary functions α (scalar) and $\boldsymbol{\beta}$ (vector), to be chosen appropriately, are single-valued on each edge $e \subseteq \mathcal{E}$.

It follows easily from the definitions (3.2.9) and (3.2.10) that

$$\{\widehat{u}\} = \widehat{u}\,, \quad [\![\widehat{u}]\!] = 0\,, \quad \{\widehat{\boldsymbol{\sigma}}\} = \widehat{\boldsymbol{\sigma}}\,, \quad \text{and} \quad [\![\widehat{\boldsymbol{\sigma}}]\!] = 0 \quad \text{on} \quad \mathcal{E}_I\,, \tag{3.2.11}$$

which implies that these numerical fluxes are *consistent* and *conservative.*

This completes the definition of our LDG method. We notice that the flux $\widehat{u}$ is independent of $\boldsymbol{\sigma}_h$, and hence the unknowns $\boldsymbol{\sigma}_h$ and $\boldsymbol{\theta}_h$ can be *locally* solved in terms of u_h, which yields the so-called primal formulation of the method. We remark that this unusual local property gives its name to the LDG method. The primal formulation is explicitly defined in the next section, and its solvability and stability will depend in a crucial way on α and $\boldsymbol{\beta}$.

3.3 The primal formulation

Throughout the rest of the paper we assume that the solution u of (3.1.1) belongs to $H^{1+\epsilon}(\Omega)$, for some $\epsilon > 0$.

We now recall that for all scalar and vector functions $v := (v_T)_{T\in\mathcal{T}_h} \in \prod_{T\in\mathcal{T}_h} L^2(\partial T)$ and $\boldsymbol{\zeta} := (\boldsymbol{\zeta}_T)_{T\in\mathcal{T}_h} \in \prod_{T\in\mathcal{T}_h} [L^2(\partial T)]^2$, respectively, the following identity holds

$$\sum_{T\in\mathcal{T}_h} \int_{\partial T} v_T\, \boldsymbol{\zeta}_T \cdot \boldsymbol{\nu}_T \, ds = \sum_{e\subseteq\mathcal{E}_I} \int_e \{v\}[\![\boldsymbol{\zeta}]\!]\, ds + \sum_{e\subseteq\mathcal{E}_I} \int_e [\![v]\!] \cdot \{\boldsymbol{\zeta}\}\, ds + \sum_{e\subseteq\mathcal{E}^\partial} \int_e [\![v]\!] \cdot \{\boldsymbol{\zeta}\}\, ds. \tag{3.3.1}$$

Now, summing up in (3.2.4) over all the triangles T of $\mathcal{T}_h$, applying the identity (3.3.1), and using the properties given in (3.2.11), we arrive to

$$\int_\Omega \mathbf{a}(\cdot, \boldsymbol{\theta}_h) \cdot \boldsymbol{\zeta}_h \, dx = \int_\Omega \boldsymbol{\sigma}_h \cdot \boldsymbol{\zeta}_h \, dx \,, \tag{3.3.2}$$

$$\int_\Omega \boldsymbol{\theta}_h \cdot \boldsymbol{\tau}_h \, dx = -\int_\Omega u_h \, \mathrm{div}_h \boldsymbol{\tau}_h \, dx + \sum_{e\subseteq\mathcal{E}_I} \int_e \hat{u}\, [\![\boldsymbol{\tau}_h]\!]\, ds + \sum_{e\subseteq\mathcal{E}^\partial} \int_e \hat{u}\, \boldsymbol{\tau}_h \cdot \boldsymbol{\nu}\, ds \,, \tag{3.3.3}$$

$$\int_\Omega \boldsymbol{\sigma}_h \cdot \nabla_h v_h \, dx = \int_\Omega f v_h \, dx + \sum_{e\subseteq\mathcal{E}_I} \int_e [\![v_h]\!] \cdot \widehat{\boldsymbol{\sigma}}\, ds + \sum_{e\subseteq\mathcal{E}^\partial} \int_e [\![v_h]\!] \cdot \widehat{\boldsymbol{\sigma}}\, ds \,, \tag{3.3.4}$$

where ∇_h and div_h stands for the elementwise gradient and divergence operators, respectively.

We now introduce the notation $\int_{\mathcal{E}_I} g\, ds := \sum_{e\subseteq\mathcal{E}_I} \int_e g\, ds$, and similarly for $\int_{\mathcal{E}^\partial}$, $\int_{\mathcal{E}_D}$ and $\int_{\mathcal{E}_N}$.

Thus, integrating by parts the first term on the right hand side of (3.3.3), using the definition (3.2.9) of the numerical flux $\hat{u}$, and applying the identity (3.3.1), we obtain

$$\begin{aligned} \int_\Omega \boldsymbol{\theta}_h \cdot \boldsymbol{\tau}_h \, dx - \int_\Omega \nabla_h u_h \cdot \boldsymbol{\tau}_h \, dx + \int_{\mathcal{E}_I} (\{\boldsymbol{\tau}_h\} - \boldsymbol{\beta}[\![\boldsymbol{\tau}_h]\!]) \cdot [\![u_h]\!]\, ds \\ + \int_{\mathcal{E}_D} u_h\, \boldsymbol{\tau}_h \cdot \boldsymbol{\nu}\, ds \;=\; \int_{\mathcal{E}_D} g_D\, \boldsymbol{\tau}_h \cdot \boldsymbol{\nu}\, ds \,. \end{aligned} \tag{3.3.5}$$

Similarly, employing the definition (3.2.10) of the numerical flux $\widehat{\boldsymbol{\sigma}}$, equation

(3.3.4) becomes

$$\int_\Omega \boldsymbol{\sigma}_h \cdot \nabla_h v_h \, dx - \int_{\mathcal{E}_I} (\{\boldsymbol{\sigma}_h\} - \boldsymbol{\beta}[\![\boldsymbol{\sigma}_h]\!]) \cdot [\![v_h]\!] \, ds - \int_{\mathcal{E}_D} v_h \, \boldsymbol{\sigma}_h \cdot \boldsymbol{\nu} \, ds$$

$$+ \int_{\mathcal{E}_I} \alpha [\![u_h]\!] \cdot [\![v_h]\!] \, ds + \int_{\mathcal{E}_D} \alpha \, u_h \, v_h \, ds = \int_\Omega f \, v_h \, dx + \int_{\mathcal{E}_D} \alpha \, g_D \, v_h \, ds + \int_{\mathcal{E}_N} g_N \, v_h \, ds \, . \tag{3.3.6}$$

Consequently, adding (3.3.6) and (3.3.2), and considering (3.3.5) instead of (3.3.3), our formulation reduces to: *Find* $(\boldsymbol{\theta}_h, u_h, \boldsymbol{\sigma}_h) \in \boldsymbol{\Sigma}_h \times V_h \times \boldsymbol{\Sigma}_h$ *such that for all* $(\boldsymbol{\zeta}_h, v_h, \boldsymbol{\tau}_h) \in \boldsymbol{\Sigma}_h \times V_h \times \boldsymbol{\Sigma}_h$

$$\begin{aligned}
&\int_\Omega \mathbf{a}(\cdot, \boldsymbol{\theta}_h) \cdot \boldsymbol{\zeta}_h \, dx + \int_{\mathcal{E}_I} \alpha [\![u_h]\!] \cdot [\![v_h]\!] \, ds + \int_{\mathcal{E}_D} \alpha \, u_h \, v_h \, ds \\
&- \int_\Omega \boldsymbol{\sigma}_h \cdot \boldsymbol{\zeta}_h \, dx + \int_\Omega \boldsymbol{\sigma}_h \cdot \nabla_h v_h \, dx - \int_{\mathcal{E}_I} (\{\boldsymbol{\sigma}_h\} - \boldsymbol{\beta}[\![\boldsymbol{\sigma}_h]\!]) \cdot [\![v_h]\!] \, ds - \int_{\mathcal{E}_D} v_h \, \boldsymbol{\sigma}_h \cdot \boldsymbol{\nu} \, ds \\
&= \int_\Omega f \, v_h \, dx + \int_{\mathcal{E}_D} \alpha \, g_D \, v_h \, ds + \int_{\mathcal{E}_N} g_N \, v_h \, ds \, , \\
&- \int_\Omega \boldsymbol{\theta}_h \cdot \boldsymbol{\tau}_h \, dx + \int_\Omega \nabla_h u_h \cdot \boldsymbol{\tau}_h \, dx - \int_{\mathcal{E}_I} (\{\boldsymbol{\tau}_h\} - \boldsymbol{\beta}[\![\boldsymbol{\tau}_h]\!]) \cdot [\![u_h]\!] \, ds \\
&- \int_{\mathcal{E}_D} u_h \, \boldsymbol{\tau}_h \cdot \boldsymbol{\nu} \, ds \; = \; - \int_{\mathcal{E}_D} g_D \, \boldsymbol{\tau}_h \cdot \boldsymbol{\nu} \, ds \, .
\end{aligned} \tag{3.3.7}$$

We introduce now the semilinear form $A : (\boldsymbol{\Sigma}_h \times V_h) \times (\boldsymbol{\Sigma}_h \times V_h) \to \mathbf{R}$, the bilinear form $B : (\boldsymbol{\Sigma}_h \times V_h) \times \boldsymbol{\Sigma}_h \to \mathbf{R}$, and the functionals $F : (\boldsymbol{\Sigma}_h \times V_h) \to \mathbf{R}$, $G : \boldsymbol{\Sigma}_h \to \mathbf{R}$, defined by

$$A((\boldsymbol{\theta}_h, u_h), (\boldsymbol{\zeta}_h, v_h)) := \int_\Omega \mathbf{a}(\cdot, \boldsymbol{\theta}_h) \cdot \boldsymbol{\zeta}_h \, dx + \int_{\mathcal{E}_I} \alpha \, [\![u_h]\!] \cdot [\![v_h]\!] \, ds + \int_{\mathcal{E}_D} \alpha \, u_h \, v_h \, ds \, ,$$

$$B((\boldsymbol{\zeta}_h, v_h), \boldsymbol{\sigma}_h) := - \int_\Omega \boldsymbol{\sigma}_h \cdot \boldsymbol{\zeta}_h \, dx + \int_\Omega \boldsymbol{\sigma}_h \cdot \nabla_h v_h \, dx$$

$$- \int_{\mathcal{E}_I} (\{\boldsymbol{\sigma}_h\} - \boldsymbol{\beta}[\![\boldsymbol{\sigma}_h]\!]) \cdot [\![v_h]\!] \, ds - \int_{\mathcal{E}_D} v_h \, \boldsymbol{\sigma}_h \cdot \boldsymbol{\nu} \, ds \, ,$$

$$F(\boldsymbol{\zeta}_h, v_h) := \int_\Omega f \, v_h \, dx + \int_{\mathcal{E}_D} \alpha \, g_D \, v_h \, ds + \int_{\mathcal{E}_N} g_N \, v_h \, ds \, ,$$

and

$$G(\boldsymbol{\tau}_h) := -\int_{\mathcal{E}_D} g_D\, \boldsymbol{\tau}_h \cdot \boldsymbol{\nu}\, ds\,,$$

for all $(\boldsymbol{\theta}_h, u_h, \boldsymbol{\sigma}_h), (\boldsymbol{\zeta}_h, v_h, \boldsymbol{\tau}_h) \in \boldsymbol{\Sigma}_h \times V_h \times \boldsymbol{\Sigma}_h$. We remark here that the hypotheses (H.1) and (H.2) on the coefficients a_i guarantee that the nonlinear part of A (given by its first term) is well defined.

Therefore, the variational formulation (3.3.7) can be written as the following dual system: *Find* $((\boldsymbol{\theta}_h, u_h), \boldsymbol{\sigma}_h) \in (\boldsymbol{\Sigma}_h \times V_h) \times \boldsymbol{\Sigma}_h$ *such that*

$$\begin{aligned} A((\boldsymbol{\theta}_h, u_h), (\boldsymbol{\zeta}_h, v_h)) \;+\; B((\boldsymbol{\zeta}_h, v_h), \boldsymbol{\sigma}_h) \;&=\; F(\boldsymbol{\zeta}_h, v_h)\,, \\ B((\boldsymbol{\theta}_h, u_h), \boldsymbol{\tau}_h) \;&=\; G(\boldsymbol{\tau}_h)\,, \end{aligned} \tag{3.3.8}$$

for all $((\boldsymbol{\zeta}_h, v_h), \boldsymbol{\tau}_h) \in (\boldsymbol{\Sigma}_h \times V_h) \times \boldsymbol{\Sigma}_h$.

The stability and unique solvability of (3.3.8) will be established below by means of an equivalent formulation (having u_h only as unknown) that arises after expressing the unknowns $\boldsymbol{\sigma}_h$ and $\boldsymbol{\theta}_h$ in terms of u_h. Alternatively, these results could also be obtained by applying a slight generalization of the classical Babuška-Brezzi theory to (3.3.8) (see Proposition 2.3 in [91] or Theorem 2.2 in [52]).

We specify now the functions α and $\boldsymbol{\beta}$ needed in the definition of the numerical fluxes. First, we follow [83] and introduce the function $\mathtt{h}$ in $L^\infty(\mathcal{E})$, related to the local meshsize, as

$$\mathtt{h}(x) := \begin{cases} \min\{h_T, h_{T'}\} & \text{if } x \in \mathrm{int}(\partial T \cap \partial T') \\ h_T & \text{if } x \in \mathrm{int}(\partial T \cap \partial \Omega). \end{cases} \tag{3.3.9}$$

Then, we define $\alpha \in L^\infty(\mathcal{E})$ as

$$\alpha := \frac{\widehat{\alpha}}{\mathtt{h}}\,, \tag{3.3.10}$$

and consider $\boldsymbol{\beta} \in [L^\infty(\mathcal{E}_I)]^2$ such that

$$\|\boldsymbol{\beta}\|_{[L^\infty(\mathcal{E}_I)]^2} \leq \widehat{\beta}\,, \tag{3.3.11}$$

where $\widehat{\alpha} > 0$ and $\widehat{\beta} \geq 0$ are independent of the meshsize.

Next, we introduce the space $V(h) := V_h + H^1(\Omega)$, and the energy norm $|||\cdot|||_h : V(h) \to \mathbf{R}$ given by

$$|||v|||_h^2 := \|\nabla_h v\|^2_{[L^2(\Omega_1)]^2} + \|\alpha^{1/2}\, [\![v]\!]\|^2_{[L^2(\mathcal{E}_I)]^2} + \|\alpha^{1/2}\, v\|^2_{L^2(\mathcal{E}_D)} \quad \forall\, v \in V(h)\,. \tag{3.3.12}$$

Further, we let $|\cdot|_h : V(h) \to \mathbf{R}$ be the semi-norm defined by

$$|v|_h^2 \,:=\, \|\alpha^{1/2}\,[\![v]\!]\|^2_{[L^2(\mathcal{E}_I)]^2} \,+\, \|\alpha^{1/2}\,v\|^2_{L^2(\mathcal{E}_D)} \quad \forall\, v \in V(h)\,. \tag{3.3.13}$$

The following preliminary results are needed to derive the equivalent formulation for (3.3.8).

LEMMA 3.3.1 *There exist constants $\bar{C}_1$, $\bar{C}_2$, $\bar{C}_3 > 0$, independent of the meshsize, such that for all $\boldsymbol{\zeta} := (\boldsymbol{\zeta}_T)_{T\in\mathcal{T}_h} \in \prod_{T\in\mathcal{T}_h} [L^2(\partial T)]^2$, there hold*

i) $\|\mathtt{h}^{1/2}\{\boldsymbol{\zeta}\}\|^2_{[L^2(\mathcal{E}_I)]^2} \;\le\; \bar{C}_1 \displaystyle\sum_{T\in\mathcal{T}_h} h_T\,\|\boldsymbol{\zeta}_T\|^2_{[L^2(\partial T)]^2}\,,$

ii) $\|\mathtt{h}^{1/2}[\![\boldsymbol{\zeta}]\!]\|^2_{L^2(\mathcal{E}_I)} \;\le\; \bar{C}_2 \displaystyle\sum_{T\in\mathcal{T}_h} h_T\,\|\boldsymbol{\zeta}_T\|^2_{[L^2(\partial T)]^2}\,,$

iii) $\|\mathtt{h}^{1/2}\boldsymbol{\zeta}\cdot\boldsymbol{\nu}\|^2_{L^2(\mathcal{E}_D)} \;\le\; \bar{C}_3 \displaystyle\sum_{T\in\mathcal{T}_h} h_T\,\|\boldsymbol{\zeta}_T\|^2_{[L^2(\partial T)]^2}\,.$

PROOF. Let $\boldsymbol{\zeta} := (\boldsymbol{\zeta}_T)_{T\in\mathcal{T}_h} \in \prod_{T\in\mathcal{T}_h} [L^2(\partial T)]^2$. According to the definitions of $\{\boldsymbol{\zeta}\}$ and $\mathtt{h}$ (cf. (3.2.8) and (3.3.9)), and using the fact that $\mathcal{T}_h$ is of *bounded variation* (cf. (3.2.2)), we have for i)

$$\|\mathtt{h}^{1/2}\{\boldsymbol{\zeta}\}\|^2_{[L^2(\mathcal{E}_I)]^2} \;=\; \frac{1}{4}\sum_{e\subseteq\mathcal{E}_I}\int_e \mathtt{h}\,|\boldsymbol{\zeta}_{T,e} \,+\, \boldsymbol{\zeta}_{T',e}|^2\,ds \le \frac{1}{2}\sum_{e\subseteq\mathcal{E}_I}\int_e \mathtt{h}\left(|\boldsymbol{\zeta}_{T,e}|^2 + |\boldsymbol{\zeta}_{T',e}|^2\right)\,ds$$

$$\le\; \frac{1}{2}\sum_{T\in\mathcal{T}_h}\int_{\partial T}\mathtt{h}\,|\boldsymbol{\zeta}_T|^2\,ds \;\le\; \bar{C}_1\sum_{T\in\mathcal{T}_h} h_T\,\|\boldsymbol{\zeta}_T\|^2_{[L^2(\partial T)]^2}\,,$$

where $\bar{C}_1$ depends on the constant l from (3.2.2).

We proceed similarly for ii) and iii), and obtain, respectively,

$$\|\mathtt{h}^{1/2}[\![\boldsymbol{\zeta}]\!]\|^2_{L^2(\mathcal{E}_I)} \;=\; \sum_{e\subseteq\mathcal{E}_I}\int_e \mathtt{h}\,|\boldsymbol{\zeta}_{T,e}\cdot\boldsymbol{\nu}_T \,+\, \boldsymbol{\zeta}_{T',e}\cdot\boldsymbol{\nu}_{T'}|^2\,ds$$

$$\le\; 2\sum_{e\subseteq\mathcal{E}_I}\int_e \mathtt{h}\left(|\boldsymbol{\zeta}_{T,e}|^2 \,+\, |\boldsymbol{\zeta}_{T',e}|^2\right)\,ds \;\le\; 4\,\bar{C}_1\sum_{T\in\mathcal{T}_h} h_T\,\|\boldsymbol{\zeta}_T\|^2_{[L^2(\partial T)]^2}$$

and

$$\|\mathtt{h}^{1/2}\,\boldsymbol{\zeta}\cdot\boldsymbol{\nu}\|^2_{L^2(\mathcal{E}_D)} \;\le\; \sum_{e\subseteq\mathcal{E}_D}\int_e \mathtt{h}\,|\boldsymbol{\zeta}_{T,e}|^2\,ds \;\le\; \sum_{T\in\mathcal{T}_h} h_T\,\|\boldsymbol{\zeta}_T\|^2_{[L^2(\partial T)]^2}\,,$$

where the last inequality also makes use of the fact that $\mathtt{h} = h_T$ on $\mathcal{E}_D$. $\square$

Lemma 3.3.2 *There exists $C_{\mathtt{inv}} > 0$, depending only on k and the shape regularity of the mesh, such that for each $d \in \{1,2\}$ there holds*

$$\|q\|^2_{[L^2(\partial T)]^d} \;\le\; C_{\mathtt{inv}}\, h_T^{-1}\, \|q\|^2_{[L^2(T)]^d} \quad \forall\, q \in [\mathbf{P}_k(T)]^d\,, \quad \forall\, T \in \mathcal{T}_h\,. \tag{3.3.14}$$

Proof. It follows from Theorem 4.7.6 (equation (4.6.4)) in [94]. We omit details. □

The above lemmas allow to prove the following important result.

Lemma 3.3.3 *Let $S : V(h) \times \mathbf{\Sigma}_h \to \mathbf{R}$ be the bilinear form defined by*

$$S(v, \boldsymbol{\tau}_h) \;:=\; \int_{\mathcal{E}_I} (\{\boldsymbol{\tau}_h\} - \boldsymbol{\beta}[\![\boldsymbol{\tau}_h]\!]) \cdot [\![v]\!]\, ds + \int_{\mathcal{E}_D} v\, \boldsymbol{\tau}_h \cdot \boldsymbol{\nu}\, ds \quad \forall\, (v, \boldsymbol{\tau}_h) \in V(h) \times \mathbf{\Sigma}_h\,. \tag{3.3.15}$$

Then there exists $C_{\mathbf{S}} > 0$, independent of the meshsize, such that

$$|\, S(v, \boldsymbol{\tau}_h)\,| \;\le\; C_{\mathbf{S}}\, |v|_h\, \|\boldsymbol{\tau}_h\|_{[L^2(\Omega)]^2} \quad \forall\, (v, \boldsymbol{\tau}_h) \in V(h) \times \mathbf{\Sigma}_h\,. \tag{3.3.16}$$

Proof. Applying triangle and the Cauchy-Schwarz inequalities, and using that $\boldsymbol{\beta} \in [L^\infty(\mathcal{E}_I)]^2$, we find that

$$\begin{aligned}
|S(v, \boldsymbol{\tau}_h)| \;&\le\; \int_{\mathcal{E}_I} |\alpha^{-1/2}\{\boldsymbol{\tau}_h\}|\, |\alpha^{1/2}[\![v]\!]|\, ds \\
&+ \|\boldsymbol{\beta}\|_{[L^\infty(\mathcal{E}_I)]^2} \int_{\mathcal{E}_I} |\alpha^{-1/2}[\![\boldsymbol{\tau}_h]\!]|\, |\alpha^{1/2}[\![v]\!]|\, ds \;+\; \int_{\mathcal{E}_D} |\alpha^{-1/2}\boldsymbol{\tau}_h \cdot \boldsymbol{\nu}|\, |\alpha^{1/2} v|\, ds \\
&\le\; \|\alpha^{-1/2}\{\boldsymbol{\tau}_h\}\|_{[L^2(\mathcal{E}_I)]^2}\, \|\alpha^{1/2}[\![v]\!]\|_{[L^2(\mathcal{E}_I)]^2} \\
&+ \widehat{\beta}\, \|\alpha^{-1/2}[\![\boldsymbol{\tau}_h]\!]\|_{[L^2(\mathcal{E}_I)]^2}\, \|\alpha^{1/2}[\![v]\!]\|_{[L^2(\mathcal{E}_I)]^2} \;+\; \|\alpha^{-1/2}\boldsymbol{\tau}_h \cdot \boldsymbol{\nu}\|_{L^2(\mathcal{E}_D)}\, \|\alpha^{1/2} v\|_{L^2(\mathcal{E}_D)}\,,
\end{aligned}$$

which yields

$$\begin{aligned}
|S(v, \boldsymbol{\tau}_h)| \;\le\; \bar{C}\, |v|_h \Bigg(\|\alpha^{-1/2}\{\boldsymbol{\tau}_h\}\|^2_{[L^2(\mathcal{E}_I)]^2} \\
+ \|\alpha^{-1/2}[\![\boldsymbol{\tau}_h]\!]\|^2_{[L^2(\mathcal{E}_I)]^2} + \|\alpha^{-1/2}\boldsymbol{\tau}_h \cdot \boldsymbol{\nu}\|^2_{L^2(\mathcal{E}_D)} \Bigg)^{1/2}
\end{aligned} \tag{3.3.17}$$

with $\bar{C} := \Big(2 \max\Big\{ 1, \widehat{\beta}^2 \Big\} \Big)^{1/2}$.

Next, we bound the remaining terms on the right hand side of (3.3.17). First, using the definition of α (cf. (3.3.10)) and part i) of Lemma 3.3.1, we deduce that

$$\|\alpha^{-1/2}\, \{\boldsymbol{\tau}_h\}\|^2_{[L^2(\mathcal{E}_I)]^2} \;=\; \widehat{\alpha}^{-1}\, \|\mathtt{h}^{1/2}\, \{\boldsymbol{\tau}_h\}\|^2_{[L^2(\mathcal{E}_I)]^2} \;\le\; \widehat{\alpha}^{-1}\, \bar{C}_1 \sum_{T \in \mathcal{T}_h} h_T\, \|\boldsymbol{\tau}_{h,T}\|^2_{[L^2(\partial T)]^2}\,,$$

which, applying Lemma 3.3.2, gives

$$\|\alpha^{-1/2}\{\boldsymbol{\tau}_h\}\|^2_{[L^2(\mathcal{E}_I)]^2} \;\le\; \widehat{\alpha}^{-1}\,\bar{C}_1\,C_{\mathtt{inv}}\,\|\boldsymbol{\tau}_h\|^2_{[L^2(\Omega_1)]^2}\,. \tag{3.3.18}$$

For the second and third terms we proceed similarly. Indeed, we apply now parts ii) and iii) of Lemma 3.3.1 and Lemma 3.3.2, and obtain, respectively,

$$\|\alpha^{-1/2}\,[\![\boldsymbol{\tau}_h]\!]\|^2_{[L^2(\mathcal{E}_I)]^2} \;\le\; \widehat{\alpha}^{-1}\,\bar{C}_2\,C_{\mathtt{inv}}\,\|\boldsymbol{\tau}_h\|^2_{[L^2(\Omega_1)]^2}\,, \tag{3.3.19}$$

and

$$\|\alpha^{-1/2}\,\boldsymbol{\tau}_h\cdot\boldsymbol{\nu}\|^2_{L^2(\mathcal{E}_D)} \;\le\; \widehat{\alpha}^{-1}\,\bar{C}_3\,C_{\mathtt{inv}}\,\|\boldsymbol{\tau}_h\|^2_{[L^2(\Omega_1)]^2}\,. \tag{3.3.20}$$

In this way, replacing (3.3.18), (3.3.19), and (3.3.20), back into (3.3.17), we derive (3.3.16) and complete the proof. □

LEMMA 3.3.4 *Let* $\mathbf{G}:\boldsymbol{\Sigma}_h\to\mathbf{R}$ *be the linear functional defined by*

$$\mathbf{G}(\boldsymbol{\tau}_h) \;:=\; \int_{\mathcal{E}_D} g_D\,\boldsymbol{\tau}_h\cdot\boldsymbol{\nu}\,ds \qquad \forall\,\boldsymbol{\tau}_h\in\boldsymbol{\Sigma}_h\,.$$

Then $\mathbf{G}$ *is bounded.*

PROOF. Given $\boldsymbol{\tau}_h\in\boldsymbol{\Sigma}_h$, the Cauchy-Schwarz inequality and (3.3.20) imply that

$$\begin{aligned}|\mathbf{G}(\boldsymbol{\tau}_h)| \;&\le\; \int_{\mathcal{E}_D}|\alpha^{1/2}g_D|\,|\alpha^{-1/2}\boldsymbol{\tau}_h\cdot\boldsymbol{\nu}|\,ds \;\le\; \|\alpha^{1/2}g_D\|_{L^2(\mathcal{E}_D)}\;\|\alpha^{-1/2}\boldsymbol{\tau}_h\cdot\boldsymbol{\nu}\|_{L^2(\mathcal{E}_D)}\\ &\le\; C\;\|\alpha^{1/2}g_D\|_{L^2(\mathcal{E}_D)}\,\|\boldsymbol{\tau}_h\|_{[L^2(\Omega)]^2}\,,\end{aligned}$$

which proves that $\mathbf{G}\in\Sigma_h'$. □

We now let $\mathbf{S}:V(h)\to\boldsymbol{\Sigma}_h$ be the linear and bounded operator induced by the bilinear form S, that is, given $v\in V(h)$, $\mathbf{S}(v)$ is the unique element in $\boldsymbol{\Sigma}_h$ (guaranteed by the Riesz representation Theorem) such that

$$\int_\Omega \mathbf{S}(v)\cdot\boldsymbol{\tau}_h\,dx \;=\; S(v,\boldsymbol{\tau}_h) \qquad \forall\,\boldsymbol{\tau}_h\in\boldsymbol{\Sigma}_h\,. \tag{3.3.21}$$

We remark, according to (3.3.16) (cf. Lemma 3.3.3), that

$$\|\,\mathbf{S}(v)\,\|_{[L^2(\Omega)]^2} \;\le\; C_{\mathbf{S}}\;|v|_h \qquad \forall\,v\in V(h)\,. \tag{3.3.22}$$

Analogously, in virtue again of the Riesz Theorem, we let $\mathcal{G}_D$ be the unique element in $\boldsymbol{\Sigma}_h$ such that

$$\int_\Omega \mathcal{G}_D\cdot\boldsymbol{\tau}_h\,dx \;=\; \mathbf{G}(\boldsymbol{\tau}_h) \qquad \forall\,\boldsymbol{\tau}_h\in\boldsymbol{\Sigma}_h\,.$$

It is important to observe here that for the solution u of (3.1.1) there holds $\mathbf{S}(u) = \mathcal{G}_D$. In fact, since $u \in H^{1+\epsilon}(\Omega)$, the Sobolev imbedding Theorem implies that $u \in C(\Omega)$ and hence $[\![u]\!] = 0$ on any interior edge of $\mathcal{T}_h$. Thus, the stated result follows from the definitions of $\mathbf{S}$ and $\mathcal{G}_D$.

We are ready now to express the unknowns $\boldsymbol{\sigma}_h$ and $\boldsymbol{\theta}_h$ in terms of u_h. Indeed, it follows from equation (3.3.5) that

$$\boldsymbol{\theta}_h = \Pi_{\boldsymbol{\Sigma}_h}\Big(\nabla_h u_h - \mathbf{S}(u_h) + \mathcal{G}_D\Big)\,, \tag{3.3.23}$$

where $\Pi_{\boldsymbol{\Sigma}_h}$ denotes the $[L^2(\Omega)]^2$-projection onto $\boldsymbol{\Sigma}_h$. But, since $\nabla_h u_h$, $\mathbf{S}(u_h)$, and $\mathcal{G}_D$ belong to $\boldsymbol{\Sigma}_h$, (3.3.23) reduces simply to

$$\boldsymbol{\theta}_h \;=\; \nabla_h u_h \;-\; \mathbf{S}(u_h) \;+\; \mathcal{G}_D\,. \tag{3.3.24}$$

Now, concerning $\boldsymbol{\sigma}_h$, we deduce from (3.3.2) and (3.3.24) that

$$\boldsymbol{\sigma}_h = \Pi_{\boldsymbol{\Sigma}_h}\Big(\mathbf{a}(\cdot,\boldsymbol{\theta}_h)\Big) = \Pi_{\boldsymbol{\Sigma}_h}\Big(\mathbf{a}(\cdot,\nabla_h u_h - \mathbf{S}(u_h) + \mathcal{G}_D)\Big)\,. \tag{3.3.25}$$

Therefore, replacing (3.3.24) and (3.3.25) into (3.3.6), we arrive to the following primal formulation: *Find* $u_h \in V_h$ *such that*

$$[A_h(u_h), v_h] \;=\; [F_h, v_h] \qquad \forall\, v_h \in V_h\,, \tag{3.3.26}$$

where the nonlinear operator $A_h : V(h) \to V(h)'$ and the functional $F_h : V(h) \to \mathbf{R}$, are defined, respectively, by

$$\begin{aligned} [A_h(w), v] \;:=\; & \int_\Omega \mathbf{a}(\cdot,\nabla_h w - \mathbf{S}(w) + \mathcal{G}_D)\cdot(\nabla_h v - \mathbf{S}(v))\,dx \\ +\; & \int_{\mathcal{E}_I} \alpha\,[\![w]\!]\cdot[\![v]\!]\,ds \;+\; \int_{\mathcal{E}_D} \alpha\,w\,v\,ds \qquad \forall\, w,\, v \in V(h)\,, \end{aligned} \tag{3.3.27}$$

and

$$[F_h, v] \;:=\; \int_\Omega f\,v\,dx + \int_{\mathcal{E}_D} \alpha\,g_D\,v\,ds + \int_{\mathcal{E}_N} g_N\,v\,ds \qquad \forall\, v \in V(h)\,. \tag{3.3.28}$$

Hereafter, $[\cdot,\cdot]$ denotes the duality pairing induced by the operators and functionals used in each case.

We remark that for *discrete* trial and test functions, the formulation (3.3.26) (together with the identities (3.3.24) and (3.3.25)) is equivalent to the original flux form of the LDG method given by (3.3.7) (or (3.3.8)). Further, we observe that this formulation has some similarities with the one described in [15] for an exterior transmission problem.

Also, it is important to emphasize, due to the discrete nature of the operator $\mathbf{S}$ (cf. (3.3.21)), that the formulation (3.3.26) is no longer consistent in the sense that it does not hold for the exact solution u of (3.1.1).

3.4 Solvability and stability of the primal formulation

In order to prove the unique solvability of (3.3.26), we need to state some properties of A_h and F_h. To this end, we recall that the assumptions (H.3) and (H.4) on the coefficients a_i imply that the nonlinear operator induced by $\mathbf{a}$ is strongly monotone and Lipschitz continuous on $[L^2(\Omega_1)]^2$ (see, e.g. [57] or [59]). This means that there exist C_0, $C_1 > 0$ such that for all $\boldsymbol{\zeta}$, $\tilde{\boldsymbol{\zeta}} \in [L^2(\Omega_1)]^2$ there hold

$$\int_\Omega \Big[\mathbf{a}(\cdot,\boldsymbol{\zeta}) - \mathbf{a}(\cdot,\tilde{\boldsymbol{\zeta}}) \Big] \cdot (\boldsymbol{\zeta} - \tilde{\boldsymbol{\zeta}})\, dx \;\geq\; C_0\, \|\boldsymbol{\zeta} - \tilde{\boldsymbol{\zeta}}\|^2_{[L^2(\Omega_1)]^2}\,, \tag{3.4.1}$$

and

$$\|\mathbf{a}(\cdot,\boldsymbol{\zeta}) - \mathbf{a}(\cdot,\tilde{\boldsymbol{\zeta}})\|_{[L^2(\Omega_1)]^2} \;\leq\; C_1\, \|\boldsymbol{\zeta} - \tilde{\boldsymbol{\zeta}}\|_{[L^2(\Omega_1)]^2}\,. \tag{3.4.2}$$

Then, the following lemma shows that the nonlinear operator A_h is also Lipschitz continuous.

LEMMA 3.4.1 *There exists $C_{\mathtt{LC}} > 0$, independent of the meshsize, such that*

$$\|A_h(w) - A_h(z)\|_{V(h)'} \;\leq\; C_{\mathtt{LC}}\, |||w - z|||_h \qquad \forall\, w,\, z \,\in\, V(h)\,.$$

PROOF. We show first that the linear part of A_h is bounded. In fact, given w, $v \in V(h)$, we apply the Cauchy-Schwarz inequality and obtain

$$\left| \int_{\mathcal{E}_I} \alpha\, [\![w]\!] [\![v]\!]\, ds + \int_{\mathcal{E}_D} \alpha\, w\, v\, ds \right| \;\leq\; \Big(\int_{\mathcal{E}_I} |\alpha^{1/2} [\![w]\!]|\, |\alpha^{1/2} [\![v]\!]|\, ds + \int_{\mathcal{E}_D} |\alpha^{1/2}\, w|\, |\alpha^{1/2}\, v|\, ds \Big)$$

$$\leq\; \Big(\|\alpha^{1/2} [\![w]\!]\|^2_{[L^2(\mathcal{E}_I)]^2} \,+\, \|\alpha^{1/2}\, w\|^2_{L^2(\mathcal{E}_D)} \Big)^{1/2} \Big(\|\alpha^{1/2} [\![v]\!]\|^2_{[L^2(\mathcal{E}_I)]^2} \,+\, \|\alpha^{1/2}\, v\|^2_{L^2(\mathcal{E}_D)} \Big)^{1/2},$$

which, by virtue of the definition of $|||\cdot|||_h$ (cf. (3.3.12)), yields

$$\left| \int_{\mathcal{E}_I} \alpha\, [\![w]\!] [\![v]\!]\, ds + \int_{\mathcal{E}_D} \alpha\, w\, v\, ds \right| \;\leq\; |||w|||_h\; |||v|||_h \quad \forall\, w,\, v \,\in\, V(h)\,. \tag{3.4.3}$$

Then, according to the definition of A_h (cf. (3.3.27)), and applying (3.4.3), (3.4.2), the Cauchy-Schwarz inequality, and the boundedness of $\mathbf{S}$ (cf. (3.3.22)), we deduce that

$$
\begin{aligned}
|[A_h(w)-A_h(z),v]| \;\le\; & \int_\Omega |\mathbf{a}(\cdot,\nabla_h w-\mathbf{S}(w)+\mathcal{G}_D)-\mathbf{a}(\cdot,\nabla_h z-\mathbf{S}(z)+\mathcal{G}_D)|\;|\nabla_h v-\mathbf{S}(v)|\,dx \\
& + \left|\int_{\mathcal{E}_I}\alpha\,[\![w-z]\!][\![v]\!]\,ds + \int_{\mathcal{E}_D}\alpha\,(w-z)\,v\,ds\right| \\
\le\; & C_1\,\|\nabla_h(w-z)-\mathbf{S}(w-z)\|_{[L^2(\Omega_1)]^2}\,\|\nabla_h v-\mathbf{S}(v)\|_{[L^2(\Omega_1)]^2} \;+\; |||w-z|||_h\,|||v|||_h \\
\le\; & C_1\Big(\|\nabla_h(w-z)\|_{[L^2(\Omega_1)]^2} + C_{\mathbf{S}}\,|||w-z|||_h\Big)\Big(\|\nabla_h v\|_{[L^2(\Omega_1)]^2} + C_{\mathbf{S}}\,|||v|||_h\Big) \\
& + |||w-z|||_h\,|||v|||_h\,,
\end{aligned}
$$

and hence, taking $C_{\mathtt{LC}} := 2\,\max\{1, C_1\,(1+C_{\mathbf{S}})^2\}$, we can write

$$
|[A_h(w)-A_h(z),v]| \;\le\; C_{\mathtt{LC}}\;|||w-z|||_h\;|||v|||_h \qquad \forall\, w,\,z,\,v\,\in\,V(h)\,,
$$

which implies the required estimate and ends the proof. $\square$

The strong monotonicity of A_h is proved next.

Lemma 3.4.2 *There exists $C_{\mathtt{SM}} > 0$, independent of the meshsize, such that*

$$
[A_h(w), w-v] - [A_h(v), w-v] \;\ge\; C_{\mathtt{SM}}\;|||w-v|||_h^2 \qquad \forall\, w,\,v\,\in\,V(h)\,.
$$

Proof. Let $w,\,v \in V(h)$. According to the definitions of A_h and $|\cdot|_h$ (cf. (3.3.27) and (3.3.13)), and applying (3.4.1), we obtain

$$
\begin{aligned}
[A_h(w),w-v]-[A_h(v),w-v] \;=\; & \int_\Omega \mathbf{a}(\cdot,\nabla_h w-\mathbf{S}(w)+\mathcal{G}_D)\cdot(\nabla_h(w-v)-\mathbf{S}(w-v))\,dx \\
& - \int_\Omega \mathbf{a}(\cdot,\nabla_h v-\mathbf{S}(v)+\mathcal{G}_D)\cdot(\nabla_h(w-v)-\mathbf{S}(w-v))\,dx \;+\; |\,w-v\,|_h^2 \\
\ge\; & C_0\,\|\nabla_h(w-v)-\mathbf{S}(w-v)\|_{[L^2(\Omega_1)]^2}^2 \;+\; |\,w-v\,|_h^2\,.
\end{aligned}
\tag{3.4.4}
$$

Then, applying the generalized Cauchy-Schwarz inequality with $\delta\, \in]0,1[$, and using again the boundedness of $\mathbf{S}$ (cf. (3.3.22)), we deduce that

$$
\|\nabla_h(w-v)-\mathbf{S}(w-v)\|_{[L^2(\Omega_1)]^2}^2 \;\ge\; (1-\delta)\,\|\nabla_h(w-v)\|_{[L^2(\Omega_1)]^2}^2 - C_{\mathbf{S}}^2\left(\frac{1}{\delta}-1\right)\,|w-v|_h^2\,.
\tag{3.4.5}
$$

Thus, it follows from (3.4.4) and (3.4.5) that

$$
[A_h(w), w-v] - [A_h(v), w-v] \;\ge\; C_1(\delta)\;\|\nabla_h(w-v)\|_{[L^2(\Omega)]^2}^2 \;+\; C_2(\delta)\,|w-v|_h^2\,,
$$

where $C_1(\delta) \,:=\, C_0\,(1-\delta)$ and $C_2(\delta) \,:=\, 1 \,-\, C_0\,C_{\mathbf{S}}^2\,\left(\dfrac{1}{\delta}-1\right)$.

We realize here that for any given $\delta_0 \,\in\, \left]\dfrac{C_0\,C_{\mathbf{S}}^2}{1+C_0\,C_{\mathbf{S}}^2}, 1\right[$ there hold $C_1(\delta_0)$, $C_2(\delta_0) \,>\, 0$, whence it suffices to take $C_{\mathtt{SM}} \,:=\, \min\{C_1(\delta_0), C_2(\delta_0)\}$ to complete the proof. $\square$

The following lemma is needed to prove the boundedness of the linear functional F_h with respect to the norm $|||\cdot|||_h$ (cf. (3.3.12)).

LEMMA 3.4.3 *There exists a constant $C_{\mathbf{L}} > 0$, independent of the meshsize, such that*

$$\|v\|_{L^2(\Omega)} \,\leq\, C_{\mathbf{L}}\,|||v|||_h \qquad \forall\, v \,\in\, V(h)\,.$$

PROOF. We proceed similarly as in [6] (see also [37]). To this end we first let ψ be a fixed function in $H_{00}^{1/2}(\Gamma_D)$ such that $\displaystyle\int_{\Gamma_D} \psi\,ds = 1$. We recall here that $H_{00}^{1/2}(\Gamma_D)$ is the space of functions in $H^{1/2}(\Gamma_D)$ whose extensions by zero to Γ_N belong to $H^{1/2}(\Gamma)$. Then, given $v \in V(h)$ we define the constant $c_v \,:=\, \displaystyle\int_\Omega v\,dx$ and let $\varphi \in H^1(\Omega)$ be the unique solution of the boundary value problem

$$-\Delta\varphi = v \quad \text{in} \quad \Omega\,, \qquad \frac{\partial\varphi}{\partial\boldsymbol{\nu}} \,=\, -\,c_v\,\psi \quad \text{on} \quad \Gamma\,, \quad \text{and} \quad \int_\Omega \varphi\,dx \,=\, 0\,.$$

Since Ω is convex, $v \,\in\, L^2(\Omega)$, and $c_v\,\psi \,\in\, H^{1/2}(\Gamma)$, we deduce (see [65]) that $\varphi \in H^2(\Omega)$ and $\|\varphi\|_{H^2(\Omega)} \,\leq\, C_\omega\,\|v\|_{L^2(\Omega)}$, with a constant $C_\omega > 0$ that depends only on Ω and the function ψ.

Thus, using Gauss' formula on each $T \in \mathcal{T}_h$, and the fact that $\dfrac{\partial\varphi}{\partial\boldsymbol{\nu}} = 0$ on Γ_N, we obtain

$$\|v\|^2_{L^2(\Omega)} \,=\, \sum_{T\in\mathcal{T}_h}\left\{-\int_T v\Delta\varphi\,dx\right\} \,=\, \sum_{T\in\mathcal{T}_h}\left\{\int_T \nabla v\cdot\nabla\varphi\,dx \,-\, \int_{\partial T} v\,\frac{\partial\varphi}{\partial\boldsymbol{\nu}_T}\,ds\right\}$$

$$=\, \int_\Omega \nabla_h v\cdot\nabla\varphi\,dx \,-\, \sum_{e\subseteq\mathcal{E}_I}\int_e ([\![v]\!]\cdot\boldsymbol{\nu}_e)\,\frac{\partial\varphi}{\partial\boldsymbol{\nu}_e}\,ds \,-\, \sum_{e\subseteq\mathcal{E}_D}\int_e v\,\frac{\partial\varphi}{\partial\boldsymbol{\nu}_e}\,ds\,,$$

where $\boldsymbol{\nu}_e \,:=\, \boldsymbol{\nu}_T$, with T being one of the elements (or the unique element, if $e \subseteq \mathcal{E}_D$) such that $e \subseteq \partial T$.

Applying now the Cauchy-Schwarz inequality, we find that

$$\|v\|^2_{L^2(\Omega)} \,\leq\, \|\nabla_h v\|_{[L^2(\Omega)]^2}\,\|\nabla\varphi\|_{[L^2(\Omega)]^2}$$

$$+ \sum_{e\subseteq\mathcal{E}_I} \|[\![v]\!]\|_{[L^2(e)]^2} \left\|\frac{\partial\varphi}{\partial\boldsymbol{\nu}_e}\right\|_{L^2(e)} + \sum_{e\subseteq\mathcal{E}_D} \|v\|_{L^2(e)} \left\|\frac{\partial\varphi}{\partial\boldsymbol{\nu}_e}\right\|_{L^2(e)}$$

$$\leq \left(\|\nabla_h v\|^2_{[L^2(\Omega)]^2} + \sum_{e\subseteq\mathcal{E}_I} h_e^{-1}\|[\![v]\!]\|^2_{[L^2(e)]^2} + \sum_{e\subseteq\mathcal{E}_D} h_e^{-1}\|v\|^2_{L^2(e)}\right)^{1/2} \mathcal{R}(\varphi)\,, \qquad (3.4.6)$$

where

$$\mathcal{R}(\varphi) := \left(\|\nabla\varphi\|^2_{[L^2(\Omega)]^2} + \sum_{e\subseteq\mathcal{E}_I\cup\mathcal{E}_D} h_e \left\|\frac{\partial\varphi}{\partial\boldsymbol{\nu}_e}\right\|^2_{L^2(e)}\right)^{1/2}. \qquad (3.4.7)$$

On the other hand, a trace inequality given in Theorem 3.10 of [1] (see also equation (2.4) in [6]) states that for each $w \in H^1(T)$ and for any $e \subseteq \partial T$ there holds

$$\|w\|^2_{L^2(e)} \leq C_{\mathtt{tr}} \left(h_e^{-1}\|w\|^2_{L^2(T)} + h_e\,|w|^2_{H^1(T)}\right) \qquad (3.4.8)$$

with a constant $C_{\mathtt{tr}} > 0$, independent of h_T. Thus, applying (3.4.8) we deduce that there exists $\bar{C}_{\mathtt{tr}} > 0$, independent of $\varphi \in H^2(\Omega)$ and the meshsize, such that

$$\left\|\frac{\partial\varphi}{\partial\boldsymbol{\nu}_e}\right\|^2_{L^2(e)} \leq \bar{C}_{\mathtt{tr}} \left(h_e^{-1}|\varphi|^2_{H^1(T)} + h_e\,|\varphi|^2_{H^2(T)}\right),$$

and hence, with $\tilde{C}_{\mathtt{tr}} := \bar{C}_{\mathtt{tr}} \max\{1, \mathtt{diam}(\Omega)^2\}$, we get

$$h_e \left\|\frac{\partial\varphi}{\partial\boldsymbol{\nu}_e}\right\|^2_{L^2(e)} \leq \tilde{C}_{\mathtt{tr}}\,\|\varphi\|^2_{H^2(T)} \qquad \forall\, e \in \mathcal{E}_I \cup \mathcal{E}_D\,. \qquad (3.4.9)$$

It follows from (3.4.7), (3.4.9), and the regularity bound for $\|\varphi\|_{H^2(\Omega)}$ that

$$\mathcal{R}(\varphi) \leq \left(1 + 2\,\tilde{C}_{\mathtt{tr}}\right)^{1/2} \|\varphi\|_{H^2(\Omega)} \leq \tilde{C}_\omega\,\|v\|_{L^2(\Omega)}\,,$$

with $\tilde{C}_\omega := C_\omega \left(1 + 2\,\tilde{C}_{\mathtt{tr}}\right)^{1/2}$. Thus, replacing this inequality back into (3.4.6), we arrive to

$$\|v\|_{L^2(\Omega)} \leq \tilde{C}_\omega \left(\|\nabla_h v\|^2_{[L^2(\Omega)]^2} + \sum_{e\subseteq\mathcal{E}_I} h_e^{-1}\|[\![v]\!]\|^2_{[L^2(e)]^2} + \sum_{e\subseteq\mathcal{E}_D} h_e^{-1}\|v\|^2_{L^2(e)}\right)^{1/2}. \qquad (3.4.10)$$

Next, since the regular triangulation $\mathcal{T}_h$ is of *bounded variation*, one can prove that there exists a constant $\widehat{l} > 0$, independent of the meshsize, such that $h_e^{-1} \leq \widehat{l}\,h_T^{-1}$ for all $T \in \mathcal{T}_h$ and for each edge $e \subseteq \partial T$. Then, the definitions of $\mathtt{h}$ and α (cf. (3.3.9) and (3.3.10)) imply that $h_e^{-1} \leq \widehat{l}\,h_T^{-1} \leq \widehat{l}\,\mathtt{h}^{-1} = \dfrac{\widehat{l}}{\widehat{\alpha}}\,\alpha$, whence (3.4.10) leads to $\|v\|_{L^2(\Omega)} \leq C_{\mathbf{L}}\,|||v|||_h$, with the constant $C_{\mathbf{L}} := \tilde{C}_\omega \left(\max\left\{1, \dfrac{\widehat{l}}{\widehat{\alpha}}\right\}\right)^{1/2}$. □

The boundedness of the functional F_h is proved in the following lemma.

Lemma 3.4.4 *There exists $C_{\mathbf{F}} > 0$, depending on $\widehat{\alpha}$, l, $\widehat{l}$, and k, but independent of the meshsize, such that*

$$|[F_h, v_h]| \,\leq\, C_{\mathbf{F}}\, \mathcal{B}(f, g_D, g_N)\; |||v_h|||_h \quad \forall\, v_h \in V_h\,, \tag{3.4.11}$$

where

$$\mathcal{B}(f, g_D, g_N) \,:=\, \Big(\|f\|^2_{L^2(\Omega)} \,+\, \|\alpha^{1/2} g_D\|^2_{L^2(\mathcal{E}_D)} \,+\, \|\alpha^{1/2} g_N\|^2_{L^2(\mathcal{E}_N)} \Big)^{1/2}.$$

Proof. Let $v_h \in V_h$. Then, (3.3.28) and the Cauchy-Schwarz inequality yield

$$\begin{aligned} |[F_h, v_h]| \,&\leq\, \|f\|_{L^2(\Omega)}\, \|v_h\|_{L^2(\Omega)} \\ &+\, \|\alpha^{1/2} g_D\|_{L^2(\mathcal{E}_D)}\, \|\alpha^{1/2} v_h\|_{L^2(\mathcal{E}_D)} \,+\, \|\alpha^{1/2} g_N\|_{L^2(\mathcal{E}_N)}\, \|\alpha^{-1/2} v_h\|_{L^2(\mathcal{E}_N)}\,. \end{aligned} \tag{3.4.12}$$

Now, using the definitions of $\mathtt{h}$ and α (cf. (3.3.9) and (3.3.10)), we obtain

$$\|\alpha^{-1/2} v_h\|^2_{L^2(\mathcal{E}_N)} \,=\, \sum_{e \subseteq \mathcal{E}_N} \frac{1}{\widehat{\alpha}} \int_e \mathtt{h}\, v_h^2\, ds \,=\, \frac{1}{\widehat{\alpha}} \sum_{e \subseteq \mathcal{E}_N} h_T \int_e v_h^2\, ds \,\leq\, \frac{1}{\widehat{\alpha}} \sum_{T \in \mathcal{T}_h} h_T\, \|v_h\|^2_{L^2(\partial T)}\,,$$

which, according to the inverse inequality (3.3.14) (cf. Lemma 3.3.2), implies that

$$\|\alpha^{-1/2} v_h\|^2_{L^2(\mathcal{E}_N)} \,\leq\, \frac{1}{\widehat{\alpha}}\, C_{\mathtt{inv}}\, \|v_h\|^2_{L^2(\Omega)}\,. \tag{3.4.13}$$

Then, applying again the Cauchy-Schwarz inequality in (3.4.12), and using (3.4.13), the fact that $\|\alpha^{1/2} v_h\|_{L^2(\mathcal{E}_D)} \,\leq\, |||v_h|||_h$, and Lemma 3.4.3, we deduce that

$$|[F_h, v_h]| \,\leq\, \mathcal{B}(f, g_D, g_N)\; \Big\{ \|v_h\|^2_{L^2(\Omega)} \,+\, \|\alpha^{1/2} v_h\|^2_{L^2(\mathcal{E}_D)} \,+\, \|\alpha^{-1/2} v_h\|^2_{L^2(\mathcal{E}_N)} \Big\}^{1/2}$$

$$\leq\, \mathcal{B}(f, g_D, g_N)\; \Big\{ \Big(1 + \frac{1}{\widehat{\alpha}}\, C_{\mathtt{inv}}\Big)\, \|v_h\|^2_{L^2(\Omega)} \,+\, |||v_h|||^2_h \Big\}^{1/2} \,\leq\, C_{\mathbf{F}}\, \mathcal{B}(f, g_D, g_N)\; |||v_h|||_h\,,$$

with $C_{\mathbf{F}} \,:=\, \Big(1 + C^2_{\mathbf{L}}\, \Big(1 + \frac{1}{\widehat{\alpha}}\, C_{\mathtt{inv}}\Big)\Big)^{1/2}$. □

We now establish the unique solvability of the primal formulation (3.3.26) and derive an associated Strang type error estimate. We recall that $C_{\mathtt{LC}}$ and $C_{\mathtt{SM}}$ are the constants (independent of the meshsize) providing the Lipschitz continuity and strong monotonicity of the nonlinear operator A_h (cf. Lemmas 3.4.1 and 3.4.2).

Theorem 3.4.1 *There exists a unique $u_h \in V_h$ solution of* (3.3.26)*, which satisfies*

$$|||u_h|||_h \,\leq\, \frac{1}{C_{\mathtt{SM}}}\, \Big\{ C_{\mathbf{F}}\, \mathcal{B}(f, g_D, g_N) \,+\, \|A_h(0)\|_{V_h'} \Big\}\,. \tag{3.4.14}$$

Moreover, the following error estimate holds

$$|||u-u_h|||_h \,\leq\, \left(1+\frac{C_{\mathtt{LC}}}{C_{\mathtt{SM}}}\right)\inf_{v_h\in V_h}|||u-v_h|||_h \,+\, \frac{1}{C_{\mathtt{SM}}}\sup_{\substack{w_h\in V_h\\ w_h\neq 0}}\frac{|[A_h(u),w_h]-[F_h,w_h]|}{|||w_h|||_h}\,, \tag{3.4.15}$$

where u is the exact solution of (3.1.1).

Proof. According to Lemmas 3.4.1 and 3.4.2, the unique solvability of (3.3.26) follows from a well known result in nonlinear functional analysis (see, e.g. Theorem 3.3.23 in Chapter III of [79], or Theorem 35.4 in [98]). Moreover, applying the strong monotonicity of A_h (cf. Lemma 3.4.2), (3.3.26), and Lemma 3.4.4, we get

$$C_{\mathtt{SM}}\,|||u_h|||_h^2 \,\leq\, [A_h(u_h)-A_h(0),u_h] \,=\, [F_h-A_h(0),u_h]$$

$$\leq\, \Big(\|F_h\|_{V_h'} \,+\, \|A_h(0)\|_{V_h'}\Big)\;|||u_h|||_h \,\leq\, \Big(C_{\mathbf{F}}\;\mathcal{B}(f,g_D,g_N) \,+\, \|A_h(0)\|_{V_h'}\Big)\;|||u_h|||_h\,,$$

which yields (3.4.14).

For the error estimate, we first observe by triangle inequality that

$$|||u-u_h|||_h \,\leq\, |||u-v_h|||_h \,+\, |||v_h-u_h|||_h \quad \forall\, v_h\in V_h\,. \tag{3.4.16}$$

Then, we use again Lemma 3.4.2 and (3.3.26) to deduce that

$$C_{\mathtt{SM}}\,|||v_h-u_h|||_h^2 \,\leq\, [A_h(v_h)\,-\,A_h(u_h),v_h-u_h]$$

$$=[A_h(v_h)\,-\,A_h(u),v_h-u_h] \,+\, \Big([A_h(u),v_h-u_h]\,-\,[F_h,v_h-u_h]\Big)\,,$$

which, applying the Lipschitz continuity of A_h (cf. Lemma 3.4.1), gives

$$C_{\mathtt{SM}}\,|||v_h-u_h|||_h^2 \,\leq\, C_{\mathtt{LC}}\,|||v_h-u|||_h\;|||v_h-u_h|||_h + |\,[A_h(u),v_h-u_h]-[F_h,v_h-u_h]\,|\,,$$

and hence

$$|||v_h-u_h|||_h \,\leq\, \frac{C_{\mathtt{LC}}}{C_{\mathtt{SM}}}\,|||u-v_h|||_h + \frac{1}{C_{\mathtt{SM}}}\sup_{\substack{w_h\in V_h\\ w_h\neq 0}}\frac{|\,[A_h(u),w_h]-[F_h,w_h]\,|}{|||w_h|||_h} \qquad \forall\, v_h\,\in\, V_h\,.$$

This inequality and (3.4.16) yield (3.4.15) and complete the proof. □

3.5 A-priori error estimates

We first recall that the assumed regularity on the solution u of (3.1.1) yields $[\![u]\!] = 0$ on any interior edge of $\mathcal{T}_h$, and that $\mathbf{S}(u) = \mathcal{G}_D$. In addition, (H.1) and (H.2) imply that $\mathbf{a}(\cdot, \nabla u(\cdot)) \in [L^2(\Omega)]^2$, and since $-\operatorname{div} \mathbf{a}(\cdot, \nabla u(\cdot)) = f$ in Ω, with $f \in L^2(\Omega)$, we deduce that $\mathbf{a}(\cdot, \nabla u(\cdot)) \in H(\operatorname{div}; \Omega_1)$, whence $[\![\mathbf{a}(\cdot, \nabla u(\cdot))]\!] = 0$ on each $e \subseteq \mathcal{E}_I$.

3.5.1 Energy norm error estimates

We begin this section with a well known approximation result (see [39] and [60] for more details), which will be used later on. Hereafter, $\mathbf{I}$ stands for a generic identity operator.

Lemma 3.5.1 *Let $\mathcal{T}_h$ be a regular triangulation, and let $s \geq 0$ and $m \in \{0, 1\}$. Given $T \in \mathcal{T}_h$, let $\Pi_T^k : H^{k+1}(T) \to \mathbf{P}_k(T)$ be the linear and bounded operator given by the $L^2(T)$-orthogonal projection, which satisfies $\Pi_T^k(p) = p$ for all $p \in \mathbf{P}_k(T)$. Then there exists $C_{\mathtt{ort}} > 0$, independent of the meshsize, such that*

$$| (\mathbf{I} - \Pi_T^k)(w) |_{H^m(T)} \leq C_{\mathtt{ort}}\, h_T^{s+1-m}\, |w|_{H^{s+1}(T)} \quad \forall\, w \in H^{s+1}(T)\,, \tag{3.5.1}$$

and

$$\| (\mathbf{I} - \Pi_T^k)(w) \|_{L^2(\partial T)} \leq C_{\mathtt{ort}}\, h_T^{s+1/2}\, |w|_{H^{s+1}(T)} \quad \forall\, w \in H^{s+1}(T)\,. \tag{3.5.2}$$

Proof. For integer s, the estimate (3.5.1) follows from Bramble-Hilbert's Lemma (see Theorem 3.1.4 in [39] or Theorem 4.4-2 in [86]), while (3.5.2) is consequence of a trace inequality (see Theorem 3.10 of [1] or equation (2.4) in [6]) and (3.5.1). In the case of a non-integer s, we refer to [60] for the corresponding proof. □

A bound for the consistency term in the Strang type error estimate (3.4.15) (cf. Theorem 3.4.1) is provided next. To this end we introduce the *discontinuous* global operator $\Pi_h^k : H^1(\Omega) \to V_h$, where, given $w \in H^1(\Omega)$, $\Pi_h^k(w)$ is the unique element in V_h such that $\Pi_h^k(w)|_T = \Pi_T^k(w|_T)$ for all $T \in \mathcal{T}_h$.

Lemma 3.5.2 *Let $s \in \mathbf{N} \cup \{0\}$ such that $0 \leq s \leq k$, and assume that $\mathbf{a}(\cdot, \nabla u(\cdot))|_T \in [H^{s+1}(T)]^2$ for all $T \in \mathcal{T}_h$. Then, there exists $C_{\mathtt{con}} > 0$, independent of the meshsize,* *but depending on $\widehat{\alpha}$, $\widehat{\beta}$, l, $\widehat{l}$ and the polynomial approximation degree k, such that for all $w \in V(h)$*

$$|[A_h(u), w] - [F_h, w]| \,\le\, C_{\mathtt{con}} \left(\sum_{T\in\mathcal{T}_h} h_T^{2(s+1)} \, \|\mathbf{a}(\cdot,\nabla u)\|^2_{[H^{s+1}(T)]^2} \right)^{1/2} .$$

PROOF. Let $w \in V(h)$. Since $\mathbf{S}(u) = \mathcal{G}_D$, $[\![u]\!] = 0$ on $\mathcal{E}_I$, $f \,=\, -\mathrm{div}\, \mathbf{a}(\cdot, \nabla u(\cdot))$ in Ω, and $u \,=\, g_D$ on Γ_D, we find that

$$[A_h(u), w] - [F_h, w] \,=\, \int_\Omega \mathbf{a}(\cdot,\nabla u)\cdot\Big(\nabla_h w - \mathbf{S}(w)\Big)\,dx \,+\, \int_{\mathcal{E}_D} \alpha\, u\, w\, ds$$

$$-\int_\Omega f\, w\, dx \,-\, \int_{\mathcal{E}_D} \alpha\, g_D\, w\, ds \,-\, \int_{\mathcal{E}_N} g_N\, w\, ds$$

$$= \int_\Omega \mathbf{a}(\cdot,\nabla u)\cdot\Big(\nabla_h w - \mathbf{S}(w)\Big)\,dx \,-\, \int_{\mathcal{E}_N} g_N\, w\, ds \,+\, \int_\Omega w\,\mathrm{div}\,\mathbf{a}(\cdot,\nabla u)\,dx\,. \tag{3.5.3}$$

Applying Gauss' formula on each element $T \in \mathcal{T}_h$, and then using (3.3.1), we obtain

$$\int_\Omega w\,\mathrm{div}\,\mathbf{a}(\cdot,\nabla u)\,dx \,=\, \sum_{T\in\mathcal{T}_h} \int_T w\,\mathrm{div}\,\mathbf{a}(\cdot,\nabla u)\,dx$$

$$= \sum_{T\in\mathcal{T}_h} \left[-\int_T \mathbf{a}(\cdot,\nabla u)\cdot\nabla w\,dx \,+\, \int_{\partial T} w\,\mathbf{a}(\cdot,\nabla u)\cdot\boldsymbol{\nu}_T\,ds \right]$$

$$= -\int_\Omega \mathbf{a}(\cdot,\nabla u)\cdot\nabla_h w\,dx + \int_{\mathcal{E}_I} [\![w]\!]\cdot\{\mathbf{a}(\cdot,\nabla u)\}\,ds + \int_{\mathcal{E}_D} w\,\mathbf{a}(\cdot,\nabla u)\cdot\boldsymbol{\nu}\,ds + \int_{\mathcal{E}_N} g_N\, w\, ds\,,$$

which, replaced back into (3.5.3), yields

$$[A_h(u), w] - [F_h, w] = -\int_\Omega \mathbf{S}(w)\cdot\mathbf{a}(\cdot,\nabla u)\,dx$$

$$+\int_{\mathcal{E}_I} \{\mathbf{a}(\cdot,\nabla u)\}\cdot[\![w]\!]\,ds + \int_{\mathcal{E}_D} w\,\mathbf{a}(\cdot,\nabla u)\cdot\boldsymbol{\nu}\,ds\,.$$

Next, using that $\int_\Omega \mathbf{S}(w)\cdot\mathbf{a}(\cdot,\nabla u)\,dx \,=\, \int_\Omega \mathbf{S}(w)\cdot\Pi_{\mathbf{\Sigma}_h}\left(\mathbf{a}(\cdot,\nabla u)\right)\,dx$, and applying the definition of $\mathbf{S}$ (cf. (3.3.21) and (3.3.15)), and the fact that $[\![\mathbf{a}(\cdot,\nabla u(\cdot))]\!] \,=\, 0$, we deduce that

$$[A_h(u), w] - [F_h, w] \,=\, \int_{\mathcal{E}_D} w\,\Big(\,(\mathbf{I} \,-\, \Pi_{\mathbf{\Sigma}_h})\,(\mathbf{a}(\cdot,\nabla u))\Big)\cdot\boldsymbol{\nu}\,ds$$

$$+\int_{\mathcal{E}_I} \Big(\{\,(\mathbf{I} \,-\, \Pi_{\mathbf{\Sigma}_h})\,(\mathbf{a}(\cdot,\nabla u))\,\} - \boldsymbol{\beta}\,[\![\,(\mathbf{I} \,-\, \Pi_{\mathbf{\Sigma}_h})\,(\mathbf{a}(\cdot,\nabla u))\,]\!]\Big)\cdot[\![w]\!]\,dx\,.$$

Then, the Cauchy-Schwarz inequality yields

$$|[A_h(u), w] \,-\, [F_h, w]| \,\le\, \hat{C}\; W_1(u)\, W_2(w)\,, \tag{3.5.4}$$

where $\hat{C}$ depends only on $\widehat{\alpha}$ and $\widehat{\beta}$,

$$W_1(u) := \Bigg(\, \|\mathtt{h}^{1/2}\,\{\,(\mathbf{I} - \Pi_{\mathbf{\Sigma}_h})\,(\mathbf{a}(\cdot,\nabla u))\,\}\|^2_{[L^2(\mathcal{E}_I)]^2}$$

$$+\,\|\mathtt{h}^{1/2}\,[\![\,(\mathbf{I} - \Pi_{\mathbf{\Sigma}_h})\,(\mathbf{a}(\cdot,\nabla u))\,]\!]\|^2_{L^2(\mathcal{E}_I)} \,+\, \|\mathtt{h}^{1/2}\,(\,(\mathbf{I} - \Pi_{\mathbf{\Sigma}_h})\,(\mathbf{a}(\cdot,\nabla u))\,)\cdot\nu\|^2_{L^2(\mathcal{E}_D)}\,\Bigg)^{1/2},$$

and

$$W_2(w) \,=\, \Big(\,\|\alpha^{1/2}[\![w]\!]\|^2_{[L^2(\mathcal{E}_I)]^2} \,+\, \|\alpha^{1/2} w\|^2_{L^2(\mathcal{E}_D)}\Big)^{1/2} \,\le\, |||w|||_h\,. \tag{3.5.5}$$

In order to bound $W_1(u)$, we notice that there holds

$$(\,\mathbf{I} - \Pi_{\mathbf{\Sigma}_h})\,(\mathbf{a}(\cdot,\nabla u)) \,=\, \Big(\mathbf{I} - \Pi_h^k\Big)\,(\mathbf{a}(\cdot,\nabla u)) \,-\, \Pi_{\mathbf{\Sigma}_h}\,\Big(\mathbf{I} - \Pi_h^k\Big)\,(\mathbf{a}(\cdot,\nabla u))\,,$$

and hence, applying the definition of Π_h^k and Lemma 3.3.1, we are lead to

$$\begin{aligned}[\,W_1(u)\,]^2 \,\le\, \bar{C} \sum_{T\in\mathcal{T}_h} h_T \Bigg\{ \|\,\Big(\mathbf{I} - \Pi_T^k\Big)\,(\mathbf{a}(\cdot,\nabla u))\|^2_{[L^2(\partial T)]^2} \\ +\quad \|\,\Pi_{\mathbf{\Sigma}_h}\,\Big(\mathbf{I} - \Pi_T^k\Big)\,(\mathbf{a}(\cdot,\nabla u))\,\|^2_{[L^2(\partial T)]^2}\Bigg\},\end{aligned} \tag{3.5.6}$$

where $\bar{C} \,:=\, 2\,(\bar{C}_1 + \bar{C}_2 + \bar{C}_3)$ (cf. Lemma 3.3.1).

According to (3.5.2) (cf. Lemma 3.5.1), we obtain

$$h_T\,\|\,\Big(\mathbf{I} - \Pi_T^k\Big)\,(\mathbf{a}(\cdot,\nabla u))\|^2_{[L^2(\partial T)]^2} \,\le\, C^2_{\mathtt{ort}}\, h_T^{2(s+1)}\,\|\mathbf{a}(\cdot,\nabla u)\|^2_{[H^{s+1}(T)]^2}\,. \tag{3.5.7}$$

Similarly, applying the inverse inequality (3.3.14) (cf. Lemma 3.3.2) and (3.5.1), we deduce that

$$\begin{aligned}h_T\,\|\,\Pi_{\mathbf{\Sigma}_h}\,\Big(\mathbf{I} - \Pi_T^k\Big)\,(\mathbf{a}(\cdot,\nabla u))\,\|^2_{[L^2(\partial T)]^2} \,\le\, C_{\mathtt{inv}}\,\|\,\Pi_{\mathbf{\Sigma}_h}\,\Big(\mathbf{I} - \Pi_T^k\Big)\,(\mathbf{a}(\cdot,\nabla u))\,\|^2_{[L^2(T)]^2} \\ \le\, C_{\mathtt{inv}}\,\|\,\Big(\mathbf{I} - \Pi_T^k\Big)\,(\mathbf{a}(\cdot,\nabla u))\,\|^2_{[L^2(T)]^2} \le\, C_{\mathtt{inv}}\,C^2_{\mathtt{ort}}\,h_T^{2(s+1)}\,\|\mathbf{a}(\cdot,\nabla u)\|^2_{[H^{s+1}(T)]^2}\,.\end{aligned} \tag{3.5.8}$$

In this way, replacing (3.5.7) and (3.5.8) back into (3.5.6), we arrive to the required bound for $W_1(u)$. This estimate and (3.5.5) complete the proof. □

The following upper bound for $|||\cdot|||_h$ will also be used below for the derivation of our a-priori error estimate.

LEMMA 3.5.3 *There exists $C_{\mathtt{upp}} > 0$, independent of the meshsize, such that*

$$|||v|||_h^2 \leq C_{\mathtt{upp}} \sum_{T\in\mathcal{T}_h} \left\{ |v|_{H^1(T)}^2 + h_T^{-1} \|v\|_{L^2(\partial T)}^2 \right\} \qquad \forall\, v \in V(h)\,.$$

PROOF. Let $v \in V(h)$. Then, using the definitions of $|||\cdot|||_h$ and $\mathtt{h}$ (cf. (3.3.12) and (3.3.9)), and the fact that $\mathtt{h} = h_T$ on $\mathcal{E}_D$, we easily find that

$$|||v|||_h^2 \leq \sum_{T\in\mathcal{T}_h} |v|_{H^1(T)}^2 + \widehat{\alpha}\, \|\mathtt{h}^{-1/2}\, [\![v]\!]\|_{[L^2(\mathcal{E}_I)]^2}^2 + \widehat{\alpha} \sum_{T\in\mathcal{T}_h} h_T^{-1} \|v\|_{L^2(\partial T)}^2\,. \tag{3.5.9}$$

Next, for the second term on the right hand side of (3.5.9), we apply part ii) of Lemma 3.3.1 and obtain

$$\begin{aligned} \|\mathtt{h}^{-1/2}\, [\![v]\!]\|_{[L^2(\mathcal{E}_I)]^2}^2 &= \|\mathtt{h}^{1/2}\, [\![\mathtt{h}^{-1}\, v]\!]\, \|_{[L^2(\mathcal{E}_I)]^2}^2 \\ \leq \bar{C}_2 \sum_{T\in\mathcal{T}_h} h_T \, \|\mathtt{h}^{-1}\, v\|_{L^2(\partial T)}^2 &\leq (1+l)^2\, \bar{C}_2 \sum_{T\in\mathcal{T}_h} h_T^{-1}\, \|v\|_{L^2(\partial T)}^2\,, \end{aligned}$$

which, replaced back into (3.5.9), provides the required estimate. □

We can establish now the a-priori error estimate in the energy norm $|||\cdot|||_h$.

THEOREM 3.5.1 *Let u_h and u be the solutions of* (3.3.26) *and* (3.1.1)*, respectively. Assume that $u|_T \in H^{k+1}(T)$ and $\mathbf{a}(\cdot,\nabla u(\cdot))|_T \in [H^{s+1}(T)]^2$, with $1 \leq s \leq k$, for all $T \in \mathcal{T}_h$. Then there exists $C_{\mathtt{err}} > 0$, independent of the meshsize, but depending on $\widehat{\alpha}$, $\widehat{\beta}$, l, $\widehat{l}$, and the polynomial approximation degree k, such that*

$$|||u - u_h|||_h^2 \leq C_{\mathtt{err}} \left(\sum_{T\in\mathcal{T}_h} h_T^{2k}\, \|u\|_{H^{k+1}(T)}^2 + \sum_{T\in\mathcal{T}_h} h_T^{2(s+1)}\, \|\mathbf{a}(\cdot,\nabla u)\|_{[H^{s+1}(T)]^2}^2 \right). \tag{3.5.10}$$

PROOF. We first note that $\Pi_h^k(u) \in V_h$. Then, according to the Strang type estimate (3.4.15) (cf. Theorem 3.4.1) and the consistency estimate provided by Lemma 3.5.2, we can write

$$|||u - u_h|||_h^2 \leq 2\,\bar{C}_{\mathtt{err}}^2 \left\{ |||\, (\mathbf{I} - \Pi_h^k)(u)\, |||_h^2 + \sum_{T\in\mathcal{T}_h} h_T^{2(s+1)}\, \|\mathbf{a}(\cdot,\nabla u)\|_{[H^{s+1}(T)]^2}^2 \right\} \tag{3.5.11}$$

with $\bar{C}_{\mathtt{err}} := \max\left\{ \left(1 + \dfrac{C_{\mathtt{LC}}}{C_{\mathtt{SM}}}\right), \dfrac{C_{\mathtt{con}}}{C_{\mathtt{SM}}} \right\}$.

On the other hand, applying Lemma 3.5.3 and the approximation error estimates given in Lemma 3.5.1, with $s = k$ and $m = 1$, we get

$$|||\, (\mathbf{I} - \Pi_h^k)(u)\, |||_h^2 \leq C_{\mathtt{upp}} \sum_{T\in\mathcal{T}_h} \left\{ |\, (\mathbf{I} - \Pi_T^k)(u)\, |_{H^1(T)}^2 + h_T^{-1}\, \|\, (\mathbf{I} - \Pi_T^k)(u)\, \|_{L^2(\partial T)}^2 \right\}$$

$$\leq \; 2\, C_{\mathtt{ort}}^2\, C_{\mathtt{upp}} \sum_{T\in\mathcal{T}_h} h_T^{2k}\; \|u\|_{H^{k+1}(T)}^2 \,. \tag{3.5.12}$$

Thus, (3.5.11) and (3.5.12) imply (3.5.10) and complete the proof. $\square$

We remark that when $k > 1$ we can choose $s = k-1$ and obtain

$$|||u-u_h|||_h \;\leq\; C_{\mathtt{err}}\; h^k \left(\sum_{T\in\mathcal{T}_h} \|u\|_{H^{k+1}(T)}^2 \;+\; \sum_{T\in\mathcal{T}_h} \|\mathbf{a}(\cdot,\nabla u)\|_{[H^k(T)]^2}^2 \right)^{1/2} , \tag{3.5.13}$$

and if $k = 1$ we have

$$|||u-u_h|||_h \;\leq\; C_{\mathtt{err}}\; h \left(\sum_{T\in\mathcal{T}_h} \|u\|_{H^{2}(T)}^2 \;+\; \sum_{T\in\mathcal{T}_h} \|\mathbf{a}(\cdot,\nabla u)\|_{[H^2(T)]^2}^2 \right)^{1/2} . \tag{3.5.14}$$

Theorem 3.5.2 *Assume the same hypotheses of Theorem* 3.5.1*. Then there exists* $\tilde{C}_{\mathtt{err}} > 0$*, independent of the meshsize, such that*

$$\|\boldsymbol{\theta}-\boldsymbol{\theta}_h\|_{[L^2(\Omega)]^2}^2 \;\leq\; \tilde{C}_{\mathtt{err}} \left(\sum_{T\in\mathcal{T}_h} h_T^{2k}\|u\|_{H^{k+1}(T)}^2 + \sum_{T\in\mathcal{T}_h} h_T^{2(s+1)}\|\mathbf{a}(\cdot,\nabla u)\|_{[H^{s+1}(T)]^2}^2 \right),$$

and

$$\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{[L^2(\Omega)]^2}^2 \;\leq\; \tilde{C}_{\mathtt{err}} \left(\sum_{T\in\mathcal{T}_h} h_T^{2k}\|u\|_{H^{k+1}(T)}^2 + \sum_{T\in\mathcal{T}_h} h_T^{2(s+1)}\|\mathbf{a}(\cdot,\nabla u)\|_{[H^{s+1}(T)]^2}^2 \right).$$

Proof. Since $\boldsymbol{\theta}_h = \nabla_h u_h - \mathbf{S}(u_h) + \mathcal{G}_D$ (cf. (3.3.24)), $\boldsymbol{\theta} = \nabla u$, and $\mathbf{S}(u) = \mathcal{G}_D$, we obtain

$$\|\boldsymbol{\theta}-\boldsymbol{\theta}_h\|_{[L^2(\Omega)]^2} \;\leq\; \|\nabla_h(u-u_h)\|_{[L^2(\Omega)]^2} \;+\; \|\mathbf{S}(u_h-u)\|_{[L^2(\Omega)]^2}$$

$$\leq\; \|\nabla_h(u-u_h)\|_{[L^2(\Omega)]^2} \;+\; C_{\mathbf{S}}\,|u_h-u|_h \;\leq\; (1+C_{\mathbf{S}})\;|||u\;-\;u_h|||_h\,,$$

where the last two inequalities made use of (3.3.22) and the definition of $|||\cdot|||_h$ (cf. (3.3.12)). This estimate and (3.5.10) give the a-priori error bound for $\boldsymbol{\theta}_h$.

In order to estimate $\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{[L^2(\Omega)]^2}$, we first recall that $\boldsymbol{\sigma} = \mathbf{a}(\cdot,\boldsymbol{\theta})$ and $\boldsymbol{\sigma}_h = \Pi_{\boldsymbol{\Sigma}_h}(\mathbf{a}(\cdot,\boldsymbol{\theta}_h))$ (cf. (3.3.25)). It follows that

$$\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{[L^2(\Omega)]^2} \;\leq\; \|\,(\mathbf{I}-\Pi_{\boldsymbol{\Sigma}_h})(\mathbf{a}(\cdot,\boldsymbol{\theta}))\,\|_{[L^2(\Omega)]^2} \;+\; \|\Pi_{\boldsymbol{\Sigma}_h}\Big(\mathbf{a}(\cdot,\boldsymbol{\theta})-\mathbf{a}(\cdot,\boldsymbol{\theta}_h)\Big)\|_{[L^2(\Omega)]^2}$$

$$\leq\; \|\,(\mathbf{I}-\Pi_{\boldsymbol{\Sigma}_h})(\mathbf{a}(\cdot,\boldsymbol{\theta}))\,\|_{[L^2(\Omega)]^2} \;+\; \|\mathbf{a}(\cdot,\boldsymbol{\theta})-\mathbf{a}(\cdot,\boldsymbol{\theta}_h)\|_{[L^2(\Omega)]^2}\,. \tag{3.5.15}$$

Then we have

$$\|\,(\mathbf{I}-\Pi_{\boldsymbol{\Sigma}_h})(\mathbf{a}(\cdot,\boldsymbol{\theta}))\,\|_{[L^2(\Omega)]^2} \;\leq\; \|\,(\mathbf{I}-\Pi_h^k)(\mathbf{a}(\cdot,\boldsymbol{\theta}))\,\|_{[L^2(\Omega)]^2} + \|\,\Pi_{\boldsymbol{\Sigma}_h}(\mathbf{I}-\Pi_h^k)(\mathbf{a}(\cdot,\boldsymbol{\theta}))\,\|_{[L^2(\Omega)]^2}$$

$$\leq \; 2 \, \| \, (\mathbf{I} - \Pi_h^k)(\mathbf{a}(\cdot, \boldsymbol{\theta})) \, \|_{[L^2(\Omega)]^2} \; = \; 2 \, \| \, (\mathbf{I} - \Pi_h^k)(\mathbf{a}(\cdot, \nabla u)) \, \|_{[L^2(\Omega)]^2} \,,$$

which, using (3.5.1) (cf. Lemma 3.5.1) with $m = 0$, implies that

$$\| \, (\mathbf{I} - \Pi_{\mathbf{\Sigma}_h})(\mathbf{a}(\cdot, \boldsymbol{\theta})) \, \|_{[L^2(\Omega)]^2} \; \leq \; 2 \, C_{\mathtt{ort}} \left(\sum_{T \in \mathcal{T}_h} h_T^{2(s+1)} \; \| \mathbf{a}(\cdot, \nabla u) \|^2_{[H^{s+1}(T)]^2} \right)^{1/2} . \tag{3.5.16}$$

On the other hand, according to the Lipschitz continuity of the nonlinear operator induced by $\mathbf{a}$ (cf. (3.4.2)), we obtain

$$\| \mathbf{a}(\cdot, \boldsymbol{\theta}) - \mathbf{a}(\cdot, \boldsymbol{\theta}_h) \|_{[L^2(\Omega)]^2} \; \leq \; C_1 \; \| \boldsymbol{\theta} - \boldsymbol{\theta}_h \|_{[L^2(\Omega)]^2} \,. \tag{3.5.17}$$

Finally, (3.5.15), (3.5.16), (3.5.17), and the a-priori error bound for $\boldsymbol{\theta}_h$ provide the corresponding estimate for $\boldsymbol{\sigma}_h$. $\square$

As before, we observe that when $k > 1$ we can take $s = k - 1$, and get

$$\| \boldsymbol{\theta} - \boldsymbol{\theta}_h \|_{[L^2(\Omega)]^2} \; \leq \; \tilde{C}_{\mathtt{err}} \, h^k \left(\sum_{T \in \mathcal{T}_h} \| u \|^2_{H^{k+1}(T)} \; + \; \sum_{T \in \mathcal{T}_h} \| \mathbf{a}(\cdot, \nabla u) \|^2_{[H^k(T)]^2} \right)^{1/2} ,$$

and

$$\| \boldsymbol{\sigma} - \boldsymbol{\sigma}_h \|_{[L^2(\Omega)]^2} \; \leq \; \tilde{C}_{\mathtt{err}} \, h^k \left(\sum_{T \in \mathcal{T}_h} \| u \|^2_{H^{k+1}(T)} \; + \; \sum_{T \in \mathcal{T}_h} \| \mathbf{a}(\cdot, \nabla u) \|^2_{[H^k(T)]^2} \right)^{1/2} .$$

Similarly, if $k = 1$ there hold

$$\| \boldsymbol{\theta} - \boldsymbol{\theta}_h \|_{[L^2(\Omega)]^2} \; \leq \; \tilde{C}_{\mathtt{err}} \, h \left(\sum_{T \in \mathcal{T}_h} \| u \|^2_{H^2(T)} \; + \; \sum_{T \in \mathcal{T}_h} \| \mathbf{a}(\cdot, \nabla u) \|^2_{[H^2(T)]^2} \right)^{1/2} ,$$

and

$$\| \boldsymbol{\sigma} - \boldsymbol{\sigma}_h \|_{[L^2(\Omega)]^2} \; \leq \; \tilde{C}_{\mathtt{err}} \, h \left(\sum_{T \in \mathcal{T}_h} \| u \|^2_{H^2(T)} \; + \; \sum_{T \in \mathcal{T}_h} \| \mathbf{a}(\cdot, \nabla u) \|^2_{[H^2(T)]^2} \right)^{1/2} .$$

3.5.2 L^2-norm error estimates

We now turn our attention to the L^2-norm for the error $(u - u_h)$. To this end, we first let $\mathcal{N}_h : V(h) \to V(h)'$ be the pure nonlinear operator forming part of A_h (cf. (3.3.27)), that is

$$[\mathcal{N}_h(w), v] \; := \; \int_\Omega \mathbf{a}(\cdot, \nabla_h w - \mathbf{S}(w) + \mathcal{G}_D) \cdot (\nabla_h v - \mathbf{S}(v)) \, dx \qquad \forall \, w \,, \; v \, \in \, V(h) \,. \tag{3.5.18}$$

Since $\mathbf{a}$ satisfies the assumption (H.4) and the operator $\mathbf{S}$ is bounded (cf. Lemma 3.3.3 and (3.3.22)), we deduce that $\mathcal{N}_h$ is Gâteaux differentiable at each $z \in V(h)$. Indeed, this derivative can be interpreted as the bounded bilinear form $D\mathcal{N}_h(z) : V(h) \times V(h) \to \mathbf{R}$ given by

$$D\mathcal{N}_h(z)(w,v) := \int_\Omega (\nabla_h w - \mathbf{S}(w))^{\mathbf{T}} D\mathbf{a}(\cdot,\tilde{\boldsymbol{\zeta}})\,(\nabla_h v - \mathbf{S}(v))\,dx \qquad \forall\, w,\ v \in V(h)\,, \tag{3.5.19}$$

where $\tilde{\boldsymbol{\zeta}} := \nabla_h z - \mathbf{S}(z) + \mathcal{G}_D$ and $D\mathbf{a}(\cdot,\tilde{\boldsymbol{\zeta}})$ is the jacobian matrix of $\mathbf{a}$ at $\tilde{\boldsymbol{\zeta}}$.

Similarly, according to the definition of A_h (cf. (3.3.27)), the Gâteaux derivative of A_h at $z \in V(h)$ reduces to the bounded bilinear form $DA_h(z) : V(h) \times V(h) \to \mathbf{R}$ defined by

$$DA_h(z)(w,v) := D\mathcal{N}_h(z)(w,v) + \int_{\mathcal{E}_I} \alpha\, [\![w]\!] \cdot [\![v]\!]\, ds + \int_{\mathcal{E}_D} \alpha\, w\, v\, ds \quad \forall\, w, v \in V(h)\,. \tag{3.5.20}$$

In what follows we assume that $D\mathbf{a}(\cdot,\tilde{\boldsymbol{\zeta}})$ is symmetric for all $z \in V(h)$ and that $D\mathcal{N}_h$ is *hemi-continuous*, that is for any $z, w \in V(h)$, the mapping $\mathbf{R} \ni \mu \to D\mathcal{N}_h(z + \mu\, w)(w,\cdot) \in V(h)'$ is continuous. Thus, applying the mean value theorem we deduce the existence of $\tilde{u} \in V(h)$, a convex combination of u and u_h, such that

$$D\mathcal{N}_h(\tilde{u})(u - u_h, v) = [\mathcal{N}_h(u) - \mathcal{N}_h(u_h), v] \qquad \forall\, v \in V(h)\,. \tag{3.5.21}$$

We observe here, according to (3.5.19), that

$$D\mathcal{N}_h(\tilde{u})(w,v) := \int_\Omega (\nabla_h w - \mathbf{S}(w))^{\mathbf{T}} D\mathbf{a}(\cdot,\tilde{\boldsymbol{\theta}})\,(\nabla_h v - \mathbf{S}(v))\,dx \qquad \forall\, w,\ v \in V(h)\,, \tag{3.5.22}$$

where $\tilde{\boldsymbol{\theta}} := \nabla_h \tilde{u} - \mathbf{S}(\tilde{u}) + \mathcal{G}_D$. Further, it follows from (3.3.27), (3.5.20) and (3.5.21) that

$$DA_h(\tilde{u})(u - u_h, v) = [A_h(u) - A_h(u_h), v] \qquad \forall\, v \in V(h)\,. \tag{3.5.23}$$

Next, we let $z \in H^1(\Omega)$ be the unique solution of the linear boundary value problem

$$\begin{aligned} -\mathrm{div}\left(D\mathbf{a}(\cdot,\tilde{\boldsymbol{\theta}})\,\nabla z\right) &= u - u_h \quad \text{in} \quad \Omega\,, \quad z = 0 \quad \text{on} \quad \Gamma_D\,, \\ \left(D\mathbf{a}(\cdot,\tilde{\boldsymbol{\theta}})\,\nabla z\right) \cdot \boldsymbol{\nu} &= 0 \quad \text{on} \quad \Gamma_N\,. \end{aligned} \tag{3.5.24}$$

Since Ω is convex and $u - u_h \in L^2(\Omega)$, we assume further regularity on $D\mathbf{a}(\cdot,\tilde{\boldsymbol{\theta}})$ so that $z \in H^2(\Omega)$ and $D\mathbf{a}(\cdot,\tilde{\boldsymbol{\theta}})\,\nabla z \in [H^1(\Omega)]^2$, with

$$\|z\|_{H^2(\Omega)} \le C_{\mathtt{reg}}\,\|u - u_h\|_{L^2(\Omega)} \quad \text{and} \quad \|D\mathbf{a}(\cdot,\tilde{\boldsymbol{\theta}})\,\nabla z\|_{[H^1(\Omega)]^2} \le C_{\mathtt{reg}}\,\|u - u_h\|_{L^2(\Omega)}\,, \tag{3.5.25}$$

and where the constant $C_{\mathtt{reg}} > 0$ is independent of u and u_h.

Then, similarly as for the derivation of (3.5.12), we apply Lemma 3.5.3 and the approximation error estimates given in Lemma 3.5.1, with $s = k = 1$ and $m = 1$, to obtain

$$|||\,(\mathbf{I} - \Pi_h^1)(z)\,|||_h \;\leq\; (\,2\,C_{\mathtt{upp}})^{1/2}\;\;C_{\mathtt{ort}}\;\;h\;\|z\|_{H^2(\Omega)} \;\leq\; \tilde{C}_{\mathtt{ort}}\;h\;\|u - u_h\|_{L^2(\Omega)}\,, \tag{3.5.26}$$

with $\tilde{C}_{\mathtt{ort}} \;=\; (\,2\,C_{\mathtt{upp}})^{1/2}\;\;C_{\mathtt{ort}}\;C_{\mathtt{reg}}$.

Now, following the method applied in Sections 3.2 and 3.3, we deduce that the LDG formulation of problem (3.5.24) reduces to: *Find $z_h \in V_h$ such that*

$$DA_h(\tilde{u})(z_h, v_h) \;=\; [\tilde{F}_h, v_h] \qquad \forall\, v_h \in V_h\,, \tag{3.5.27}$$

where $\tilde{F}_h : V(h) \to \mathbf{R}$ is defined by

$$[\tilde{F}_h, v] \;:=\; \int_\Omega (u - u_h)\,v\,dx \qquad \forall\, v\;\in\;V(h)\,. \tag{3.5.28}$$

As a consequence of the assumption (H.3) on $\mathbf{a}$, and proceeding similarly as in the proof of Lemma 3.4.2, one can show that $DA_h(\tilde{u})$ is $V(h)$-elliptic with respect to $|||\cdot|||_h$, and hence problem (3.5.27) has a unique solution $z_h \in V_h$. Furthermore, applying the linear version of the consistency estimate provided by Lemma 3.5.2 with $s = 0$, and using the second estimate in (3.5.25), we find that

$$\begin{aligned} |\,DA_h(\tilde{u})(z, w) - [\tilde{F}_h, w]\,| \;&\leq\; C_{\mathtt{con}}\;h\;\|D\mathbf{a}(\cdot, \tilde{\boldsymbol{\theta}})\,\nabla z\|_{[H^1(\Omega)]^2}\;|||w|||_h \\ &\leq\; \tilde{C}_{\mathtt{con}}\;h\;\|u - u_h\|_{L^2(\Omega)}\;|||w|||_h \qquad \forall\, w\;\in\;V(h)\,, \end{aligned} \tag{3.5.29}$$

with $\tilde{C}_{\mathtt{con}} \;=\; C_{\mathtt{con}}\,C_{\mathtt{reg}}$.

We are now in a position to provide the estimate for $\|u - u_h\|_{L^2(\Omega)}$.

THEOREM 3.5.3 *Assume the same hypotheses of Theorem* 3.5.1*, and that* $D\mathbf{a}(\cdot,\cdot)$ *and the solution* z *of* (3.5.24) *satisfy the conditions stated in the present subsection. Then there exists* $\bar{C}_{\mathtt{err}} > 0$*, independent of the meshsize, such that for* $k > 1$ *and* $k = 1$*, respectively, there hold*

$$\|u - u_h\|_{L^2(\Omega)} \;\leq\; \bar{C}_{\mathtt{err}}\,h^{k+1}\,\left(\sum_{T\in\mathcal{T}_h} \|u\|^2_{H^{k+1}(T)} \;+\; \sum_{T\in\mathcal{T}_h} \|\mathbf{a}(\cdot, \nabla u)\|^2_{[H^k(T)]^2}\right)^{1/2}\,,$$

and

$$\|u - u_h\|_{L^2(\Omega)} \,\le\, \bar{C}_{\tt err}\, h^2 \left(\sum_{T\in\mathcal{T}_h} \|u\|^2_{H^2(T)} \,+\, \sum_{T\in\mathcal{T}_h} \|\mathbf{a}(\cdot,\nabla u)\|^2_{[H^2(T)]^2} \right)^{1/2} .$$

Proof. We use (3.5.28) with $v := (u - u_h) \in V(h)$, and get

$$\|u-u_h\|^2_{L^2(\Omega)} \,=\, [\tilde{F}_h, u-u_h] \,=\, DA_h(\tilde{u})(z, u-u_h) - \Big(DA_h(\tilde{u})(z, u-u_h) \,-\, [\tilde{F}_h, u-u_h] \Big) ,$$

which, according to (3.5.29), yields

$$\|u-u_h\|^2_{L^2(\Omega)} \,\le\, |\, DA_h(\tilde{u})(z, u-u_h)\,| + \tilde{C}_{\tt con}\, h\, \|u-u_h\|_{L^2(\Omega)}\, |||u-u_h|||_h\,. \tag{3.5.30}$$

Now, since $DA_h(\tilde{u})$ is symmetric, we can write

$$DA_h(\tilde{u})(z, u-u_h) \,=\, DA_h(\tilde{u})(u-u_h, (\mathbf{I}-\Pi^1_h)(z)) \,+\, DA_h(\tilde{u})(u-u_h, \Pi^1_h(z))\,,$$

which, because of (3.5.23), reduces to

$$DA_h(\tilde{u})(z, u-u_h) \,=\, [A_h(u) - A_h(u_h), (\mathbf{I}-\Pi^1_h)(z)] \,+\, [A_h(u) - A_h(u_h), \Pi^1_h(z)]\,,$$

and using the Lipschitz-continuity of A_h and the estimate (3.5.26), gives

$$|\, DA_h(\tilde{u})(z, u-u_h)\,| \,\le\, C_{\tt LC}\, \tilde{C}_{\tt ort}\, h\, |||u-u_h|||_h\, \|u-u_h\|_{L^2(\Omega)} + |\, [A_h(u) - A_h(u_h), \Pi^1_h(z)]\,|\,. \tag{3.5.31}$$

On the other hand, it is not difficult to see, using Gauss' formula, that z also satisfies $[A_h(u), z] \,=\, [F_h, z]$, and from the primal formulation (3.3.26) we know that $[A_h(u_h), \Pi^1_h(z)] \,=\, [F_h, \Pi^1_h(z)]$. It follows that

$$[A_h(u) - A_h(u_h), \Pi^1_h(z)] \,=\, [F_h, (\mathbf{I}-\Pi^1_h)(z)] \,-\, [A_h(u), (\mathbf{I}-\Pi^1_h)(z)]\,,$$

and hence, applying again Lemma 3.5.2 (with $s = k-1$) and the estimate (3.5.26), we conclude that

$$\begin{aligned} &|\, [A_h(u) - A_h(u_h), \Pi^1_h(z)]\,| \\ &\le\, C_{\tt con}\, \tilde{C}_{\tt ort}\, h^{k+1} \left(\sum_{T\in\mathcal{T}_h} \|\mathbf{a}(\cdot,\nabla u)\|^2_{[H^k(\Omega)]^2} \right)^{1/2} \|u-u_h\|_{L^2(\Omega)}\,. \end{aligned} \tag{3.5.32}$$

Finally, (3.5.30), (3.5.31), and (3.5.32) imply that

$$\begin{aligned} \|u-u_h\|_{L^2(\Omega)} \,&\le\, \Big(\tilde{C}_{\tt con} \,+\, C_{\tt LC}\, \tilde{C}_{\tt ort} \Big)\, h\, |||u-u_h|||_h \\ &+\, C_{\tt con}\, \tilde{C}_{\tt ort}\, h^{k+1} \left(\sum_{T\in\mathcal{T}_h} \|\mathbf{a}(\cdot,\nabla u)\|^2_{[H^k(\Omega)]^2} \right)^{1/2} , \end{aligned}$$

which, together with the estimate for $|||u-u_h|||_h$ (cf. Theorem 3.5.1), completes the proof. □

3.6 The non-convex case

Let Ω be a non-convex domain that can be decomposed as the disjoint union of two convex polygonal regions Ω_1 and Ω_2, and let Γ be the boundary of Ω. Instead of (3.1.1), throughout this section we consider the pure Dirichlet problem. This means that given $f \in L^2(\Omega)$ and $g_D \in H^{1/2}(\Gamma)$, we now look for $u \in H^1(\Omega)$ such that

$$-\mathrm{div}\,\mathbf{a}(\cdot, \nabla u(\cdot)) = f \quad \text{in} \quad \Omega \quad \text{and} \quad u = g_D \quad \text{on} \quad \Gamma\,. \tag{3.6.1}$$

It is well known (see [65]) that the solution u of (3.6.1) belongs to $H^{1+\gamma}(\Omega)$, with $\gamma \in (1/2, 1]$. In this case we show below that the corresponding orders of convergence for $|||u - u_h|||_h$ and $||u - u_h||_{L^2(\Omega)}$ become γ and 2γ, respectively.

From now on we assume that the triangulation of the whole domain $\mathcal{T}_h$ is the union of corresponding triangulations $\mathcal{T}_{h,1}$ and $\mathcal{T}_{h,2}$ of Ω_1 and Ω_2, respectively, which are not necessarily matching on their common boundary.

Then, proceeding as in Section 3.2, with the same notations introduced there, considering Γ_N empty and setting $\Gamma_D := \Gamma$, our LDG formulation becomes: Find $(\boldsymbol{\theta}_h, \boldsymbol{\sigma}_h, u_h) \in \boldsymbol{\Sigma}_h \times \boldsymbol{\Sigma}_h \times V_h$ such that for all $(\boldsymbol{\zeta}_h, \boldsymbol{\tau}_h, v_h) \in \boldsymbol{\Sigma}_h \times \boldsymbol{\Sigma}_h \times V_h$:

$$\begin{aligned}
&\int_\Omega \mathbf{a}(\cdot, \boldsymbol{\theta}_h) \cdot \boldsymbol{\zeta}_h \, dx + \int_{\mathcal{E}_I} \alpha [\![u_h]\!] \cdot [\![v_h]\!] \, ds + \int_{\mathcal{E}_D} \alpha \, u_h \, v_h \, ds \\
&- \int_\Omega \boldsymbol{\sigma}_h \cdot \boldsymbol{\zeta}_h \, dx + \int_\Omega \boldsymbol{\sigma}_h \cdot \nabla_h v_h \, dx - \int_{\mathcal{E}_I} (\{\boldsymbol{\sigma}_h\} - \boldsymbol{\beta} [\![\boldsymbol{\sigma}_h]\!]) \cdot [\![v_h]\!] \, ds - \int_{\mathcal{E}_D} v_h \, \boldsymbol{\sigma}_h \cdot \boldsymbol{\nu} \, ds \\
&= \int_\Omega f \, v_h \, dx + \int_{\mathcal{E}_D} \alpha \, g_D \, v_h \, ds \,, \\
&- \int_\Omega \boldsymbol{\theta}_h \cdot \boldsymbol{\tau}_h \, dx + \int_\Omega \nabla_h u_h \cdot \boldsymbol{\tau}_h \, dx - \int_{\mathcal{E}_I} (\{\boldsymbol{\tau}_h\} - \boldsymbol{\beta} [\![\boldsymbol{\tau}_h]\!]) \cdot [\![u_h]\!] \, ds \\
&- \int_{\mathcal{E}_D} u_h \, \boldsymbol{\tau}_h \cdot \boldsymbol{\nu} \, ds \; = \; - \int_{\mathcal{E}_D} g_D \, \boldsymbol{\tau}_h \cdot \boldsymbol{\nu} \, ds \,,
\end{aligned} \tag{3.6.2}$$

where the functions $\alpha \in L^\infty(\mathcal{E})$ and $\boldsymbol{\beta} \in [L^\infty(\mathcal{E}_I)]^2$ are defined similarly as in the convex case (cf. (3.2.9) and (3.2.10) with $\mathcal{E}_N = \phi$).

Now, in order to establish the unique solvability and a-priori estimate for (3.6.2), we provide next an equivalent formulation by eliminating the unknowns $\boldsymbol{\sigma}_h$ and $\boldsymbol{\theta}_h$ in terms of u_h. To this end, we first introduce the space $V(h) := V_h + H^1(\Omega)$, and define, respectively, the energy norm $|||\cdot|||_h : V(h) \to \mathbf{R}$ and the associated seminorm $|\cdot|_h : V(h) \to \mathbf{R}$, as the ones given in (3.3.12) and (3.3.13). Then, Lemmas 3.3.1 and 3.3.2 imply the analogue of Lemma 3.3.3, which means that there exists $C_{\mathbf{S}} > 0$, independent of the meshsize, such that

$$|\, S(v, \boldsymbol{\tau}_h)\,| \;\leq\; C_{\mathbf{S}}\, |v|_h \, \|\boldsymbol{\tau}_h\|_{[L^2(\Omega)]^2} \quad \forall\, (v, \boldsymbol{\tau}_h) \in V(h) \times \boldsymbol{\Sigma}_h\,, \tag{3.6.3}$$

where $S : V(h) \times \boldsymbol{\Sigma}_h \to \mathbf{R}$ is the bilinear form defined by

$$S(v, \boldsymbol{\tau}_h) \;:=\; \int_{\mathcal{E}_I} (\{\boldsymbol{\tau}_h\} - \boldsymbol{\beta}[\![\boldsymbol{\tau}_h]\!]) \cdot [\![v]\!]\, ds + \int_{\mathcal{E}_D} v\, \boldsymbol{\tau}_h \cdot \boldsymbol{\nu}\, ds \quad \forall\, (v, \boldsymbol{\tau}_h) \in V(h) \times \boldsymbol{\Sigma}_h\,. \tag{3.6.4}$$

Next, according to Lemma 3.3.4, we let $\mathbf{G} : \boldsymbol{\Sigma}_h \to \mathbf{R}$ be the linear functional defined by

$$\mathbf{G}(\boldsymbol{\tau}_h) \;:=\; \int_{\mathcal{E}_D} g_D\, \boldsymbol{\tau}_h \cdot \boldsymbol{\nu}\, ds \qquad \forall\, \boldsymbol{\tau}_h \in \boldsymbol{\Sigma}_h\,,$$

and let $\mathcal{G}_D$ be the unique element in $\boldsymbol{\Sigma}_h$ such that

$$\int_{\Omega} \mathcal{G}_D \cdot \boldsymbol{\tau}_h\, dx \;=\; \mathbf{G}(\boldsymbol{\tau}_h) \qquad \forall\, \boldsymbol{\tau}_h \in \boldsymbol{\Sigma}_h\,.$$

Similarly as in (3.3.21), we also let $\mathbf{S} : V(h) \to \boldsymbol{\Sigma}_h$ be the linear and bounded operator induced by the bilinear form S, which satisfies $\mathbf{S}(u) = \mathcal{G}_D$. Thus, since $\nabla_h u_h$, $\mathbf{S}(u_h)$, and $\mathcal{G}_D$ belong to $\boldsymbol{\Sigma}_h$, we are able again to express $\boldsymbol{\theta}_h$ and $\boldsymbol{\sigma}_h$ in terms of u_h through the equations (3.3.24) and (3.3.25), respectively. Therefore, the corresponding primal formulation becomes: *Find* $u_h \in V_h$ *such that*

$$[A_h(u_h), v_h] \;=\; [F_h, v_h] \qquad \forall\, v_h \in V_h\,, \tag{3.6.5}$$

where the nonlinear operator $A_h : V(h) \to V(h)'$ and the functional $F_h : V(h) \to \mathbf{R}$, are defined, respectively, by

$$\begin{aligned} [A_h(w), v] \;:=\; & \int_{\Omega} \mathbf{a}(\cdot, \nabla_h w - \mathbf{S}(w) + \mathcal{G}_D) \cdot (\nabla_h v - \mathbf{S}(v))\, dx \\ & + \int_{\mathcal{E}_I} \alpha\, [\![w]\!] \cdot [\![v]\!]\, ds \;+\; \int_{\mathcal{E}_D} \alpha\, w\, v\, ds \qquad \forall\, w,\, v \in V(h)\,, \end{aligned} \tag{3.6.6}$$

and

$$[F_h, v] \;:=\; \int_{\Omega} f\, v\, dx + \int_{\mathcal{E}_D} \alpha\, g_D\, v\, ds \qquad \forall\, v \in V(h)\,. \tag{3.6.7}$$

On the other hand, proceeding analogously as in Section 3.4, we can show that the nonlinear operator A_h is Lipschitz continuous and strongly monotone. In addition, we also have the estimates provided by the following two lemmas.

LEMMA 3.6.1 *There exists a constant $C_{\mathbf{L}} > 0$, independent of the meshsize, such that*

$$\|v\|_{L^2(\Omega)} \;\leq\; C_{\mathbf{L}}\; |||v|||_h \qquad \forall\, v \,\in\, V(h)\,.$$

PROOF. We proceed similarly as in [6]. Given $v \in V(h)$ and $i \in \{1,2\}$, we let $\varphi^i \in H^1(\Omega_i)$ be the unique solution of the boundary value problem

$$-\Delta\varphi^i = v \quad \text{in} \quad \Omega_i \quad \text{and} \quad \varphi^i = 0 \quad \text{on} \quad \partial\Omega_i\,.$$

Since Ω_i is convex and $v|_{\Omega_i} \in L^2(\Omega_i)$, we deduce (see [65]) that $\varphi^i \in H^2(\Omega_i) \cap H^1_0(\Omega_i)$ and $||\varphi^i||_{H^2(\Omega_i)} \leq C^i_\omega \, \|v\|_{L^2(\Omega_i)}$, with a constant $C^i_\omega > 0$ that depends only on Ω_i. The rest of the proof follows as in Lemma 3.4.3. We omit further details. □

It is important to point out, as indicated in Remark 2.1 of [72], that Lemma 3.6.1 can also be proved using Theorem 3.1 in [87] and the inf-sup condition for Raviart-Thomas spaces.

LEMMA 3.6.2 *There exists $C_{\mathbf{F}} > 0$, depending on $\widehat{\alpha}$, l, $\widehat{l}$, and k, but independent of the meshsize, such that*

$$|[F_h, v_h]| \;\leq\; C_{\mathbf{F}}\; \mathcal{B}(f, g_D)\; |||v_h|||_h \quad \forall\, v_h \in V_h\,, \tag{3.6.8}$$

where

$$\mathcal{B}(f, g_D) \;:=\; \Big(\, \|f\|^2_{L^2(\Omega)} \;+\; \|\alpha^{1/2} g_D\|^2_{L^2(\mathcal{E}_D)} \Big)^{1/2}\,.$$

The unique solvability of the primal formulation (3.6.5) and the associated Strang type error estimate, which follow as in Theorem 3.4.1, are established next.

THEOREM 3.6.1 *There exists a unique $u_h \in V_h$ solution of* (3.6.5)*, which satisfies*

$$|||u_h|||_h \;\leq\; \frac{1}{C_{\mathtt{SM}}} \,\Big\{\, C_{\mathbf{F}}\, \mathcal{B}(f, g_D) \;+\; \|A_h(0)\|_{V_h'} \Big\}\,. \tag{3.6.9}$$

Moreover, the following error estimate holds

$$|||u - u_h|||_h \;\leq\; \left(1 + \frac{C_{\mathtt{LC}}}{C_{\mathtt{SM}}}\right) \inf_{v_h \in V_h} |||u - v_h|||_h \;+\; \frac{1}{C_{\mathtt{SM}}} \sup_{\substack{w_h \in V_h \\ w_h \neq 0}} \frac{|[A_h(u), w_h] - [F_h, w_h]|}{|||w_h|||_h}\,, \tag{3.6.10}$$

where u is the exact solution of (3.6.1).

It is important to remark that the assumed regularity on the exact solution u yields $[\![u]\!] = 0$ on any interior edge of $\mathcal{T}_h$. In addition, since $-\mathrm{div}\,\mathbf{a}(\cdot, \nabla u(\cdot)) = f$ in Ω, with $f \in L^2(\Omega)$, we deduce that $\mathbf{a}(\cdot, \nabla u(\cdot)) \in H(\mathrm{div}\,, \Omega)$, whence $[\![\mathbf{a}(\cdot, \nabla u(\cdot))]\!] = 0$ on each $e \subseteq \mathcal{E}_I$.

The bound for the consistency term in the Strang type error estimate (3.6.10) can be proved similarly as for Lemma 3.5.2. In this case, however, we need to apply Lemma 3.5.1. The corresponding result is stated as follows.

LEMMA 3.6.3 *Assume that* $\mathbf{a}(\cdot, \nabla u(\cdot))|_T \in [H^{\gamma+1/2}(T)]^2$, *for all* $T \in \mathcal{T}_h$. *Then there exists* $C_{\mathtt{con}} > 0$, *independent of the meshsize, but depending on* $\widehat{\alpha}$, $\widehat{\beta}$, l *and* $\widehat{l}$, *such that for all* $w \in V(h)$

$$|[A_h(u), w] - [F_h, w]| \,\le\, C_{\mathtt{con}} \left(\sum_{T\in\mathcal{T}_h} h_T^{2(\gamma+1/2)} \, \|\mathbf{a}(\cdot, \nabla u)\|^2_{[H^{\gamma+1/2}(T)]^2} \right)^{1/2} |||w|||_h \,.$$

We now introduce the *discontinuous* global operator $\Pi_h : H^1(\Omega) \to V_h$, where, given $w \in H^1(\Omega)$, $\Pi_h^k(w)$ is the unique element in V_h such that $\Pi_h^k(w)|_T = \Pi_T^k(w|_T)$ for all $T \in \mathcal{T}_h$. Then, following the analysis used in the proof of Theorem 3.5.1, and using again the upper bound for $|||\cdot|||_h$ provided by Lemma 3.5.3 (which is also valid in the present situation), we can prove the following a-priori error estimate in the energy norm $|||\cdot|||_h$.

THEOREM 3.6.2 *Let* u *and* u_h *be the solutions of* (3.6.1) *and* (3.6.5), *respectively. Assume that* $\mathbf{a}(\cdot, \nabla u(\cdot))|_T \in [H^{\gamma+1/2}(T)]^2$, *for all* $T \in \mathcal{T}_h$. *Then there exists* $C_{\mathtt{err}} > 0$, *independent of the meshsize, but depending on* $\widehat{\alpha}$, $\widehat{\beta}$, l *and* $\widehat{l}$, *such that*

$$|||u - u_h|||_h^2 \,\le\, C_{\mathtt{err}} \left(\sum_{T\in\mathcal{T}_h} h_T^{2\gamma} \, \|u\|^2_{H^{1+\gamma}(T)} + \sum_{T\in\mathcal{T}_h} h_T^{2(\gamma+1/2)} \, \|\mathbf{a}(\cdot, \nabla u)\|^2_{[H^{\gamma+1/2}(T)]^2} \right) . \tag{3.6.11}$$

Similarly, following now the proof of Theorem 3.5.2, we can show the following result.

THEOREM 3.6.3 *Assume the same hypotheses of Theorem* 3.6.2. *Then there exists* $\tilde{C}_{\mathtt{err}} > 0$, *independent of the meshsize, such that*

$$\|\boldsymbol{\theta} - \boldsymbol{\theta}_h\|^2_{[L^2(\Omega)]^2} \,\le\, \tilde{C}_{\mathtt{err}} \left(\sum_{T\in\mathcal{T}_h} h_T^{2\gamma} \|u\|^2_{H^{1+\gamma}(T)} + \sum_{T\in\mathcal{T}_h} h_T^{2(\gamma+1/2)} \|\mathbf{a}(\cdot, \nabla u)\|^2_{[H^{\gamma+1/2}(T)]^2} \right) ,$$

and

$$\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|^2_{[L^2(\Omega)]^2} \,\le\, \tilde{C}_{\texttt{err}} \left(\sum_{T\in\mathcal{T}_h} h_T^{2\gamma} \|u\|^2_{H^{1+\gamma}(T)} + \sum_{T\in\mathcal{T}_h} h_T^{2(\gamma+1/2)} \|\mathbf{a}(\cdot,\nabla u)\|^2_{[H^{\gamma+1/2}(T)]^2} \right) .$$

In order to provide the estimate for $||u-u_h||_{L^2(\Omega)}$, we now let $z \in H^1(\Omega)$ be the unique solution of the auxiliary problem:

$$-\mathrm{div}\, \Big(D\mathbf{a}(\cdot,\tilde{\boldsymbol{\theta}})\,\nabla z \Big) \,=\, u-u_h \quad \text{in} \quad \Omega\,, \quad \text{and} \quad z \,=\, 0 \quad \text{on} \quad \Gamma. \tag{3.6.12}$$

Hence, we assume enough regularity on $D\mathbf{a}(\cdot,\tilde{\boldsymbol{\theta}})$, so that $z \in H^{1+\gamma}(\Omega)$ and $\Big(D\mathbf{a}(\cdot,\tilde{\boldsymbol{\theta}})\,\nabla z\Big) \in [H^{\gamma+1/2}(\Omega)]^2$, with the estimates (see [65])

$$||z||_{H^{1+\gamma}(\Omega)} \le C_{\texttt{reg}}\,\|u-u_h\|_{L^2(\Omega)} \quad \text{and} \quad \|D\mathbf{a}(\cdot,\tilde{\boldsymbol{\theta}})\,\nabla z\|_{[H^{\gamma+1/2}(\Omega)]^2} \,\le\, C_{\texttt{reg}}\,\|u-u_h\|_{L^2(\Omega)}\,,$$

where the constant $C_{\texttt{reg}} > 0$ is independent of u and u_h.

Finally, we are able to establish the following result, whose proof is analogous to the one of Theorem 3.5.3.

THEOREM 3.6.4 *Assume the same hypotheses of Theorem* 3.6.2*, and that* $D\mathbf{a}(\cdot,\tilde{\boldsymbol{\theta}})$ *and the solution* z *of* (3.6.12) *satisfy the conditions indicated above. Then there exists* $\bar{C}_{\texttt{err}} > 0$*, independent of the meshsize, such that*

$$\|u-u_h\|_{L^2(\Omega)} \,\le\, \bar{C}_{\texttt{err}}\, h^{2\gamma} \left(\sum_{T\in\mathcal{T}_h} \|u\|^2_{H^{1+\gamma}(T)} \,+\, \sum_{T\in\mathcal{T}_h} \|\mathbf{a}(\cdot,\nabla u)\|^2_{[H^{\gamma+1/2}(T)]^2} \right)^{1/2} .$$

3.7 Numerical results

In this section we provide some numerical results illustrating the performance of the present LDG method in convex and non-convex domains. We remark that our computations rely on the primal formulation (3.3.7) instead of the equivalent one (3.3.26), which is used only for the analysis and the derivation of the a-priori error estimates.

The notation to be used here is described next. N denotes the number of degrees of freedom defining the product space $\boldsymbol{\Sigma}_h \times V_h \times \boldsymbol{\Sigma}_h$. In addition, given $u \in H^1(\Omega)$ and $(\boldsymbol{\theta}_h, u_h, \boldsymbol{\sigma}_h) \in \boldsymbol{\Sigma}_h \times V_h \times \boldsymbol{\Sigma}_h$, the unique solutions of (3.1.1) and (3.3.7), we use the following errors

$$\mathbf{e}_0(\boldsymbol{\theta}) := ||\boldsymbol{\theta} - \boldsymbol{\theta}_h||_{[L^2(\Omega)]^2}\,, \quad \mathbf{e}_h(u) := |||u - u_h|||_h\,,$$

$$\mathbf{e}_0(\boldsymbol{\sigma}) := ||\boldsymbol{\sigma} - \boldsymbol{\sigma}_h||_{[L^2(\Omega)]^2}\,, \quad \text{and} \quad \mathbf{e}_0(u) := ||u - u_h||_{L^2(\Omega)}\,.$$

Further, the experimental rates of convergence are denoted, respectively, by $r_0(\boldsymbol{\theta})$, $r_h(u)$, $r_0(\boldsymbol{\sigma})$, and $r_0(u)$. In particular, if $\mathbf{e}_h(u)$ and $\mathbf{e}_{\tilde{h}}(u)$ stand for the error at two consecutive triangulations with N and $\tilde{N}$ degrees of freedom, respectively, then we set

$$r_h(u) \,:=\, -2\,\frac{\log\left(\mathbf{e}_h(u)/\mathbf{e}_{\tilde{h}}(u)\right)}{\log\left(N/\tilde{N}\,\right)}\,,$$

and similarly for $r_0(\boldsymbol{\theta})$, $r_0(\boldsymbol{\sigma})$, and $r_0(u)$.

Our numerical results were obtained in a *Compaq Alpha ES40 parallel computer* using a Matlab code. The LDG scheme (3.3.7), which becomes a nonlinear algebraic system with N unknowns, is solved by Newton's method with the initial guess given by the solution of the associated Poisson problem, and a tolerance of 10^{-10} for the corresponding residual.

We present six examples for the nonlinear boundary value problem (3.1.1). In all cases we take the parameters $\widehat{\alpha} = 1$ and $\boldsymbol{\beta} = (1,1)^{\mathbf{T}}$ in the primal formulation (3.3.7). Also, for the computations we consider a sequence of structured meshes in which each one is obtained from a global refinement of the previous mesh by dividing every triangle into four similar triangles. The first four examples use the nonlinear coefficient $\mathbf{a}(\cdot,\boldsymbol{\zeta}) := \left(2 + \frac{1}{1+\|\boldsymbol{\zeta}\|}\right)\boldsymbol{\zeta}$, while Examples 5 and 6 consider $\mathbf{a}(\cdot,\boldsymbol{\zeta}) := \left(1 + \mathrm{e}^{-\|\boldsymbol{\zeta}\|^2}\right)\boldsymbol{\zeta}$, for all $\boldsymbol{\zeta} \in \mathbf{R}^2$. It is not difficult to show that both mappings satisfy the hypotheses (H.1), (H.2), (H.3), and (H.4).

In Example 1 we consider the convex domain $\Omega :=]-1,1[^2$, with $\Gamma_D = \Big([-1,1] \times \{-1\}\Big) \cup \Big(\{1\} \times [-1,1]\Big)$, and $\Gamma_N = \Big([-1,1] \times \{1\}\Big) \cup \Big(\{-1\} \times [-1,1]\Big)$. Then we choose the data f, g_D and g_N so that the solution u is given by the smooth function $u(x_1,x_2) = \cos\left(\dfrac{\pi}{2}\,x_1\right)\,\cos\left(\dfrac{\pi}{2}\,x_2\right)$.

In the second example we consider the L-shaped domain $\Omega :=]-1,1[^2 \backslash ([0,1] \times [-1,0])$, which is clearly non-convex. Then we take $\Gamma_D = \partial\Omega$, and choose the data f and g_D so that the solution u is the non-smooth function given (in polar coordinates) by $u(r,\alpha) = r^{2/3}\,\sin\left(\dfrac{2}{3}\,\alpha\right)$. Because of the power of r, we notice that the partial derivatives of u are singular at the origin.

In Example 3 we consider again the L-shaped domain $\Omega :=]-1,1[^2 \backslash ([0,1]\times[-1,0])$ with $\Gamma_D = \partial\Omega$, and choose the data f and g_D so that the solution u is the smooth function given by $u(x_1,x_2) = \cos\left(\frac{\pi}{2}\,x_2\right) + \chi(x_1)\,x_1^{4.5}$, where χ is the characteristic function on $[0,1]$. It can be proved that this solution belongs to $H^5(\Omega)$.

Example 4 refers to a case not fully covered by our theory for non-convex domains. Indeed, we take the L-shaped domain $\Omega :=]-1,1[^2 \backslash ([0,1]\times[-1,0])$, and consider mixed boundary conditions with $\Gamma_D = ([-1,1]\times\{-1\}) \cup (\{1\}\times[-1,1])$, and $\Gamma_N = ([-1,1]\times\{1\}) \cup (\{-1\}\times[-1,1])$. The data f, g_D and g_N are chosen such that the solution u is given again by $u(x_1,x_2) = \cos\left(\frac{\pi}{2}\,x_2\right) + \chi(x_1)\,x_1^{4.5}$.

Finally, in Examples 5 and 6 we take the other nonlinear mapping $\mathbf{a}$. Example 5 considers the same geometry, boundary conditions, and solution of Example 1, while Example 6 considers those of Example 2. The data of all the examples are summarized in Table 3.7.0 shown below.

In Tables 3.7.1 up to 3.7.6 we give the individual errors and the corresponding experimental rates of convergence for the uniform refinements of the 6 examples. We performed the computations with $\mathbf{P}_1 - \mathbf{P}_1 - \mathbf{P}_1$ and $\mathbf{P}_2 - \mathbf{P}_1 - \mathbf{P}_1$ approximations for the unknowns u, $\boldsymbol{\theta}$, and $\boldsymbol{\sigma}$, respectively. In addition, Figures 3.7.1 up to 3.7.6 display the individual errors versus the degrees of freedom in a log-log scale. All the errors are computed on each triangle using a 7 points Gaussian quadrature rule. The number of Newton iterations needed to attain the prescribed tolerance is ≤ 4 in all the examples presented here.

Table 3.7.0: Summary of data for the 6 examples.

Ex.	Mapping $\mathbf{a}$	Domain Ω	Data on $\partial\Omega$	Solution u
1	$\left(2+\frac{1}{1+\lVert\boldsymbol{\zeta}\rVert}\right)\boldsymbol{\zeta}$	Square	Mixed	$\cos\left(\frac{\pi}{2}\,x_1\right)\,\cos\left(\frac{\pi}{2}\,x_2\right)$
2	$\left(2+\frac{1}{1+\lVert\boldsymbol{\zeta}\rVert}\right)\boldsymbol{\zeta}$	*L*-shaped	Dirichlet	$r^{2/3}\,\sin\left(\frac{2}{3}\,\alpha\right)$
3	$\left(2+\frac{1}{1+\lVert\boldsymbol{\zeta}\rVert}\right)\boldsymbol{\zeta}$	*L*-shaped	Dirichlet	$\cos\left(\frac{\pi}{2}\,x_2\right) + \chi(x_1)\,x_1^{4.5}$
4	$\left(2+\frac{1}{1+\lVert\boldsymbol{\zeta}\rVert}\right)\boldsymbol{\zeta}$	*L*-shaped	Mixed	$\cos\left(\frac{\pi}{2}\,x_2\right) + \chi(x_1)\,x_1^{4.5}$
5	$\left(1+\mathrm{e}^{-\lVert\boldsymbol{\zeta}\rVert^2}\right)\boldsymbol{\zeta}$	Square	Mixed	$\cos\left(\frac{\pi}{2}\,x_1\right)\,\cos\left(\frac{\pi}{2}\,x_2\right)$
6	$\left(1+\mathrm{e}^{-\lVert\boldsymbol{\zeta}\rVert^2}\right)\boldsymbol{\zeta}$	*L*-shaped	Dirichlet	$r^{2/3}\,\sin\left(\frac{2}{3}\,\alpha\right)$

We notice that the orders of convergence predicted by the theory (at least) are achieved in every case. This is certainly valid for a smooth solution in a convex region, as seen in Tables 3.7.1, 3.7.5 and Figures 3.7.1, 3.7.5, where the orders for $(\mathbf{e}_h(u), \mathbf{e}_0(\boldsymbol{\theta}), \mathbf{e}_0(\boldsymbol{\sigma}), \mathbf{e}_0(u))$ are (h, h, h, h^2) and (h^2, h^2, h^2, h^3), respectively, with $\mathbf{P}_1 - \mathbf{P}_1 - \mathbf{P}_1$ and $\mathbf{P}_2 - \mathbf{P}_1 - \mathbf{P}_1$ approximations. On the other hand, for the non-convex region given by the L-shaped domain we find that the maximum interior angle is $w = 3\pi/2$, and hence the elliptic regularity constant γ becomes $2/3$. Then, the results provided by Theorems 3.6.2, 3.6.3, and 3.6.4 predict orders $(h^\gamma, h^\gamma, h^\gamma, h^{2\gamma})$ for $(\mathbf{e}_h(u), \mathbf{e}_0(\boldsymbol{\theta}), \mathbf{e}_0(\boldsymbol{\sigma}), \mathbf{e}_0(u))$, independently of the polynomial degrees used. This is confirmed by the non-smooth solution of Examples 2 and 6, as observed in Tables 3.7.2, 3.7.6 and Figures 3.7.2, 3.7.6, where the orders lie around $(h^{2/3}, h^{2/3}, h^{2/3}, h^{4/3})$ for $\mathbf{P}_1 - \mathbf{P}_1 - \mathbf{P}_1$ and $\mathbf{P}_2 - \mathbf{P}_1 - \mathbf{P}_1$ approximations. However, in Example 3, where the domain Ω is again the L-shaped region but the solution is now smooth, we obtain almost the same orders of convergence as in the convex case. This can be seen in Table 3.7.3 and Figure 3.7.3, and it is particularly observed with the $\mathbf{P}_2 - \mathbf{P}_1 - \mathbf{P}_1$ polynomial approximation. Further, as noticed from Table 3.7.4 and Figure 3.7.4, a similar phenomenon takes place with the same smooth solution in Example 4, but now with mixed boundary conditions. The numerical results provided by this example support the conjecture that the present method might behave quite well even in a case not fully covered by the theoretical results.

Table 3.7.1: $\mathbf{P}_1 - \mathbf{P}_1 - \mathbf{P}_1$ and $\mathbf{P}_2 - \mathbf{P}_1 - \mathbf{P}_1$ approximations (Example 1).

N	$\mathbf{e}_h(u)$	$r_h(u)$	$\mathbf{e}_0(\boldsymbol{\theta})$	$r_0(\boldsymbol{\theta})$	$\mathbf{e}_0(\boldsymbol{\sigma})$	$r_0(\boldsymbol{\sigma})$	$\mathbf{e}_0(u)$	$r_0(u)$
120	0.9855	——	0.6809	——	1.6143	——	0.1322	——
480	0.6238	0.6596	0.3395	1.0038	0.8186	0.9796	0.0475	1.4775
1920	0.3383	0.8829	0.1724	0.9780	0.4185	0.9678	0.0126	1.9145
7980	0.0172	0.9449	0.0876	0.9506	0.2142	0.9404	0.0031	1.9565
30720	0.0087	1.0190	0.0443	1.0113	0.1088	1.0053	0.0008	2.0708
122880	0.0044	0.9965	0.0223	0.9903	0.0548	0.9878	0.0002	2.0078
144	8.6093	——	0.5199	——	1.2164	——	1.0539	——
576	1.7314	2.3139	0.1235	2.0733	0.2979	2.0295	0.1032	2.7785
2304	0.4085	2.0836	0.0301	2.0389	0.0729	2.0316	0.0121	2.9326
9216	0.1009	2.0173	0.0074	2.0245	0.0179	2.0224	0.0015	3.0039
36864	0.0251	2.0056	0.0018	2.0130	0.0045	2.0124	0.0002	2.9862
147456	0.0063	2.0016	0.0005	2.0064	0.0011	2.0063	0.0000	2.9920

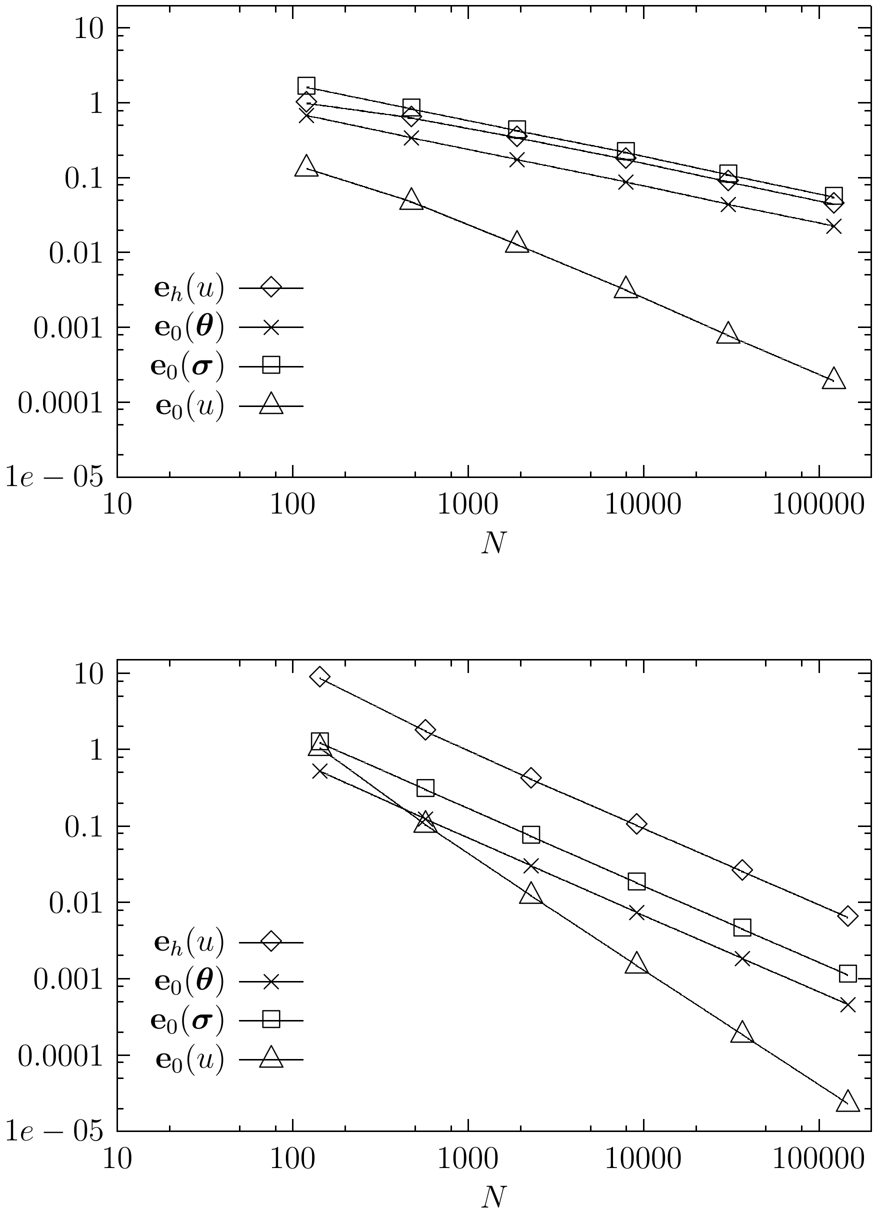

Figure 3.7.1: $\mathbf{P}_1-\mathbf{P}_1-\mathbf{P}_1$ and $\mathbf{P}_2-\mathbf{P}_1-\mathbf{P}_1$ approximations (Example 1).

Table 3.7.2: $\mathbf{P}_1 - \mathbf{P}_1 - \mathbf{P}_1$ and $\mathbf{P}_2 - \mathbf{P}_1 - \mathbf{P}_1$ approximations (Example 2).

N	$\mathbf{e}_h(u)$	$r_h(u)$	$\mathbf{e}_0(\boldsymbol{\theta})$	$r_0(\boldsymbol{\theta})$	$\mathbf{e}_0(\boldsymbol{\sigma})$	$r_0(\boldsymbol{\sigma})$	$\mathbf{e}_0(u)$	$r_0(u)$
90	0.3024	——	0.1618	——	0.3815	——	0.0350	——
360	0.2241	0.4324	0.1267	0.3530	0.2924	0.3840	0.0135	1.3744
1440	0.1506	0.5728	0.0837	0.5971	0.1899	0.6227	0.0046	1.5522
5760	0.0983	0.6160	0.0541	0.6302	0.1206	0.6551	0.0015	1.6050
23040	0.0632	0.6368	0.0346	0.6452	0.0758	0.6687	0.0004	1.6168
92160	0.0403	0.6487	0.0220	0.6533	0.0475	0.6749	0.0001	1.6132
108	0.1877	——	0.1652	——	0.3917	——	0.0219	——
432	0.1218	0.6247	0.1062	0.6374	0.2462	0.6699	0.0085	1.3583
1728	0.0753	0.6933	0.0673	0.6584	0.1526	0.6906	0.0032	1.4012
6912	0.0461	0.7066	0.0425	0.6626	0.0944	0.6922	0.0012	1.3880
27648	0.0282	0.7119	0.0268	0.6635	0.0585	0.6898	0.0005	1.3707
110592	0.0172	0.7116	0.0169	0.6638	0.0364	0.6865	0.0002	1.3577

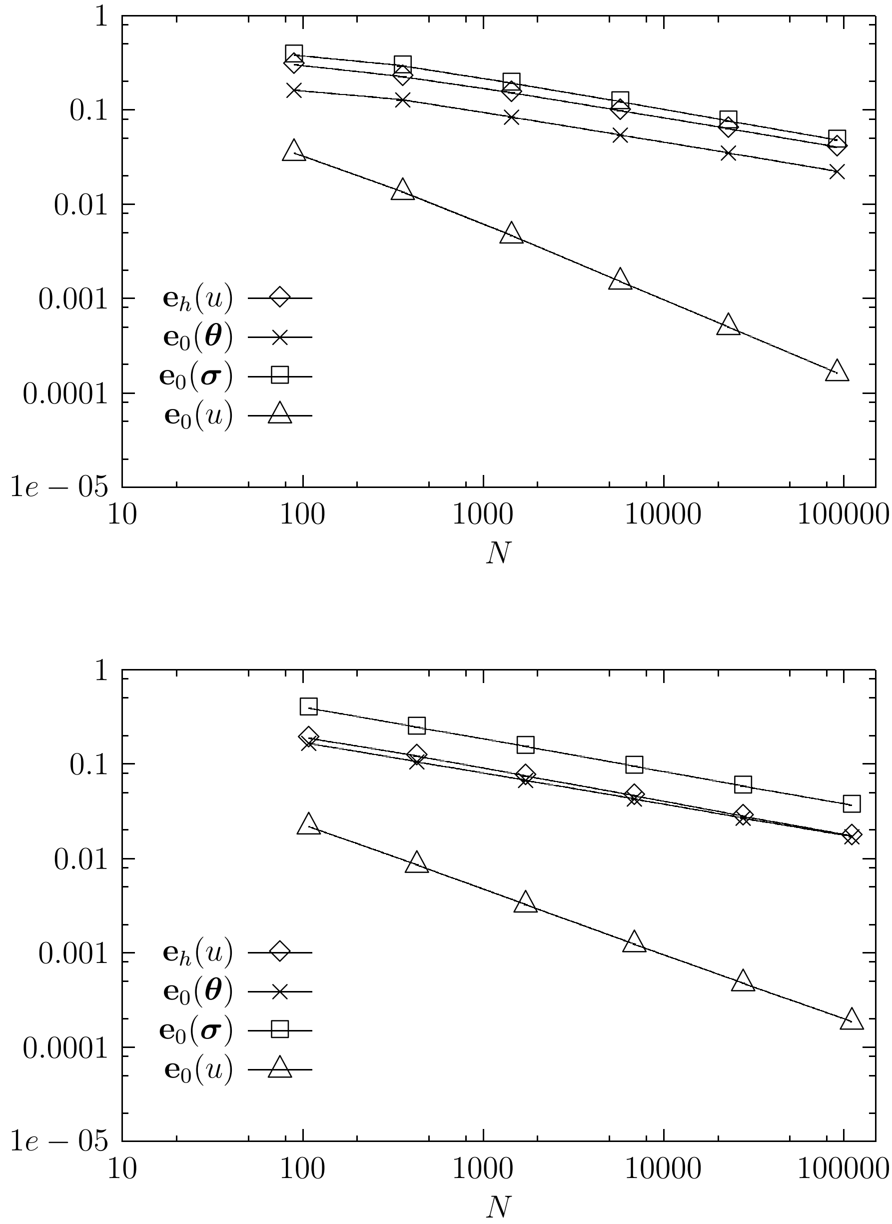

Figure 3.7.2: $\mathbf{P}_1 - \mathbf{P}_1 - \mathbf{P}_1$ and $\mathbf{P}_2 - \mathbf{P}_1 - \mathbf{P}_1$ approximations (Example 2).

Table 3.7.3: $\mathbf{P}_1 - \mathbf{P}_1 - \mathbf{P}_1$ and $\mathbf{P}_2 - \mathbf{P}_1 - \mathbf{P}_1$ approximations (Example 3).

N	$\mathbf{e}_h(u)$	$r_h(u)$	$\mathbf{e}_0(\boldsymbol{\theta})$	$r_0(\boldsymbol{\theta})$	$\mathbf{e}_0(\boldsymbol{\sigma})$	$r_0(\boldsymbol{\sigma})$	$\mathbf{e}_0(u)$	$r_0(u)$
90	2.3929	——	0.6304	——	1.4082	——	0.2992	——
360	1.0608	1.1736	0.2295	1.4580	0.5154	1.4501	0.0635	2.2356
1440	0.5150	1.0423	0.0769	1.5778	0.1785	1.5295	0.0154	2.0422
5760	0.2601	0.9855	0.0408	0.9145	0.0939	0.9277	0.0039	1.9583
23040	0.1314	0.9849	0.0246	0.7268	0.0552	0.7661	0.0010	1.9631
92160	0.0661	0.9909	0.0138	0.8378	0.0305	0.8568	0.0003	1.9780
108	7.6869	——	0.4829	——	1.0562	——	0.9189	——
432	1.9057	2.0121	0.1539	1.6503	0.3184	1.7298	0.1171	2.9716
1728	0.4887	1.9472	0.0408	1.8977	0.0846	1.8967	0.0151	2.9275
6912	0.0127	1.9596	0.0103	2.0038	0.0213	2.0055	0.0019	2.9763
27648	0.0033	1.9568	0.0026	2.0025	0.0053	2.0023	0.0003	2.9665
110592	0.0008	1.9745	0.0006	2.0039	0.0013	2.0035	0.0000	2.9810

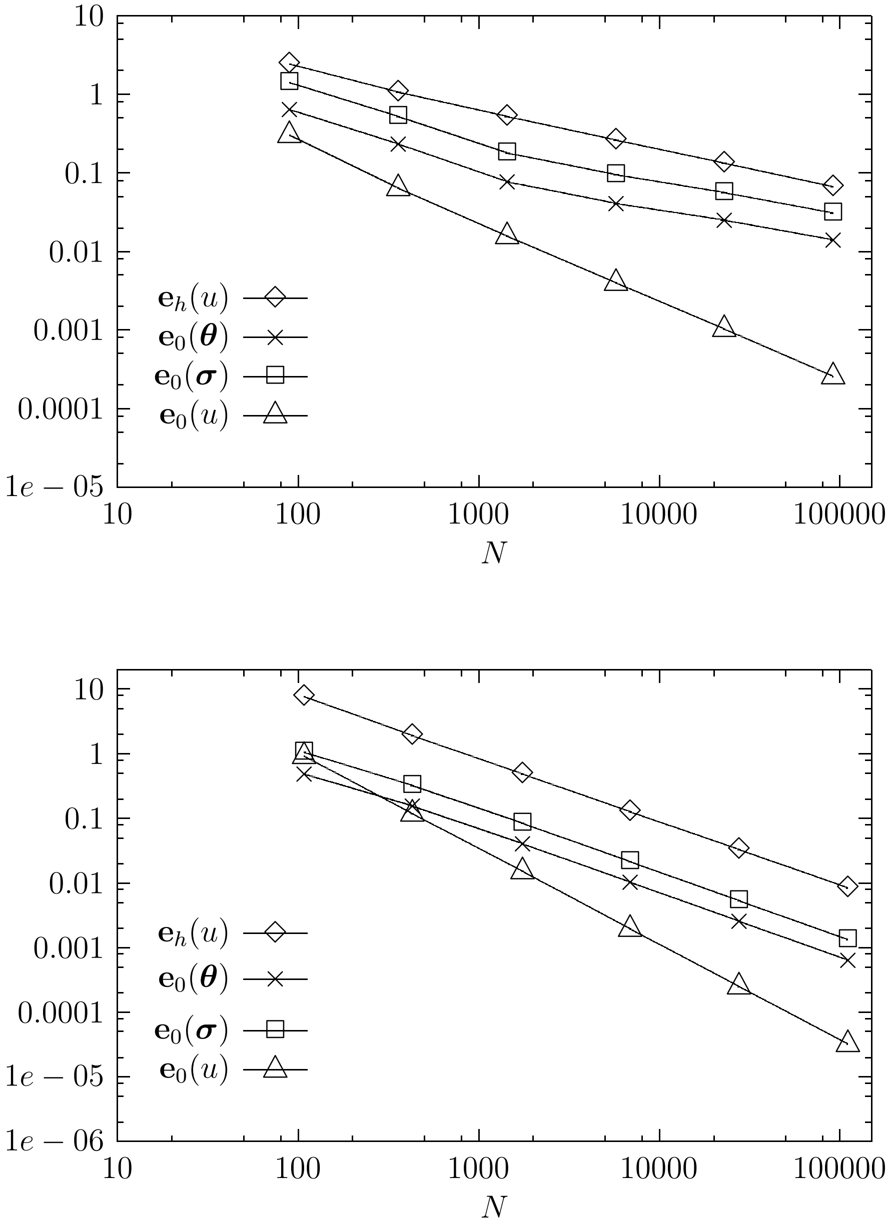

Figure 3.7.3: $\mathbf{P}_1 - \mathbf{P}_1 - \mathbf{P}_1$ and $\mathbf{P}_2 - \mathbf{P}_1 - \mathbf{P}_1$ approximations (Example 3).

Table 3.7.4: $\mathbf{P}_1 - \mathbf{P}_1 - \mathbf{P}_1$ and $\mathbf{P}_2 - \mathbf{P}_1 - \mathbf{P}_1$ approximations (Example 4).

N	$\mathbf{e}_h(u)$	$r_h(u)$	$\mathbf{e}_0(\boldsymbol{\theta})$	$r_0(\boldsymbol{\theta})$	$\mathbf{e}_0(\boldsymbol{\sigma})$	$r_0(\boldsymbol{\sigma})$	$\mathbf{e}_0(u)$	$r_0(u)$
90	1.7706	——	0.7701	——	1.7361	——	0.2112	——
360	1.0018	0.8216	0.2484	1.6323	0.5604	1.6312	0.0596	1.8251
1440	0.5127	0.9663	0.0788	1.6561	0.1829	1.6157	0.0155	1.9411
5760	0.2596	0.9818	0.0423	0.8984	0.0969	0.9164	0.0040	1.9534
23040	0.1312	0.9848	0.0252	0.7455	0.0564	0.7817	0.0010	1.9630
92160	0.0604	0.9900	0.0140	0.8535	0.0308	0.8707	0.0003	1.9768
108	7.0228	——	0.5276	——	1.1650	——	0.8467	——
432	1.9960	1.8149	0.1604	1.7179	0.3350	1.7979	0.1234	2.7785
1728	0.5128	1.9445	0.0417	1.9268	0.0868	1.9332	0.0159	2.9326
6912	0.1309	1.9861	0.0104	2.0197	0.0216	2.0246	0.0020	3.0039
27648	0.0333	1.9767	0.0026	2.0100	0.0054	2.0112	0.0003	2.9862
110592	0.0084	1.9858	0.0006	2.0075	0.0013	2.0079	0.0000	2.9920

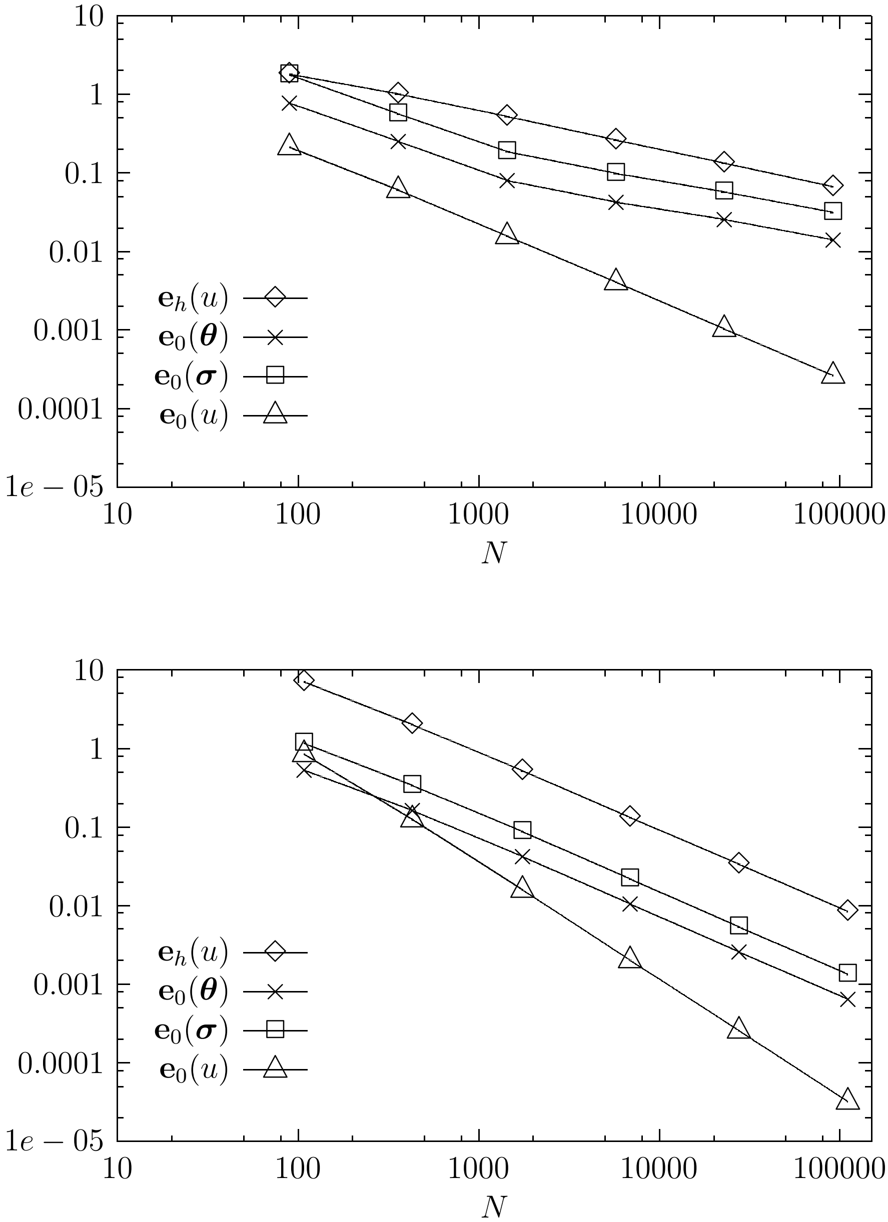

Figure 3.7.4: $\mathbf{P}_1 - \mathbf{P}_1 - \mathbf{P}_1$ and $\mathbf{P}_2 - \mathbf{P}_1 - \mathbf{P}_1$ approximations (Example 4).

Table 3.7.5: $\mathbf{P}_1 - \mathbf{P}_1 - \mathbf{P}_1$ and $\mathbf{P}_2 - \mathbf{P}_1 - \mathbf{P}_1$ approximations (Example 5).

N	$\mathbf{e}_h(u)$	$r_h(u)$	$\mathbf{e}_0(\boldsymbol{\theta})$	$r_0(\boldsymbol{\theta})$	$\mathbf{e}_0(\boldsymbol{\sigma})$	$r_0(\boldsymbol{\sigma})$	$\mathbf{e}_0(u)$	$r_0(u)$
120	0.9330	——	0.7128	——	0.7428	——	0.1329	——
480	0.6122	0.6080	0.3496	1.0279	0.4123	0.8492	0.0531	1.3240
1920	0.3337	0.8755	0.1771	0.9809	0.2075	0.9906	0.0143	1.8908
7980	0.1710	0.9388	0.0896	0.9568	0.1086	0.9090	0.0036	1.9474
30720	0.0861	1.0169	0.0452	1.0145	0.0559	0.9848	0.0009	2.0658
122880	0.0432	0.9955	0.0227	0.9920	0.0283	0.9804	0.0002	2.0063
144	5.4104	——	0.5748	——	0.5611	——	0.6192	——
576	1.2972	2.0604	0.1533	1.9064	0.1910	1.5546	0.0780	2.9892
2304	0.2836	2.1933	0.0347	2.1429	0.0456	2.0661	0.0085	3.2057
9216	0.0665	2.0931	0.0084	2.0451	0.0111	2.0387	0.0010	3.1273
36864	0.0163	2.0282	0.0021	2.0280	0.0028	2.0131	0.0001	3.0445
147456	0.0041	2.0098	0.0005	2.0133	0.0007	2.0059	0.0000	3.0159

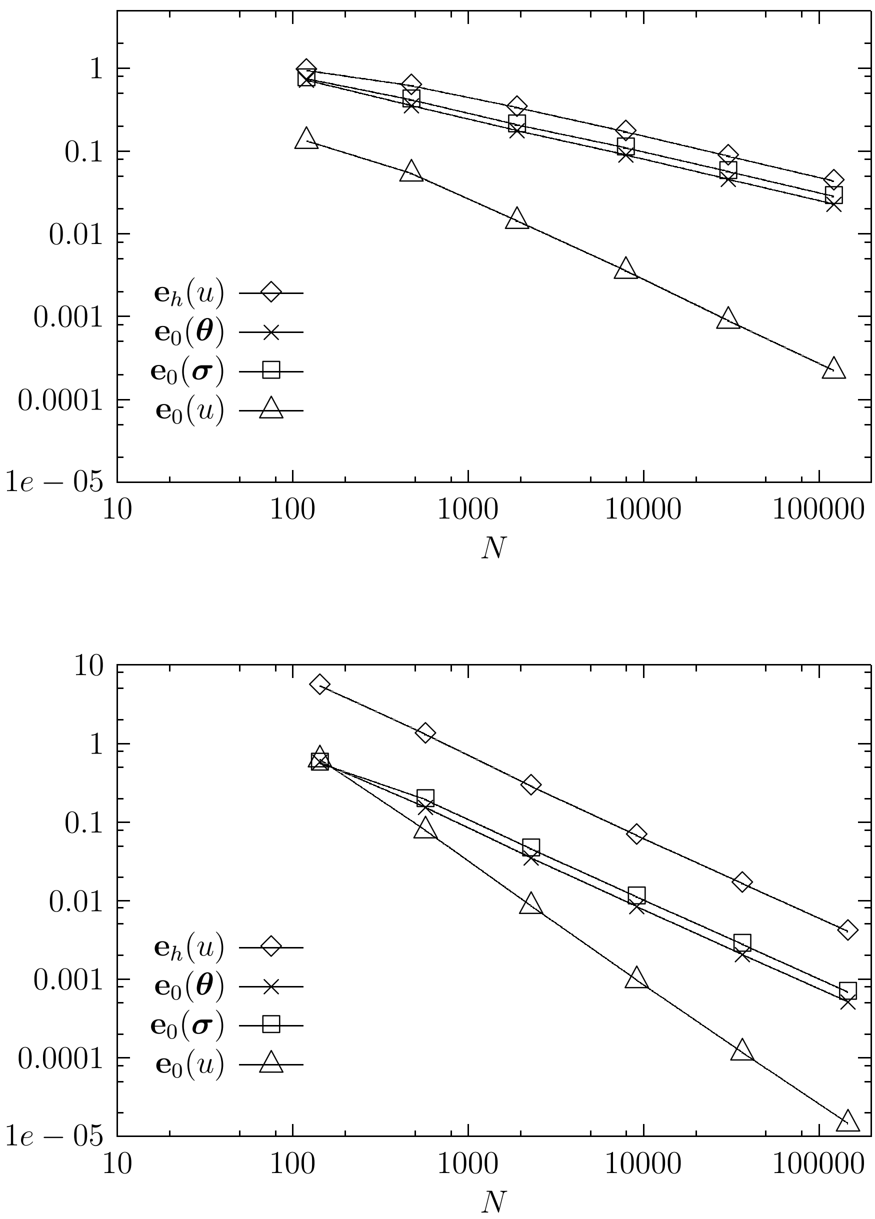

Figure 3.7.5: $\mathbf{P}_1 - \mathbf{P}_1 - \mathbf{P}_1$ and $\mathbf{P}_2 - \mathbf{P}_1 - \mathbf{P}_1$ approximations (Example 5).

Table 3.7.6: $\mathbf{P}_1 - \mathbf{P}_1 - \mathbf{P}_1$ and $\mathbf{P}_2 - \mathbf{P}_1 - \mathbf{P}_1$ approximations (Example 6).

N	$\mathbf{e}_h(u)$	$r_h(u)$	$\mathbf{e}_0(\boldsymbol{\theta})$	$r_0(\boldsymbol{\theta})$	$\mathbf{e}_0(\boldsymbol{\sigma})$	$r_0(\boldsymbol{\sigma})$	$\mathbf{e}_0(u)$	$r_0(u)$
90	0.3123	——	0.1440	——	0.1392	——	0.0364	——
360	0.2194	0.5092	0.1247	0.2074	0.1104	0.3355	0.0132	1.4679
1440	0.1473	0.5748	0.0858	0.5406	0.0728	0.6010	0.0045	1.5571
5760	0.0961	0.6162	0.0561	0.6114	0.0494	0.5593	0.0015	1.6007
23040	0.0618	0.6369	0.0354	0.6656	0.0327	0.5934	0.0005	1.6546
92160	0.0394	0.6480	0.0221	0.6788	0.0211	0.6320	0.0001	1.6559
108	0.3915	——	0.1438	——	0.1417	——	0.0471	——
432	0.2071	0.9185	0.0998	0.5271	0.0869	0.7050	0.0126	1.9071
1728	0.0817	1.3417	0.0687	0.5394	0.0586	0.5701	0.0037	1.7753
6912	0.0412	0.9883	0.0446	0.6233	0.0410	0.5146	0.0014	1.4095
27648	0.0259	0.6698	0.0275	0.6979	0.0268	0.6107	0.0005	1.5078
110592	0.0156	0.7277	0.0169	0.7027	0.0168	0.6761	0.0002	1.4164

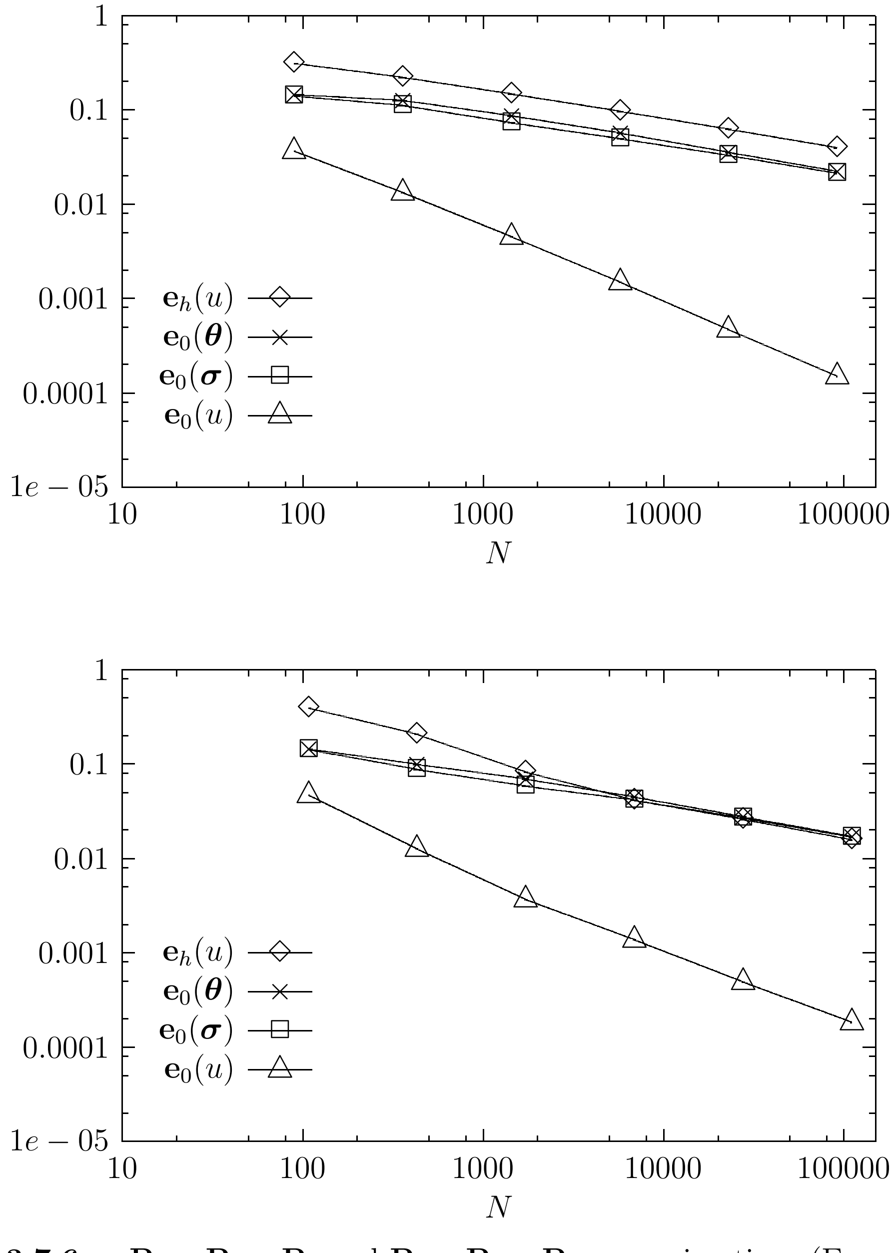

Figure 3.7.6: $\mathbf{P}_1 - \mathbf{P}_1 - \mathbf{P}_1$ and $\mathbf{P}_2 - \mathbf{P}_1 - \mathbf{P}_1$ approximations (Example 6).

Acknowledgements. The authors are grateful to Professor Bernardo Cockburn for valuable discussions and for leading to reference [72].

Chapter 4

An a-posteriori error estimate for the local discontinuous Galerkin method applied to linear and nonlinear diffusion problems

In this chapter we present a modified residual-based reliable a-posteriori error estimator for the local discontinuous Galerkin approximations of linear and nonlinear diffusion problems in polygonal regions of $\mathbf{R}^2$. Our analysis, which applies to convex and non-convex domains, is based on Helmholtz decompositions of the error and a suitable auxiliary polynomial function interpolating the Dirichlet datum. Several examples confirming the reliability of the estimator and providing numerical evidences for its efficiency are given. Furthermore, the associated adaptive method, which considers meshes with and without hanging nodes, is shown to be much more efficient than a uniform refinement to compute the discrete solutions. In particular, the experiments illustrate the ability of the adaptive algorithm to localize the singularities of each problem.

4.1 Introduction

The local discontinuous Galerkin method (LDG) has been frequently used in recent years to solve diffusion dominated and purely elliptic linear problems (see, e.g. [38], [40], [45], [83] and [84]). Moreover, the analysis in [83] was extended in [32] to a class of nonlinear diffusion problems with mixed boundary conditions. As it is well known, the main advantages of the LDG are the high order of approximation provided, the high degree of parallelism involved, and its great suitability for h, p, and hp refinements. In particular, the latter advantage makes the LDG very attractive to be used in combination with adaptive algorithms, and hence the need of corresponding a-posteriori error estimators providing local information on where to refine becomes evident. However, up to the authors' knowledge, no much has been done in connection with a-posteriori error analysis for discontinuous Galerkin methods. We may mention [23] and [88], where residual estimators for the L^2-norm of the error and implicit estimators based on local problems for the energy norm of the error are provided. More recently, a residual-based reliable a-posteriori error estimate for a mesh dependent energy norm of the error is presented in [22] for a general family of discontinuous Galerkin methods. The procedure from [22] follows known techniques from mixed finite element methods and nonconforming schemes, so that it relies on a Helmholtz decomposition of the gradient of the error and applies to nonconvex polyhedra domains in two and three dimensions. Furthermore, the approach in [22] is valid for any other conservative method.

In this chapter we derive a new explicit and reliable a-posteriori error estimate for the LDG applied to second order linear elliptic equations in divergence form. In addition, we extend the results to the nonlinear diffusion problems studied in [32]. Similarly as in [22], our analysis also makes use of Helmholtz decompositions, but in contrast to that work, which requires certain polynomial behaviour of the Dirichlet datum, we consider here a suitable piecewise polynomial function interpolating that datum. The rest of the chapter is organized as follows. In Section 4.2 we introduce the linear model problem and describe the main aspects of the local discontinuous Galerkin scheme, which includes the definition of the numerical fluxes and the associated primal formulation. The corresponding a-posteriori error analysis is presented in Section 4.3. Then, in Section 4.4 we extend our results from Section 4.3 to the class of nonlinear elliptic problems studied in [32]. Finally, several numerical results confirming the reliability of the estimator, providing evidences for its efficiency, and illustrating the good performance of the associated adaptive algorithm, are reported in Section 4.5.

Throughout this chapter, c and C, with or without subscripts, denote positive constants, independent of the parameters and functions involved, which may take different values at different ocurrences.

4.2 The linear model problem and its primal formulation

We let Ω be a bounded and simply connected domain in $\mathbf{R}^2$ with polygonal boundary Γ, and let $\Gamma_D \neq \phi$ and Γ_N be parts of Γ such that $\Gamma = \bar{\Gamma}_D \cup \bar{\Gamma}_N$ and $\Gamma_D \cap \Gamma_N = \phi$. Then, given $f \in [L^2(\Omega)]^2$, $g_D \in H^{1/2}(\Gamma_D)$, $g_N \in L^2(\Gamma_N)$, and a symmetric matrix valued function $\boldsymbol{\kappa} \in C(\bar{\Omega})$, we look for $u \in H^1(\Omega)$ such that

$$\begin{gathered} -\mathrm{div}\,(\boldsymbol{\kappa}\,\nabla u(\cdot)) = f \quad \text{in} \quad \Omega\,, \\ u = g_D \quad \text{on} \quad \Gamma_D \quad \text{and} \quad (\boldsymbol{\kappa}\,\nabla u(\cdot)) \cdot \boldsymbol{\nu} = g_N \quad \text{on} \quad \Gamma_N\,, \end{gathered} \tag{4.2.1}$$

where div is the usual divergence operator and $\boldsymbol{\nu}$ denotes the unit outward normal to $\partial\Omega$. We assume that $\boldsymbol{\kappa}$ induces a strongly elliptic differential operator, that is there exists $C > 0$ such that

$$C\,|\boldsymbol{\zeta}|^2 \;\leq\; (\boldsymbol{\kappa}(x)\,\boldsymbol{\zeta}) \cdot \boldsymbol{\zeta} \quad \forall\,\boldsymbol{\zeta} \in \mathbf{R}^2\,, \quad \forall\, x \in \Omega\,. \tag{4.2.2}$$

A detailed definition of the primal formulation of (4.2.1) (in the general nonlinear case) was given in [32], and hence we just recall here the main aspects of it. Indeed, we follow [40] (see also [56], [59], [83]) and introduce the gradient $\boldsymbol{\theta} := \nabla u$ in Ω and the flux $\boldsymbol{\sigma} := \boldsymbol{\kappa}\,\boldsymbol{\theta}$ in Ω as additional unknowns. In this way, (4.2.1) can be reformulated as the following problem in $\bar{\Omega}$: *Find* $(\boldsymbol{\theta}, \boldsymbol{\sigma}, u)$ *in appropriate spaces such that, in the distributional sense,*

$$\begin{gathered} \boldsymbol{\theta} = \nabla u \quad \text{in} \quad \Omega\,, \quad \boldsymbol{\sigma} = \boldsymbol{\kappa}\,\boldsymbol{\theta} \quad \text{in} \quad \Omega\,, \quad -\mathrm{div}\,\boldsymbol{\sigma} = f \quad \text{in} \quad \Omega\,, \\ u = g_D \quad \text{on} \quad \Gamma_D\,, \quad \text{and} \quad \boldsymbol{\sigma} \cdot \boldsymbol{\nu} = g_N \quad \text{on} \quad \Gamma_N. \end{gathered} \tag{4.2.3}$$

We now let $\mathcal{T}_h$ be a shape-regular triangulation of $\bar{\Omega}$ (with possible hanging nodes) made up of straight triangles T with diameter h_T and unit outward normal to ∂T given by $\boldsymbol{\nu}_T$. As usual, the index h also denotes $h := \max\limits_{T \in \mathcal{T}_h} h_T$. In addition, we define the edges of $\mathcal{T}_h$ as follows. An *interior edge of* $\mathcal{T}_h$ *is the* (non-empty) interior of

$\partial T \cap \partial T'$, where T and T' are two adjacent elements of $\mathcal{T}_h$, not necessarily matching. Similarly, a *boundary edge* of $\mathcal{T}_h$ is the (non-empty) interior of $\partial T \cap \partial \Omega$, where T is a boundary element of $\mathcal{T}_h$. We denote by Γ_I the union of all interior edges of $\mathcal{T}_h$, and set $\mathcal{E} := \Gamma_I \cup \Gamma_D \cup \Gamma_N$ the union of all edges of $\mathcal{T}_h$. Further, for each edge $e \in \mathcal{E}$, h_e represents its diameter. Also, in what follows we assume that $\mathcal{T}_h$ is of *bounded variation*, that is there exists a constant $l > 1$, independent of the meshsize h, such that $l^{-1} \leq \frac{h_T}{h_{T'}} \leq l$ for each pair T, $T' \in \mathcal{T}_h$ sharing an interior edge.

Then, we want to approximate the exact solution $(\boldsymbol{\theta}, \boldsymbol{\sigma}, u)$ by discrete functions $(\boldsymbol{\theta}_h, \boldsymbol{\sigma}_h, u_h)$ in the finite element space $\boldsymbol{\Sigma}_h \times \boldsymbol{\Sigma}_h \times V_h$, where

$$\boldsymbol{\Sigma}_h := \Big\{ \boldsymbol{\theta}_h \in [L^2(\Omega)]^2 : \quad \boldsymbol{\theta}_h|_T \in [\mathbf{P}_r(T)]^2 \quad \forall\, T \in \mathcal{T}_h \Big\},$$

$$V_h := \Big\{ v_h \in L^2(\Omega) : \quad v_h|_T \in \mathbf{P}_k(T) \quad \forall\, T \in \mathcal{T}_h \Big\},$$

with $k \geq 1$ and $r = k$ or $r = k-1$. Hereafter, given an integer $s \geq 0$, we denote by $\mathbf{P}_s(T)$ the space of polynomial functions of degree at most $s \geq 1$ on T. From now on, the space $\boldsymbol{\Sigma}_h$ is provided with the usual product norm of $[L^2(\Omega)]^2$, which is denoted by $\| \cdot \|_{[L^2(\Omega)]^2}$. The norm for V_h will be defined at the end of the current section.

The idea of the LDG method is to enforce the local conservation laws with the traces of $\boldsymbol{\sigma}$ and u on the boundary of each $T \in \mathcal{T}_h$ being replaced by suitable numerical approximations $\widehat{\boldsymbol{\sigma}}$ and $\widehat{u}$, respectively, which are named *numerical fluxes*. In other words, proceeding as in [83], we consider the following formulation: *Find* $(\boldsymbol{\theta}_h, \boldsymbol{\sigma}_h, u_h) \in \boldsymbol{\Sigma}_h \times \boldsymbol{\Sigma}_h \times V_h$ *such that for each* $T \in \mathcal{T}_h$ *we have*

$$\begin{aligned}
&\int_T \boldsymbol{\kappa}\, \boldsymbol{\theta}_h \cdot \boldsymbol{\zeta}_h \, dx - \int_T \boldsymbol{\sigma}_h \cdot \boldsymbol{\zeta}_h \, dx && = 0 && \forall\, \boldsymbol{\zeta}_h \in \boldsymbol{\Sigma}_h \,, \\
&\int_T \boldsymbol{\theta}_h \cdot \boldsymbol{\tau}_h \, dx + \int_T u_h \operatorname{div} \boldsymbol{\tau}_h \, dx - \int_{\partial T} \widehat{u}\, \boldsymbol{\tau}_h \cdot \boldsymbol{\nu}_T \, ds && = 0 && \forall\, \boldsymbol{\tau}_h \in \boldsymbol{\Sigma}_h \,, \\
&\int_T \boldsymbol{\sigma}_h \cdot \nabla v_h \, dx - \int_{\partial T} v_h\, \widehat{\boldsymbol{\sigma}} \cdot \boldsymbol{\nu}_T \, ds && = \int_T f v_h \, dx && \forall\, v_h \in V_h \,,
\end{aligned} \tag{4.2.4}$$

where $\widehat{u}$ and $\widehat{\boldsymbol{\sigma}}$, which usually depend on u_h, $\boldsymbol{\sigma}_h$, and the boundary conditions, are chosen so that some compatibility conditions are satisfied. In fact, following [32],

[83] and [38], we define $\widehat{u}$ and $\widehat{\boldsymbol{\sigma}}$ for each $T \in \mathcal{T}_h$, as follows:

$$\widehat{u}_{T,e} := \begin{cases} \{u_h\} + \boldsymbol{\beta} \cdot [\![u_h]\!] & \text{if } e \in \Gamma_I\,, \\ g_D & \text{if } e \in \Gamma_D\,, \\ u_h & \text{if } e \in \Gamma_N\,, \end{cases} \tag{4.2.5}$$

and

$$\widehat{\boldsymbol{\sigma}}_{T,e} := \begin{cases} \{\boldsymbol{\sigma}_h\} - \boldsymbol{\beta}[\![\boldsymbol{\sigma}_h]\!] - \alpha[\![u_h]\!] & \text{if } e \in \Gamma_I\,, \\ \boldsymbol{\sigma}_h - \alpha(u_h - g_D)\boldsymbol{\nu} & \text{if } e \in \Gamma_D\,, \quad , \\ \widehat{\boldsymbol{\sigma}} \cdot \boldsymbol{\nu} = g_N & \text{on } \Gamma_N\,, \end{cases} \tag{4.2.6}$$

where the auxiliary functions α (scalar) and $\boldsymbol{\beta}$ (vector), to be chosen appropriately, are single-valued on each edge $e \in \mathcal{E}$, and the averages $\{\,\}$ and jumps $[\![\,]\!]$ are defined in the usual way (see e.g. [8], [38] or [32]).

We now specify the norm associated to V_h. First, according to [32], we introduce the function $\mathtt{h}$ in $L^\infty(\mathcal{E})$, related to the local meshsize, as

$$\mathtt{h}(x) := \begin{cases} \min\{h_T, h_{T'}\} & \text{if } x \in \mathrm{int}(\partial T \cap \partial T') \\ h_T & \text{if } x \in \mathrm{int}(\partial T \cap \partial \Omega). \end{cases} \tag{4.2.7}$$

Then, we define $\alpha \in L^\infty(\mathcal{E})$ as $\alpha := \frac{\widehat{\alpha}}{\mathtt{h}}$, and consider $\boldsymbol{\beta} \in [L^\infty(\Gamma_I)]^2$ such that $\|\boldsymbol{\beta}\|_{[L^\infty(\Gamma_I)]^2} \le \widehat{\beta}$, where $\widehat{\alpha} > 0$ and $\widehat{\beta} \ge 0$ are independent of the meshsize. Next, we introduce the space $V(h) := V_h + H^1(\Omega)$, and define the energy norm $|||\cdot|||_h : V(h) \to \mathbf{R}$ and the associated semi-norm $|\cdot|_h : V(h) \to \mathbf{R}$, by

$$|||v|||_h^2 := \|\boldsymbol{\kappa}^{1/2}\,\nabla_h v\|^2_{[L^2(\Omega)]^2} + \|\alpha^{1/2}\,[\![v]\!]\|^2_{[L^2(\Gamma_I)]^2} + \|\alpha^{1/2}\,v\|^2_{L^2(\Gamma_D)} \quad \forall\, v \in V(h)\,, \tag{4.2.8}$$

and

$$|v|_h^2 := \|\alpha^{1/2}\,[\![v]\!]\|^2_{[L^2(\Gamma_I)]^2} + \|\alpha^{1/2}\,v\|^2_{L^2(\Gamma_D)} \quad \forall\, v \in V(h)\,, \tag{4.2.9}$$

where $\nabla_h v$ stands for the elementwise gradient of v on the triangles of $\mathcal{T}_h$.

Thus, proceeding as in [83] (see also [32]), we find that the primal formulation of (4.2.4) reduces to: *Find $u_h \in V_h$ such that*

$$[A_h(u_h), v_h] \;=\; [F_h, v_h] \qquad \forall\, v_h \in V_h\,, \tag{4.2.10}$$

where the linear operator $A_h : V(h) \to V(h)'$ and the functional $F_h : V(h) \to \mathbf{R}$, are given, respectively, by

$$[A_h(w), v] \;:=\; \int_\Omega \boldsymbol{\kappa}\,(\nabla_h w - \mathbf{S}(w))\cdot(\nabla_h v - \mathbf{S}(v))\,dx \;+\; \int_{\Gamma_I} \alpha\,[\![w]\!]\cdot[\![v]\!]\,ds + \int_{\Gamma_D} \alpha\,w\,v\,ds\,, \tag{4.2.11}$$

and

$$[F_h, v] \;:=\; \int_\Omega f\,v\,dx + \int_{\Gamma_D} \alpha\,g_D\,v\,ds + \int_{\Gamma_N} g_N\,v\,ds - \int_\Omega \boldsymbol{\kappa}\,\mathcal{G}_D\cdot(\nabla_h v - \mathbf{S}(v))\,dx\,, \tag{4.2.12}$$

for all $w,\, v \in V(h)$, where the linear operator $\mathbf{S} : V(h) \to \boldsymbol{\Sigma}_h$ and $\mathcal{G}_D \in \boldsymbol{\Sigma}_h$ are defined as in Section 3 of [32]. Hereafter, $[\cdot,\cdot]$ denotes the duality pairing induced by the operators and functionals used in each case. The existence and uniqueness of solution of (4.2.10), as well as the a-priori analysis of the error, are shown in [83] (see also [32]).

4.3 The a-posteriori error analysis

We begin with the Helmholtz decomposition of $\boldsymbol{\kappa}\,\nabla_h(u-u_h)$ (see also Lemma 2.1 in [35] or Lemma 3.1 in [22]), which constitutes the key result to obtain the a-posteriori estimator for the error $|||u-u_h|||_h$. Hereafter, we denote $\mathrm{curl}\, v := \left(-\frac{\partial v}{\partial y}, \frac{\partial v}{\partial x}\right)$ for any $v \in H^1(\Omega)$.

Lemma 4.3.1 *There exist $\psi \in H^1(\Omega)$ with $\psi = 0$ on Γ_D, and $\chi \in H^1(\Omega)$ with $\mathrm{curl}\,\chi\cdot\boldsymbol{\nu} = 0$ on Γ_N, such that*

$$\boldsymbol{\kappa}\,\nabla_h(u-u_h) = \boldsymbol{\kappa}\,\nabla\psi + \mathrm{curl}\,\chi\,. \tag{4.3.1}$$

Furthermore, there exists a constant $C_{\mathtt{sta}} > 0$, independent of h, such that there holds the stability estimate

$$||\boldsymbol{\kappa}^{1/2}\nabla\psi||_{[L^2(\Omega)]^2} + ||\mathrm{curl}\,\chi||_{[L^2(\Omega)]^2} \;\le\; C_{\mathtt{sta}}\,||\boldsymbol{\kappa}^{1/2}\nabla_h(u-u_h)||_{[L^2(\Omega)]^2}\,. \tag{4.3.2}$$

Proof. Let $\psi \in H^1_{\Gamma_D}(\Omega) := \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_D\}$ be the unique weak solution of the boundary value problem:

$$-\,\mathrm{div}\,(\boldsymbol{\kappa}\,\nabla\psi) \;=\; -\,\mathrm{div}\,(\boldsymbol{\kappa}\,\nabla_h(u-u_h)) \quad \text{in} \quad \Omega\,,$$
$$\psi \;=\; 0 \quad \text{on} \quad \Gamma_D\,, \qquad \boldsymbol{\kappa}\nabla\psi\cdot\boldsymbol{\nu} \;=\; \boldsymbol{\kappa}\nabla_h(u-u_h)\cdot\boldsymbol{\nu} \quad \text{on} \quad \Gamma_N\,.$$

Since $\operatorname{div}(\boldsymbol{\kappa}\nabla_h(u-u_h)-\boldsymbol{\kappa}\nabla\psi)=0$ in Ω in the sense of distributions, and Ω is simply connected, the rest of the proof is a consequence of Theorem 3.1 in Chapter I of [64]. We omit further details. $\square$

The following two lemmas provide technical results that will be used below to derive an equivalent expression to $||\boldsymbol{\kappa}^{1/2}\nabla_h(u-u_h)||_{[L^2(\Omega)]^2}$ in terms of residual terms.

LEMMA 4.3.2 *Let Π_0 be a modified piecewise constant projection from $H^1(\Omega)$ onto $L^2(\Omega)$, so that for all $z\in H^1(\Omega)$, $(\Pi_0 z)|_T := \dfrac{1}{|T|}\displaystyle\int_T z\,dx$ for each T with $\partial T\cap\Gamma_D=\phi$, and $(\Pi_0 z)|_T := 0$ on each $T\in\mathcal{T}_h$ with an edge on Γ_D. Then, given the function $\psi\in H^1_{\Gamma_D}(\Omega)$ from* Lemma 4.3.1*, there holds*

$$\sum_{T\in\mathcal{T}_h}\int_T \boldsymbol{\kappa}\,\nabla(u-u_h)\cdot\nabla\psi\,dx = \sum_{T\in\mathcal{T}_h}\int_T (f+\operatorname{div}(\boldsymbol{\kappa}\nabla u_h))\,(\psi-\Pi_0\psi)\,dx$$
$$+\sum_{T\in\mathcal{T}_h}\int_{\partial T\setminus\Gamma_D}(\psi-\Pi_0\psi)\,(\widehat{\boldsymbol{\sigma}}\cdot\boldsymbol{\nu}_T-\boldsymbol{\kappa}\nabla u_h\cdot\boldsymbol{\nu}_T)\,ds \tag{4.3.3}$$

PROOF. Since $(\Pi_0\psi)|_T$ is constant for any $T\in\mathcal{T}_h$ and $\psi\in H^1_{\Gamma_D}(\Omega)$, we obtain, after integrating by parts, that

$$\sum_{T\in\mathcal{T}_h}\int_T \boldsymbol{\kappa}\nabla(u-u_h)\cdot\nabla\psi\,dx = \sum_{T\in\mathcal{T}_h}\int_T \boldsymbol{\kappa}\nabla(u-u_h)\cdot\nabla(\psi-\Pi_0\psi)\,dx$$

$$=\sum_{T\in\mathcal{T}_h}\left\{-\int_T \operatorname{div}(\boldsymbol{\kappa}\,\nabla(u-u_h))(\psi-\Pi_0\psi)\,dx\right.$$

$$\left.+\int_{\partial T\setminus\Gamma_D}(\psi-\Pi_0\psi)\,\boldsymbol{\kappa}\nabla(u-u_h)\cdot\boldsymbol{\nu}_T\,ds\right\}$$

$$=\sum_{T\in\mathcal{T}_h}\left\{\int_T (f+\operatorname{div}(\boldsymbol{\kappa}\,\nabla u_h))\,(\psi-\Pi_0\psi)\,dx\right.$$
$$\left.+\int_{\partial T\setminus\Gamma_D}(\psi-\Pi_0\psi)(\boldsymbol{\kappa}\,\nabla u\cdot\boldsymbol{\nu}_T-\boldsymbol{\kappa}\,\nabla u_h\cdot\boldsymbol{\nu}_T)\,ds\right\}. \tag{4.3.4}$$

Since $\boldsymbol{\kappa}\,\nabla u \in H(\mathrm{div}\,;\Omega)$, $\psi \in H^1(\Omega)$ and $\widehat{\boldsymbol{\sigma}}$ is continuous across edges (i.e. *conservative*) we find that

$$\sum_{T\in\mathcal{T}_h}\int_{\partial T\setminus\Gamma_D}\boldsymbol{\kappa}\,\nabla u\cdot\boldsymbol{\nu}_T\,\psi\,ds = \int_{\Gamma_I}\{\psi\}[\![\boldsymbol{\kappa}\nabla u]\!]\,ds$$

$$+\int_{\Gamma_I}[\![\psi]\!]\cdot\{\boldsymbol{\kappa}\,\nabla u\}\,ds + \sum_{e\in\Gamma_N}\int_e \psi\,\boldsymbol{\kappa}\,\nabla u\cdot\boldsymbol{\nu}_T\,ds$$

$$= \int_{\Gamma_I}\{\psi\}[\![\widehat{\boldsymbol{\sigma}}]\!]\,ds + \int_{\Gamma_I}[\![\psi]\!]\cdot\{\widehat{\boldsymbol{\sigma}}\}\,ds + \sum_{e\in\Gamma_N}\int_e\psi\,\widehat{\boldsymbol{\sigma}}\cdot\boldsymbol{\nu}_T\,ds = \sum_{T\in\mathcal{T}_h}\int_{\partial T\setminus\Gamma_D}\psi\,\widehat{\boldsymbol{\sigma}}\cdot\boldsymbol{\nu}_T\,ds\,. \tag{4.3.5}$$

Next, integrating by parts, and then using (4.2.1) and the local conservation law stating that $\displaystyle\int_T f\,dx + \int_{\partial T}\widehat{\boldsymbol{\sigma}}\cdot\boldsymbol{\nu}_T\,ds = 0$ (which is obtained from the third equation of (4.2.4) with $v_h \equiv 1$), we find that

$$\begin{aligned}\int_{\partial T}\boldsymbol{\kappa}\nabla u\cdot\boldsymbol{\nu}_T\,\Pi_0\psi\,ds &= \int_T \mathrm{div}\,(\boldsymbol{\kappa}\nabla u)\,\Pi_0\psi\,dx + \int_T\boldsymbol{\kappa}\,\nabla u\cdot\nabla(\Pi_0\psi)\,dx\\ &= -\Pi_0\psi\int_T f\,dx = -\Pi_0\psi\left(-\int_{\partial T}\widehat{\boldsymbol{\sigma}}\cdot\boldsymbol{\nu}_T\,ds\right) = \int_{\partial T}\Pi_0\psi\,\widehat{\boldsymbol{\sigma}}\cdot\boldsymbol{\nu}_T\,ds\,,\end{aligned} \tag{4.3.6}$$

for any $T\in\mathcal{T}_h$ with $\partial T\cap\Gamma_D = \phi$. In this way, (4.3.5) and (4.3.6) yield

$$\sum_{T\in\mathcal{T}_h}\int_{\partial T\setminus\Gamma_D}\boldsymbol{\kappa}\nabla u\cdot\boldsymbol{\nu}_T\,(\psi-\Pi_0\psi)\,ds = \sum_{T\in\mathcal{T}_h}\int_{\partial T\setminus\Gamma_D}\widehat{\boldsymbol{\sigma}}\cdot\boldsymbol{\nu}_T\,(\psi-\Pi_0\psi)\,ds\,,$$

which, replaced back into (4.3.4), completes the proof. □

In what follows, given a part S of Γ, say an edge e, Γ_D, Γ_N, or Γ itself, we let $H_{00}^{1/2}(S)$ be the subspace of functions in $H^{1/2}(S)$ whose extensions by zero on the rest of Γ belong to $H^{1/2}(\Gamma)$. Then we let $H_{00}^{-1/2}(S)$ be its dual and denote by $\langle\cdot,\cdot\rangle_S$ the associated duality pairing with respect to the $L^2(S)$-inner product. Certainly, $H_{00}^{1/2}(\Gamma) = H^{1/2}(\Gamma)$, $H_{00}^{-1/2}(\Gamma) = H^{-1/2}(\Gamma)$, and the corresponding duality with respect to the $L^2(\Gamma)$-inner product is denoted by $\langle\cdot,\cdot\rangle_\Gamma$. Also, when S is an edge $e\in\Gamma$ and $T\in\mathcal{T}_h$ is such that $e\subseteq\partial T$, we may equivalently define $H_{00}^{1/2}(e)$ as the subspace of functions in $H^{1/2}(e)$ whose extensions by zero on the rest of ∂T belong to $H^{1/2}(\partial T)$. In this case, $\|\cdot\|_{H_{00}^{1/2}(e)}$ and $\|\cdot\|_{H^{1/2}(\partial T)}$ are equivalent for a function in $H_{00}^{1/2}(e)$ and its extension to ∂T, respectively.

Lemma 4.3.3 *Let $\tilde{\varphi}_h$ be an auxiliary function such that $\tilde{\varphi}_h \in H^1(\Omega) \cap C(\bar{\Omega})$ and $\tilde{\varphi}_h(\bar{\mathbf{x}}) = g_D(\bar{\mathbf{x}})$ for each vertex $\bar{\mathbf{x}}$ of $\mathcal{T}_h$ lying on Γ_D. Then, with the function $\chi \in H^1(\Omega)$ from* Lemma 4.3.1, *there holds*

$$\sum_{T\in\mathcal{T}_h} \int_T \nabla(u-u_h)\cdot \operatorname{curl}\chi \, dx \;=\; \int_\Omega (\nabla\tilde{\varphi}_h - \nabla_h u_h)\cdot \operatorname{curl}\chi \, dx + \sum_{e\in\Gamma_D} \langle \operatorname{curl}\chi\cdot\boldsymbol{\nu}, g_D - \tilde{\varphi}_h\rangle_e \,.$$

Proof. Because of the assumptions on $\tilde{\varphi}_h$, we observe that $(u-\tilde{\varphi}_h)|_e \in H_{00}^{1/2}(e)$ for each $e \in \Gamma_D$, and $(u-\tilde{\varphi}_h)|_{\Gamma_N} \in H_{00}^{1/2}(\Gamma_N)$, whence

$$\langle \mu, u-\tilde{\varphi}_h\rangle_\Gamma \;=\; \langle \mu, u-\tilde{\varphi}_h\rangle_{\Gamma_N} \;+\; \sum_{e\in\Gamma_D} \langle \mu, u-\tilde{\varphi}_h\rangle_e \qquad \forall\, \mu \in H^{-1/2}(\Gamma)\,. \tag{4.3.7}$$

Then, we add and substract $\tilde{\varphi}_h$ inside the gradient, and apply the integration by parts formula, to obtain

$$\begin{aligned}\sum_{T\in\mathcal{T}_h} \int_T \nabla(u-\tilde{\varphi}_h)\cdot \operatorname{curl}\chi \, dx &= \int_\Omega \nabla(u-\tilde{\varphi}_h)\cdot \operatorname{curl}\chi \, dx \\ &= \;\langle \operatorname{curl}\chi\cdot\boldsymbol{\nu}, u-\tilde{\varphi}_h\rangle_\Gamma \;=\; \sum_{e\in\Gamma_D} \langle \operatorname{curl}\chi\cdot\boldsymbol{\nu}, g_D-\tilde{\varphi}_h\rangle_e\,,\end{aligned} \tag{4.3.8}$$

where the last equality is consequence of (4.3.7) together with the fact that $u = g_D$ on Γ_D and $\operatorname{curl}\chi\cdot\boldsymbol{\nu} = 0$ on Γ_N. □

We remark that the introduction of the auxiliary function $\tilde{\varphi}_h \in H^1(\Omega)\cap C(\bar{\Omega})$ allows the above application of the integration by parts formula on the whole domain Ω. This would not be possible with u_h instead of $\tilde{\varphi}_h$ since (4.3.8) would fail in this case.

Now, the equivalent expression to $||\boldsymbol{\kappa}^{1/2}\,\nabla_h(u-u_h)||_{[L^2(\Omega)]^2}$ in terms of residual terms is given next.

Theorem 4.3.1 *With the same notations of the previous lemmata, there holds*

$$\begin{aligned}||\boldsymbol{\kappa}^{1/2}\nabla_h(u-u_h)||^2_{[L^2(\Omega)]^2} = \sum_{T\in\mathcal{T}_h} \Bigg\{ &\int_T (f + \operatorname{div}(\boldsymbol{\kappa}\nabla u_h))\,(\psi - \Pi_0\psi)\,dx \\ + \int_{\partial T\setminus\Gamma_D} (\psi-\Pi_0\psi)\,(\widehat{\boldsymbol{\sigma}}\cdot\boldsymbol{\nu}_T - \boldsymbol{\kappa}\nabla u_h\cdot\boldsymbol{\nu}_T)\,ds &+ \int_T (\nabla\tilde{\varphi}_h - \nabla u_h)\cdot\operatorname{curl}\chi\,dx \Bigg\} \\ &+ \sum_{e\in\Gamma_D} \langle \operatorname{curl}\chi\cdot\boldsymbol{\nu}, g_D - \tilde{\varphi}_h\rangle_e\,.\end{aligned} \tag{4.3.9}$$

PROOF. According to the Helmholtz decomposition provided by Lemma 4.3.1, we obtain

$$||\boldsymbol{\kappa}^{1/2}\nabla_h(u-u_h)||^2_{[L^2(\Omega)]^2} \;=\; \sum_{T\in\mathcal{T}_h}\int_T \boldsymbol{\kappa}\nabla(u-u_h)\cdot\nabla(u-u_h)\,dx$$

$$= \sum_{T\in\mathcal{T}_h}\left\{\int_T \boldsymbol{\kappa}\nabla(u-u_h)\cdot\nabla\psi\,dx + \int_T \nabla(u-u_h)\cdot\mathrm{curl}\,\chi\,dx\right\}, \qquad (4.3.10)$$

and hence the proof is completed by applying Lemmas 4.3.2 and 4.3.3, respectively, to the terms on the right hand side of (4.3.10). □

The following theorem gives an explicit reliable a-posteriori estimate for the error $|||u-u_h|||_h$.

THEOREM 4.3.2 *There exists a constant $C_{\texttt{rel}} > 0$, independent of the meshsize, such that*

$$|||u-u_h|||_h^2 \;\le\; C_{\texttt{rel}}\,\bar{\boldsymbol{\eta}}^2 \;:=\; C_{\texttt{rel}}\sum_{T\in\mathcal{T}_h}\bar{\eta}_T^2\,,$$

where, for each $T\in\mathcal{T}_h$, we define

$$\bar{\eta}_T^2 \;:=\; h_T^2\,||f+\mathrm{div}\,(\boldsymbol{\kappa}\nabla u_h)||^2_{L^2(T)} \;+\; h_T\,||\widehat{\boldsymbol{\sigma}}\cdot\boldsymbol{\nu}_T \;-\; \boldsymbol{\kappa}\nabla u_h\cdot\boldsymbol{\nu}_T||^2_{L^2(\partial T\setminus\Gamma_D)}$$

$$+\,||\alpha^{1/2}[\![u_h]\!]||^2_{[L^2(\partial T\cap\Gamma_I)]^2} \;+\; ||\alpha^{1/2}(g_D-u_h)||^2_{L^2(\partial T\cap\Gamma_D)}$$

$$+\,||\nabla\tilde{\varphi}_h-\nabla u_h||^2_{[L^2(T)]^2} \;+\; \sum_{e\subseteq\partial T\cap\Gamma_D}||g_D-\tilde{\varphi}_h||^2_{H^{1/2}_{00}(e)}\,.$$

PROOF. Since u is the exact solution of (4.2.1), we first notice that

$$|||u-u_h|||_h^2 \;=\; ||\boldsymbol{\kappa}^{1/2}\,\nabla_h(u-u_h)||^2_{[[L^2(\Omega)]^2]^2} + ||\alpha^{1/2}[\![u_h]\!]||^2_{[L^2(\Gamma_I)]^2} + ||\alpha^{1/2}(g_D-u_h)||^2_{L^2(\Gamma_D)}\,, \qquad (4.3.11)$$

which already provides the third and fourth terms defining $\bar{\eta}_T^2$.

Now, in order to estimate the first term on the right hand side of (4.3.11), we make use below of the equivalent relation (4.3.9) given in Theorem 4.3.1. In addition, according to a well known approximation result (see, e.g., Lemma 5.1 in [32]) and (4.2.2), we have

$$||\psi-\Pi_0\psi||_{L^2(T)} \;\le\; C_{\texttt{int}}\,h_T\,||\nabla\psi||_{[L^2(T)]^2} \;\le\; \widetilde{C}_{\texttt{int}}\,h_T\,||\boldsymbol{\kappa}^{1/2}\nabla\psi||_{[L^2(T)]^2} \qquad \forall\, T\in\mathcal{T}_h\,. \qquad (4.3.12)$$

For the first term on the right hand side of (4.3.9) we apply the Cauchy Schwarz inequality and (4.3.12), and find that

$$\left| \int_T (f + \mathrm{div}\,(\boldsymbol{\kappa}\nabla u_h))\,(\psi - \Pi_0\psi)\,dx \right| \;\leq\; ||f + \mathrm{div}\,(\boldsymbol{\kappa}\nabla u_h)||_{L^2(T)}\,||\psi - \Pi_0\psi||_{L^2(T)}$$

$$\leq\; \widetilde{C}_{\mathtt{int}}\,h_T\;||f + \mathrm{div}\,(\boldsymbol{\kappa}\nabla u_h)||_{L^2(T)}\,||\boldsymbol{\kappa}^{1/2}\nabla\psi||_{[L^2(T)]^2}\,. \tag{4.3.13}$$

Next, for the second term in (4.3.9) we use again the Cauchy-Schwarz inequality, the trace inequality (see equation (2.4) in [6] or equation (4.8) in [32]) given by

$$||w||^2_{L^2(e)} \;\leq\; C_{\mathtt{tr}}\,\left(h_e^{-1}||w||^2_{L^2(T)} + h_e\,||\nabla w||^2_{[L^2(T)]^2}\right) \qquad \forall\, w \in H^1(T)\,, \forall\, e \subseteq \partial T\,, \tag{4.3.14}$$

and then (4.3.12), to obtain

$$\begin{aligned} &\left| \int_{\partial T\setminus\Gamma_D} (\psi - \Pi_0\psi)\,(\widehat{\boldsymbol{\sigma}}\cdot\boldsymbol{\nu}_T - \boldsymbol{\kappa}\nabla u_h\cdot\boldsymbol{\nu}_T)\,ds \right| \\ &\leq\; ||\psi - \Pi_0\psi||_{L^2(\partial T\setminus\Gamma_D)}\;||\widehat{\boldsymbol{\sigma}}\cdot\boldsymbol{\nu}_T - \boldsymbol{\kappa}\nabla u_h\cdot\boldsymbol{\nu}_T||_{L^2(\partial T\setminus\Gamma_D)} \\ &\leq\; C\,h_T^{1/2}\,||\widehat{\boldsymbol{\sigma}}\cdot\boldsymbol{\nu}_T - \boldsymbol{\kappa}\nabla u_h\cdot\boldsymbol{\nu}_T||_{L^2(\partial T\setminus\Gamma_D)}\;||\boldsymbol{\kappa}^{1/2}\nabla\psi||_{[L^2(T)]^2}\,. \end{aligned} \tag{4.3.15}$$

The third term in (4.3.9) is easily bounded by $||\nabla\tilde{\varphi}_h - \nabla_h u_h||_{[L^2(\Omega)]^2}\;||\mathrm{curl}\,\chi||_{[L^2(\Omega)]^2}$.

Finally, for the last term in (4.3.9) we use that $\|\mathrm{curl}\,\chi\cdot\boldsymbol{\nu}\|_{H_{00}^{-1/2}(e)} \leq c\,\|\mathrm{curl}\,\chi\cdot\boldsymbol{\nu}_T\|_{H^{-1/2}(\partial T)}$, where $T \in \mathcal{T}_h$ is such that $e \subseteq \partial T$, and consider the following trace inequality (see [64])

$$||\boldsymbol{\tau}\cdot\boldsymbol{\nu}_T|||_{H^{-1/2}(\partial T)} \leq \widetilde{C}_{\mathtt{tr}}\left(||\boldsymbol{\tau}||_{[L^2(T)]^2} + h_T\,||\mathrm{div}\,\boldsymbol{\tau}||_{L^2(T)}\right) \quad \forall\,\boldsymbol{\tau} \in H(\mathrm{div}\,;T)\,, \quad \forall\, T \in \mathcal{T}_h\,,$$

where $\widetilde{C}_{\mathtt{tr}} > 0$ is independent of the meshsize. Indeed, taking $\boldsymbol{\tau} := \mathrm{curl}\,\chi$, we obtain

$$||\mathrm{curl}\,\chi\cdot\boldsymbol{\nu}_T||_{H^{-1/2}(\partial T)} \;\leq\; \widetilde{C}_{\mathtt{tr}}\,||\mathrm{curl}\,\chi||_{[L^2(T)]^2}\,,$$

and hence

$$\left|\sum_{e\in\Gamma_D}\langle\mathrm{curl}\,\chi\cdot\boldsymbol{\nu}, g_D - \tilde{\varphi}_h\rangle_e\right| \;\leq\; \sum_{e\in\Gamma_D}\|\mathrm{curl}\,\chi\cdot\boldsymbol{\nu}\|_{H_{00}^{-1/2}(e)}\;\|g_D - \tilde{\varphi}_h\|_{H_{00}^{1/2}(e)}$$

$$\leq\; C\,\left(\sum_{T\in\mathcal{T}_h}\;\sum_{e\subseteq\partial T\cap\Gamma_D}\|g_D - \tilde{\varphi}_h\|^2_{H_{00}^{1/2}(e)}\right)^{1/2}\;||\mathrm{curl}\,\chi||_{[L^2(\Omega)]^2}\,.$$

In this way, the above upper bounds for each term defining (4.3.9), and the stability estimate given in Lemma 4.3.1, complete the proof. □

We now recall from [32] that $\mathbf{S}$ is bounded, $\mathbf{S}(u) = \mathcal{G}_D$, and $\boldsymbol{\theta}_h = \nabla_h u_h - \mathbf{S}(u_h) + \mathcal{G}_D$, which yields $\boldsymbol{\theta} - \boldsymbol{\theta}_h = \nabla_h(u - u_h) + \mathbf{S}(u - u_h)$ and $\|\boldsymbol{\theta} - \boldsymbol{\theta}_h\|_{[L^2(\Omega)]^2} \le C\,|||u - u_h|||_h$. It follows that the a-posteriori estimate provided in Theorem 4.3.2, which does not depend explicitly on $\boldsymbol{\theta}_h$, is also valid for the error $\|\boldsymbol{\theta} - \boldsymbol{\theta}_h\|_{[L^2(\Omega)]^2}$.

Alternatively, in the next two theorems we show that using a further Helmholtz decomposition, a-posteriori estimates for the errors $||\boldsymbol{\theta} - \boldsymbol{\theta}_h||_{[L^2(\Omega)]^2}$ and $||\boldsymbol{\sigma} - \boldsymbol{\sigma}_h||_{[L^2(\Omega)]^2}$, depending explicitly on $\boldsymbol{\theta}_h$ and $\boldsymbol{\sigma}_h$, respectively, can also be obtained. Moreover, we will observe below in Section 4.4 that these estimates arise naturally when the corresponding a-posteriori error estimates of the associated nonlinear problem are particularized to the present linear case.

Theorem 4.3.3 *Let $\tilde{\varphi}_h$ be as stated before. Then, there exists $C_{\mathtt{th}} > 0$, independent of the meshsize, such that*

$$||\boldsymbol{\theta} - \boldsymbol{\theta}_h||^2_{[L^2(\Omega)]^2} \le C_{\mathtt{th}}\,\tilde{\boldsymbol{\eta}}^2 := C_{\mathtt{th}} \sum_{T\in\mathcal{T}_h} \tilde{\eta}_T^2\,,$$

where, for each $T \in \mathcal{T}_h$, we define

$$\tilde{\eta}_T^2 := h_T^2\,||f + \operatorname{div}(\boldsymbol{\kappa}\,\boldsymbol{\theta}_h)||^2_{L^2(T)} + h_T\,||\widehat{\boldsymbol{\sigma}}\cdot\boldsymbol{\nu}_T - \boldsymbol{\kappa}\,\boldsymbol{\theta}_h\cdot\boldsymbol{\nu}_T||^2_{L^2(\partial T\setminus\Gamma_D)}$$

$$+\,||\nabla\tilde{\varphi}_h - \nabla u_h||^2_{[L^2(T)]^2} + \sum_{e\subseteq\partial T\cap\Gamma_D} \|g_D - \tilde{\varphi}_h\|^2_{H^{1/2}_{00}(e)} + ||\nabla u_h - \boldsymbol{\theta}_h||^2_{[L^2(T)]^2}\,.$$

Proof. Similarly as in Lemma 4.3.1, we deduce the existence of $q \in H^1_{\Gamma_D}(\Omega)$ and $\psi \in H^1(\Omega)$, with $\operatorname{curl}\psi\cdot\boldsymbol{\nu} = 0$ on Γ_N, such that the following Helmholtz decomposition holds

$$\boldsymbol{\kappa}\,(\boldsymbol{\theta} - \boldsymbol{\theta}_h) = \boldsymbol{\kappa}\,\nabla q + \operatorname{curl}\psi\,,$$

and there exists $c_{\mathtt{sta}} > 0$, independent of h, such that

$$||\boldsymbol{\kappa}^{1/2}\nabla q||_{[L^2(\Omega)]^2} + ||\operatorname{curl}\psi||_{[L^2(\Omega)]^2} \le c_{\mathtt{sta}}\,||\boldsymbol{\kappa}^{1/2}\,(\boldsymbol{\theta} - \boldsymbol{\theta}_h)||_{[L^2(\Omega)]^2}\,. \qquad (4.3.16)$$

Hence, employing again the operator Π_0 defined in Lemma 4.3.2, we can write

$$||\boldsymbol{\kappa}^{1/2}\,(\boldsymbol{\theta} - \boldsymbol{\theta}_h)||^2_{[L^2(\Omega)]^2} = \int_\Omega \boldsymbol{\kappa}\,(\boldsymbol{\theta} - \boldsymbol{\theta}_h)\cdot\nabla q\,dx + \int_\Omega (\boldsymbol{\theta} - \boldsymbol{\theta}_h)\cdot\operatorname{curl}\psi\,dx$$

$$= \sum_{T\in\mathcal{T}_h} \int_T \boldsymbol{\kappa}\,(\boldsymbol{\theta}-\boldsymbol{\theta}_h)\cdot\nabla(q-\Pi_0 q)\,dx + \int_\Omega \nabla_h(u-u_h)\cdot\mathrm{curl}\,\psi\,dx + \int_\Omega (\nabla_h u_h-\boldsymbol{\theta}_h)\cdot\mathrm{curl}\,\psi\,dx\,. \tag{4.3.17}$$

Now, applying integration by parts in the first term of (4.3.17), noting that $\boldsymbol{\sigma} := \boldsymbol{\kappa}\,\boldsymbol{\theta}$, and following the same arguments of the proof of Lemma 4.3.2, we find that

$$\sum_{T\in\mathcal{T}_h} \int_T \boldsymbol{\kappa}\,(\boldsymbol{\theta}-\boldsymbol{\theta}_h)\cdot\nabla(q-\Pi_0 q)\,dx \;=\; \sum_{T\in\mathcal{T}_h} \int_T (f+\mathrm{div}\,(\boldsymbol{\kappa}\,\boldsymbol{\theta}_h))\,(q-\Pi_0 q)\,dx$$

$$+ \sum_{T\in\mathcal{T}_h} \int_{\partial T\setminus\Gamma_D} (q-\Pi_0 q)\,(\widehat{\boldsymbol{\sigma}}\cdot\boldsymbol{\nu}_T - \boldsymbol{\kappa}\,\boldsymbol{\theta}_h\cdot\boldsymbol{\nu}_T)\,ds\,,$$

which, using the trace inequality (4.3.14) and the approximation result (4.3.12), yields the first two terms defining $\tilde{\eta}_T^2$.

The remaining terms in (4.3.17) are bounded by applying Lemma 4.3.3 and Cauchy-Schwarz inequality. Finally, collecting all the bounds and using (4.3.16), we conclude the proof. □

THEOREM 4.3.4 *There exists $C_{\mathtt{fl}} > 0$, independent of the meshsize, such that*

$$||\boldsymbol{\sigma}-\boldsymbol{\sigma}_h||^2_{[L^2(\Omega)]^2} \;\le\; C_{\mathtt{fl}}\,\widehat{\boldsymbol{\eta}}^2 \;:=\; C_{\mathtt{fl}} \sum_{T\in\mathcal{T}_h} \widehat{\eta}_T^2\,,$$

where, for each $T\in\mathcal{T}_h$, we define

$$\widehat{\eta}_T^2 \;:=\; \tilde{\eta}_T^2 \;+\; ||\boldsymbol{\kappa}\,\boldsymbol{\theta}_h - \boldsymbol{\sigma}_h||^2_{[L^2(T)]^2}\,,$$

with $\tilde{\eta}_T$ given in Theorem 4.3.3. □

PROOF. It suffices to consider the relation $(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h) \;=\; \boldsymbol{\kappa}\,(\boldsymbol{\theta}-\boldsymbol{\theta}_h) + (\boldsymbol{\kappa}\,\boldsymbol{\theta}_h-\boldsymbol{\sigma}_h)$, and then apply triangle inequality and the a-posteriori estimate provided by Theorem 4.3.3. □

The following lemma bounds the expression $\displaystyle\sum_{e\subseteq\partial T\cap\Gamma_D} \|g_D-\tilde{\varphi}_h\|^2_{H^{1/2}_{00}(e)}$ by means of local computable quantities.

LEMMA 4.3.4 *Assume that the null extension of $(g_D-\tilde{\varphi}_h)$ to ∂T belongs to $H^1(\partial T)$ for each $T\in\mathcal{T}_h$. Then there exists a constant $C>0$, independent of the meshsize, such that*

$$\sum_{e\subseteq\partial T\cap\Gamma_D} \|g_D-\tilde{\varphi}_h\|^2_{H^{1/2}_{00}(e)} \;\le\; C\,\log(1+\kappa_T) \sum_{e\subseteq\partial T\cap\Gamma_D} h_e \left\|\frac{\partial}{\partial s}(g_D-\tilde{\varphi}_h)\right\|^2_{L^2(e)}\,, \tag{4.3.18}$$

where $\kappa_T := \max\left\{\frac{h_e}{h_{e'}} \quad : \quad e, e' \subseteq \partial T\,, e \neq e'\right\}$, *and* s *denotes the arc length variable.*

PROOF. It is a direct application of Theorem 1 in [34]. $\square$

We now recall that under some regularity assumptions on the exact solution $(u, \boldsymbol{\theta}, \boldsymbol{\sigma})$, the errors $|||u - u_h|||_h$, $||\boldsymbol{\theta} - \boldsymbol{\theta}_h||_{[L^2(\Omega)]^2}$, and $||\boldsymbol{\sigma} - \boldsymbol{\sigma}_h||_{[L^2(\Omega)]^2}$ converge with the same theoretical rate. Hence, our global a-posteriori error estimate is summarized in the following theorem.

THEOREM 4.3.5 *There exists a constant* $C > 0$, *independent of the meshsize, such that*

$$|||u - u_h|||_h^2 + ||\boldsymbol{\theta} - \boldsymbol{\theta}_h||^2_{[L^2(\Omega)]^2} + ||\boldsymbol{\sigma} - \boldsymbol{\sigma}_h||^2_{[L^2(\Omega)]^2} \leq C\,\boldsymbol{\eta}^2 := C \sum_{T \in \mathcal{T}_h} \eta_T^2\,, \tag{4.3.19}$$

where, for each $T \in \mathcal{T}_h$, *we define*

$$\begin{aligned}
\eta_T^2 &:= h_T^2\,||f + \operatorname{div}(\boldsymbol{\kappa}\nabla u_h)||^2_{L^2(T)} + h_T\,||\widehat{\boldsymbol{\sigma}} \cdot \boldsymbol{\nu}_T - \boldsymbol{\kappa}\nabla u_h \cdot \boldsymbol{\nu}_T||^2_{L^2(\partial T \setminus \Gamma_D)} \\
&+ h_T^2\,||f + \operatorname{div}(\boldsymbol{\kappa}\boldsymbol{\theta}_h)||^2_{L^2(T)} + h_T\,||\widehat{\boldsymbol{\sigma}} \cdot \boldsymbol{\nu}_T - \boldsymbol{\kappa}\boldsymbol{\theta}_h \cdot \boldsymbol{\nu}_T||^2_{L^2(\partial T \setminus \Gamma_D)} \\
&+ ||\boldsymbol{\kappa}\,\boldsymbol{\theta}_h - \boldsymbol{\sigma}_h||^2_{[L^2(T)]^2} + ||\nabla u_h - \boldsymbol{\theta}_h||^2_{[L^2(T)]^2} + ||\nabla\tilde{\varphi}_h - \nabla u_h||^2_{[L^2(T)]^2} \\
&+ ||\alpha^{1/2}(g_D - u_h)||^2_{L^2(\partial T \cap \Gamma_D)} + ||\alpha^{1/2}[\![u_h]\!]||^2_{[L^2(\partial T \cap \Gamma_I)]^2} \\
&+ \log(1 + \kappa_T) \sum_{e \subseteq \partial T \cap \Gamma_D} h_e \left\| \frac{\partial}{\partial s}(g_D - \tilde{\varphi}_h) \right\|^2_{L^2(e)}\,.
\end{aligned} \tag{4.3.20}$$

PROOF. It follows from Theorems 4.3.2, 4.3.3 and 4.3.4, and Lemma 4.3.4. $\square$

A suitable choice of the auxiliary function $\tilde{\varphi}_h$ will be addressed at the end of Section 4.4.

4.4 The nonlinear case

In this section we extend the results from Section 4.3 to the nonlinear diffusion problem studied in [32]. For this purpose, we let Ω, Γ_D, Γ_N and Γ be as in Section 4.2. Also, we let $a_i : \Omega \times \mathbf{R}^2 \to \mathbf{R}$, $i = 1, 2$, be nonlinear mappings satisfying suitable hypotheses, such as (H.1)-(H.4) in [32], and denote by $\mathbf{a} : \Omega \times \mathbf{R}^2 \to \mathbf{R}^2$ the vector function defined by $\mathbf{a}(x,\boldsymbol{\zeta}) := (a_1(x,\boldsymbol{\zeta}), a_2(x,\boldsymbol{\zeta}))^{\mathbf{T}}$ for all $(x,\boldsymbol{\zeta}) \in \Omega \times \mathbf{R}^2$. Then, given $f \in [L^2(\Omega)]^2$, $g_D \in H^{1/2}(\Gamma_D)$, and $g_N \in L^2(\Gamma_N)$, we look for $u \in H^1(\Omega)$ such that

$$
\begin{gathered}
-\mathrm{div}\, \mathbf{a}(\cdot, \nabla u(\cdot)) = f \quad \text{in} \quad \Omega\,, \\
u = g_D \quad \text{on} \quad \Gamma_D \quad \text{and} \quad \mathbf{a}(\cdot, \nabla u(\cdot)) \cdot \boldsymbol{\nu} = g_N \quad \text{on} \quad \Gamma_N\,.
\end{gathered}
\tag{4.4.1}
$$

Similarly as for the linear case, we introduce the gradient $\boldsymbol{\theta} := \nabla u$ and the nonlinear flux $\boldsymbol{\sigma} := \mathbf{a}(\cdot, \boldsymbol{\theta})$ as additional unknowns. Then, following the same analysis and notations from [32], we find that the primal formulation associated to (4.4.1) becomes: *Find $u_h \in V_h$ such that*

$$
[A_h(u_h), v_h] \;=\; [F_h, v_h] \qquad \forall\, v_h \in V_h\,, \tag{4.4.2}
$$

where the nonlinear operator $A_h : V(h) \to V(h)'$ and the functional $F_h : V_h \to \mathbf{R}$ are defined, respectively, by

$$
\begin{aligned}
[A_h(w), v] \;:=\; & \int_\Omega \mathbf{a}(\cdot, \nabla_h w - \mathbf{S}(w) + \mathcal{G}_D) \cdot (\nabla_h v - \mathbf{S}(v))\, dx \\
& + \int_{\Gamma_I} \alpha\, [\![w]\!] \cdot [\![v]\!]\, ds \;+\; \int_{\Gamma_D} \alpha\, w\, v\, ds
\end{aligned}
\tag{4.4.3}
$$

and

$$
[F_h, v] \;:=\; \int_\Omega f\, v\, dx + \int_{\Gamma_D} \alpha\, g_D\, v\, ds + \int_{\Gamma_N} g_N\, v\, ds\,, \tag{4.4.4}
$$

for all $w,\, v \,\in\, V(h)$. The corresponding discrete unknowns associated to $\boldsymbol{\theta}$ and $\boldsymbol{\sigma}$, which are used in the formulation of the LDG method, are denoted also by $\boldsymbol{\theta}_h \in \boldsymbol{\Sigma}_h$ and $\boldsymbol{\sigma}_h \in \boldsymbol{\Sigma}_h$, respectively.

From now on we define the energy norm $|||\cdot|||_h : V(h) \to \mathbf{R}$ by

$$
|||v|||_h^2 := \|\nabla_h v\|^2_{[L^2(\Omega)]^2} \;+\; \|\alpha^{1/2}\, [\![v]\!]\|^2_{[L^2(\Gamma_I)]^2} \;+\; \|\alpha^{1/2}\, v\|^2_{L^2(\Gamma_D)} \quad \forall\, v \in V(h)\,, \tag{4.4.5}
$$

and let $|\cdot|_h : V(h) \to \mathbf{R}$ be the semi-norm defined by (4.2.9).

Next, we let $\mathcal{N}_h : V(h) \to V(h)'$ be the pure nonlinear operator forming part of A_h (cf. (4.4.3)), that is

$$
[\mathcal{N}_h(w), v] \;:=\; \int_\Omega \mathbf{a}(\cdot, \nabla_h w - \mathbf{S}(w) + \mathcal{G}_D) \cdot (\nabla_h v - \mathbf{S}(v))\, dx \qquad \forall\, w, v \in V(h)\,. \tag{4.4.6}
$$

Since $\mathbf{a}$ satisfies the assumption (H.4) in [32] and the operator $\mathbf{S}$ is bounded (cf. Lemma 3.3 and (3.16) in [32]), we deduce that $\mathcal{N}_h$ is Gâteaux differentiable at each $z \in V(h)$, and its derivative can be interpreted as the bounded bilinear form $D\mathcal{N}_h(z) : V(h) \times V(h) \to \mathbf{R}$ given by

$$D\mathcal{N}_h(z)(w,v) \,:=\, \int_\Omega D\mathbf{a}(\cdot,\tilde{\boldsymbol{\zeta}})\,(\nabla_h v - \mathbf{S}(v)) \,\cdot\, (\nabla_h w - \mathbf{S}(w))\,dx \qquad \forall\, w,v \in V(h)\,, \tag{4.4.7}$$

where $\tilde{\boldsymbol{\zeta}} := \nabla_h z - \mathbf{S}(z) + \mathcal{G}_D$ and $D\mathbf{a}(\cdot,\tilde{\boldsymbol{\zeta}})$ is the jacobian matrix of $\mathbf{a}$ at $\tilde{\boldsymbol{\zeta}}$.

Hence, according to the definition of A_h (cf. (4.4.3)), the Gâteaux derivative of A_h at $z \in V(h)$ becomes the bounded bilinear form $DA_h : V(h) \times V(h) \to \mathbf{R}$ defined by

$$DA_h(z)(w,v) \,:=\, D\mathcal{N}_h(z)(w,v) + \int_{\Gamma_I} \alpha\, [\![w]\!] \cdot [\![v]\!]\, ds + \int_{\Gamma_D} \alpha\, w\, v\, ds \qquad \forall\, w,v \in V(h)\,. \tag{4.4.8}$$

In what follows we assume that $D\mathbf{a}(\cdot,\tilde{\boldsymbol{\zeta}})$ is symmetric for all $z \in V(h)$ and that $D\mathcal{N}_h$ is *hemicontinuous*, that is for any $z, w \in V(h)$, the mapping $\mathbf{R} \ni \mu \to D\mathcal{N}_h(z + \mu w)(w,\cdot) \in V(h)'$ is continuous. Then, applying the mean value theorem we deduce the existence of $\tilde{u} \in V(h)$, a convex combination of u and u_h, such that

$$D\mathcal{N}_h(\tilde{u})(u - u_h, v) \,=\, [\mathcal{N}_h(u) - \mathcal{N}_h(u_h), v] \qquad \forall\, v \in V(h)\,, \tag{4.4.9}$$

which, according to (4.4.3) and (4.4.8), implies that

$$DA_h(\tilde{u})(u - u_h, v) \,=\, [A_h(u) - A_h(u_h), v] \qquad \forall\, v \in V(h)\,. \tag{4.4.10}$$

In order to provide the a-posteriori estimate for the error $|||u - u_h|||_h$, we follow Lemma 4.3.3 and consider again an auxiliary function $\tilde{\varphi}_h \in H^1(\Omega) \cap C(\bar{\Omega})$ such that $\tilde{\varphi}_h(\bar{\mathbf{x}}) = g_D(\bar{\mathbf{x}})$ for each vertex $\bar{\mathbf{x}}$ of $\mathcal{T}_h$ lying on Γ_D. Nevertheless, despite the existence of several similarities with Section 4.3, the nonlinearity involved here does not allow to establish an analogue of (4.3.11) for $|||u - u_h|||_h$. Consequently, we employ a different procedure, which yields other terms defining the corresponding a-posteriori error estimate. This means that the local indicator $\bar{\eta}_T^2$ given in Theorem 4.3.2 for the linear case does not coincide with the particularization to that situation of the indicator obtained below for the present nonlinear problem. Indeed, we have the following result.

THEOREM 4.4.1 *There exists a constant $\tilde{C}_{\texttt{rel}} > 0$, independent of the meshsize, such that*

$$|||u - u_h|||_h^2 \,\leq\, \tilde{C}_{\texttt{rel}}\, \bar{\boldsymbol{\vartheta}}^2 \,:=\, \tilde{C}_{\texttt{rel}} \sum_{T \in \mathcal{T}_h} \bar{\vartheta}_T^2\,,$$

where, for each $T \in \mathcal{T}_h$, we define

$$\bar{\vartheta}_T^2 \;:=\; h_T^2 \,||f + \operatorname{div} \mathbf{a}(\cdot, \boldsymbol{\theta}_h)||^2_{L^2(T)} \;+\; h_T \,||\widehat{\boldsymbol{\sigma}} \cdot \boldsymbol{\nu}_T \;-\; \mathbf{a}(\cdot, \boldsymbol{\theta}_h) \cdot \boldsymbol{\nu}_T||^2_{L^2(\partial T \setminus \Gamma_D)}$$

$$+ \;||\nabla u_h - \boldsymbol{\theta}_h||^2_{[L^2(T)]^2} \;+\; ||\alpha^{1/2} [\![u_h]\!]||^2_{[L^2(\partial T \cap \Gamma_I)]^2} \;+\; ||\alpha^{1/2}(g_D - u_h)||^2_{L^2(\partial T \cap \Gamma_D)}$$

$$+ \;||\nabla \tilde{\varphi}_h - \nabla u_h||^2_{[L^2(T)]^2} \;+ \sum_{e \subseteq \partial T \cap \Gamma_D} \|g_D - \tilde{\varphi}_h\|^2_{H^{1/2}_{00}(e)} \,.$$

PROOF. Using that A_h is strongly monotone in $V(h)$ (cf. Lemma 4.2 in [32]) and (4.4.10), we obtain

$$C_{\mathtt{SM}} \;|||u - u_h|||_h^2 \;\leq\; [A_h(u) - A_h(u_h), u - u_h] \;=\; DA_h(\tilde{u})(u - u_h, u - u_h)$$

$$=\; D\mathcal{N}_h(\tilde{u})(u - u_h, u - u_h) \;+\; ||\alpha^{1/2} [\![u_h]\!]||^2_{[L^2(\Gamma_I)]^2} \;+\; ||\alpha^{1/2}(g_D - u_h)||^2_{L^2(\Gamma_D)} \,, \tag{4.4.11}$$

where, according to (4.4.7),

$$D\mathcal{N}_h(\tilde{u})(u - u_h, u - u_h) \;=\; \int_\Omega D\mathbf{a}(\cdot, \tilde{\boldsymbol{\theta}}) \,\boldsymbol{\zeta} \cdot \boldsymbol{\zeta} \, dx \,, \tag{4.4.12}$$

with $\tilde{\boldsymbol{\theta}} := \nabla_h \tilde{u} - \mathbf{S}(\tilde{u}) + \mathcal{G}_D$ and $\boldsymbol{\zeta} := \nabla_h(u - u_h) - \mathbf{S}(u - u_h) \in [L^2(\Omega)]^2$.

Then, applying the Helmholtz decomposition (cf. Lemma 4.3.1) to $\boldsymbol{\kappa} := D\mathbf{a}(\cdot, \tilde{\boldsymbol{\theta}})$, we deduce the existence of $p \in H^1_{\Gamma_D}(\Omega)$ and $\psi \in H^1(\Omega)$, with $\operatorname{curl} \psi \cdot \boldsymbol{\nu} = 0$ on Γ_N, such that

$$D\mathbf{a}(\cdot, \tilde{\boldsymbol{\theta}}) \boldsymbol{\zeta} \;=\; D\mathbf{a}(\cdot, \tilde{\boldsymbol{\theta}}) \,\nabla p \;+\; \operatorname{curl} \psi \,.$$

By replacing the above relation back into (4.4.12) and using the simmetry of $D\mathbf{a}(\cdot, \tilde{\boldsymbol{\theta}})$ and the fact that $\mathbf{S}(p) = 0$, we get

$$D\mathcal{N}_h(\tilde{u})(u - u_h, u - u_h) \;=\; \int_\Omega \boldsymbol{\zeta} \cdot [D\mathbf{a}(\cdot, \tilde{\boldsymbol{\theta}}) \nabla p \;+\; \operatorname{curl} \psi] \, dx$$

$$=\; \int_\Omega D\mathbf{a}(\cdot, \tilde{\boldsymbol{\theta}}) \,\boldsymbol{\zeta} \cdot \nabla p \, dx \;+\; \int_\Omega \boldsymbol{\zeta} \cdot \operatorname{curl} \psi \, dx \;=\; D\mathcal{N}_h(\tilde{u})(u - u_h, p) \;+\; \int_\Omega \boldsymbol{\zeta} \cdot \operatorname{curl} \psi \, dx \,. \tag{4.4.13}$$

At this point we recall again from [32] that $\mathbf{S}(u) = \mathcal{G}_D$ and $\boldsymbol{\theta}_h = \nabla_h u_h - \mathbf{S}(u_h) + \mathcal{G}_D$. Then, using (4.4.9) and (4.4.6), and noting that $\boldsymbol{\zeta} = (\boldsymbol{\theta} - \boldsymbol{\theta}_h)$, we obtain from (4.4.13) that

$$D\mathcal{N}_h(\tilde{u})(u - u_h, u - u_h) \;=\; \int_\Omega \Big(\mathbf{a}(\cdot, \nabla u) - \mathbf{a}(\cdot, \boldsymbol{\theta}_h) \Big) \cdot \nabla_h (p - \Pi_0 p) \, dx + \int_\Omega (\boldsymbol{\theta} - \boldsymbol{\theta}_h) \cdot \operatorname{curl} \psi \, dx \,, \tag{4.4.14}$$

where Π_0 is the operator defined in Lemma 4.3.2.

We now proceed analogously as in the proof of Lemma 4.3.2. Indeed, after integrating by parts over each element, the first term in (4.4.14) reduces to

$$\sum_{T\in\mathcal{T}_h} \left\{ -\int_T \operatorname{div}\left(\mathbf{a}(\cdot,\nabla u) - \mathbf{a}(\cdot,\boldsymbol{\theta}_h) \right) (p - \Pi_0 p)\, dx \right.$$

$$\left. + \int_{\partial T} (p - \Pi_0 p)\, (\mathbf{a}(\cdot,\nabla u) - \mathbf{a}(\cdot,\boldsymbol{\theta}_h)) \cdot \boldsymbol{\nu}_T\, ds \right\}$$

$$= \sum_{T\in\mathcal{T}_h} \left\{ \int_T \left(f + \operatorname{div}\mathbf{a}(\cdot,\boldsymbol{\theta}_h) \right) (p - \Pi_0 p)\, dx \right.$$

$$\left. + \int_{\partial T\setminus\Gamma_D} (p - \Pi_0 p)\, (\widehat{\boldsymbol{\sigma}}\cdot\boldsymbol{\nu}_T - \mathbf{a}(\cdot,\boldsymbol{\theta}_h)\cdot\boldsymbol{\nu}_T)\, ds \right\}.$$

On the other hand, for the second term in the right hand side of (4.4.14), we apply Lemma 4.3.3 and obtain that

$$\int_\Omega (\boldsymbol{\theta} - \boldsymbol{\theta}_h)\cdot \operatorname{curl}\psi\, dx \;=\; \sum_{T\in\mathcal{T}_h}\int_T \nabla(u - u_h)\cdot\operatorname{curl}\psi\, dx \;+\; \int_\Omega (\nabla_h u_h - \boldsymbol{\theta}_h)\cdot\operatorname{curl}\psi\, dx$$

$$= \int_\Omega (\nabla\tilde{\varphi}_h - \nabla_h u_h)\cdot\operatorname{curl}\psi\, dx + \sum_{e\in\Gamma_D}\langle \operatorname{curl}\psi\cdot\boldsymbol{\nu}, g_D - \tilde{\varphi}_h\rangle_e + \int_\Omega (\nabla_h u_h - \boldsymbol{\theta}_h)\cdot\operatorname{curl}\psi\, dx\,.$$

In this way, replacing the resulting expression for (4.4.14) in (4.4.11), and applying the stability relation (4.3.2), we complete the proof. $\square$

As announced before, we remark now that in the linear case, that is when $\mathbf{a}(\cdot,\boldsymbol{\zeta}) = \boldsymbol{\kappa}\,\boldsymbol{\zeta}$, the above theorem yields the local indicator

$$\bar{\vartheta}_T^2 \;:=\; h_T^2\, ||f + \boldsymbol{\kappa}\,\boldsymbol{\theta}_h||^2_{L^2(T)} \;+\; h_T\, ||\widehat{\boldsymbol{\sigma}}\cdot\boldsymbol{\nu}_T \;-\; \boldsymbol{\kappa}\,\boldsymbol{\theta}_h\cdot\boldsymbol{\nu}_T||^2_{L^2(\partial T\setminus\Gamma_D)}$$

$$+\; ||\nabla u_h - \boldsymbol{\theta}_h||^2_{[L^2(T)]^2} \;+\; ||\alpha^{1/2}[\![u_h]\!]||^2_{[L^2(\partial T\cap\Gamma_I)]^2} \;+\; ||\alpha^{1/2}(g_D - u_h)||^2_{L^2(\partial T\cap\Gamma_D)}$$

$$+\; ||\nabla\tilde{\varphi}_h - \nabla u_h||^2_{[L^2(T)]^2} \;+\; \sum_{e\subseteq\partial T\cap\Gamma_D} ||g_D - \tilde{\varphi}_h||^2_{H^{1/2}_{00}(e)}\,,$$

which differs from $\bar{\eta}_T^2$ (cf. Theorem 4.3.2) in the first two terms, and includes additionally $||\nabla u_h - \boldsymbol{\theta}_h||^2_{[L^2(T)]^2}$, which makes it more expensive.

Our next goal is the derivation of a-posteriori estimates for the errors $||\boldsymbol{\theta} - \boldsymbol{\theta}_h||_{[L^2(\Omega)]^2}$ and $||\boldsymbol{\sigma} - \boldsymbol{\sigma}_h||_{[L^2(\Omega)]^2}$ depending explicitly on $\boldsymbol{\theta}_h$ and $\boldsymbol{\sigma}_h$, respectively. For this purpose, we now introduce the nonlinear operator $N : [L^2(\Omega)]^2 \to ([L^2(\Omega)]^2)'$, defined by

$$[N(\boldsymbol{\theta}), \boldsymbol{\xi}] := \int_\Omega \mathbf{a}(\cdot, \boldsymbol{\theta}) \cdot \boldsymbol{\xi} \, dx \qquad \forall\, \boldsymbol{\theta}, \boldsymbol{\xi} \in [L^2(\Omega)]^2 \,, \tag{4.4.15}$$

which, because of the properties of $\mathbf{a}$ given in [32], is Gâteaux differentiable at each $\boldsymbol{\zeta} \in [L^2(\Omega)]^2$. In fact, this derivative can be characterized as the uniformly bounded and uniformly elliptic bilinear form $DN : [L^2(\Omega)]^2 \times [L^2(\Omega)]^2 \to \mathbf{R}$ defined by

$$DN(\boldsymbol{\zeta})(\boldsymbol{\theta}, \boldsymbol{\xi}) \;:=\; \int_\Omega D\mathbf{a}(\cdot, \boldsymbol{\zeta})\, \boldsymbol{\xi} \cdot \boldsymbol{\theta} \, dx \qquad \forall\, \boldsymbol{\theta}, \boldsymbol{\xi} \in [L^2(\Omega)]^2 \,. \tag{4.4.16}$$

Then, assuming that DN is also *hemi-continuous*, we deduce the existence of an element $\tilde{\boldsymbol{\theta}} \in [L^2(\Omega)]^2$, a convex combination of $\boldsymbol{\theta}$ and $\boldsymbol{\theta}_h$, such that

$$DN(\tilde{\boldsymbol{\theta}})(\boldsymbol{\theta} - \boldsymbol{\theta}_h, \boldsymbol{\zeta}) = [N(\boldsymbol{\theta}) - N(\boldsymbol{\theta}_h), \boldsymbol{\zeta}] \quad \forall\, \boldsymbol{\zeta} \in [L^2(\Omega)]^2 \,. \tag{4.4.17}$$

THEOREM 4.4.2 *There exists $C_{\mathtt{eth}} > 0$, independent of the meshsize, such that*

$$||\boldsymbol{\theta} - \boldsymbol{\theta}_h||^2_{[L^2(\Omega)]^2} \;\leq\; C_{\mathtt{eth}}\, \tilde{\boldsymbol{\vartheta}}^2 \;:=\; C_{\mathtt{eth}} \sum_{T \in \mathcal{T}_h} \tilde{\vartheta}_T^2 \,,$$

where, for each $T \in \mathcal{T}_h$, we define

$$\begin{aligned} \tilde{\vartheta}_T^2 \;:=\;& h_T^2 ||f + \operatorname{div} \mathbf{a}(\cdot, \boldsymbol{\theta}_h)||^2_{L^2(T)} \;+\; h_T\, ||\hat{\boldsymbol{\sigma}} \cdot \boldsymbol{\nu}_T \;-\; \mathbf{a}(\cdot, \boldsymbol{\theta}_h) \cdot \boldsymbol{\nu}_T||^2_{L^2(\partial T \setminus \Gamma_D)} \\ &+ ||\nabla \tilde{\varphi}_h - \nabla u_h||^2_{[L^2(T)]^2} \;+ \sum_{e \subseteq \partial T \cap \Gamma_D} ||g_D - \tilde{\varphi}_h||^2_{H^{1/2}_{00}(e)} +\; ||\nabla u_h - \boldsymbol{\theta}_h||^2_{[L^2(T)]^2} \,. \end{aligned} \tag{4.4.18}$$

PROOF. Since N is strongly monotone in $[L^2(\Omega)]^2$ (see [32]), we can write

$$C_0\, ||\boldsymbol{\theta} - \boldsymbol{\theta}_h||^2 \;\leq\; [N(\boldsymbol{\theta}) - N(\boldsymbol{\theta}_h), \boldsymbol{\theta} - \boldsymbol{\theta}_h] \;=\; DN(\tilde{\boldsymbol{\theta}})(\boldsymbol{\theta} - \boldsymbol{\theta}_h, \boldsymbol{\theta} - \boldsymbol{\theta}_h) \,. \tag{4.4.19}$$

Then, applying Helmholtz decomposition (cf. Lemma 4.3.1), we deduce the existence of $p \in H^1_{\Gamma_D}(\Omega)$ and $\psi \in H^1(\Omega)$, with $\operatorname{curl} \psi \cdot \boldsymbol{\nu} = 0$ on Γ_N, such that

$$D\mathbf{a}(\cdot, \tilde{\boldsymbol{\theta}})\,(\boldsymbol{\theta} - \boldsymbol{\theta}_h) \;=\; D\mathbf{a}(\cdot, \tilde{\boldsymbol{\theta}})\, \nabla p \;+\; \operatorname{curl} \psi \,, \tag{4.4.20}$$

which, using (4.4.15) and (4.4.16), leads to

$$\begin{aligned} DN(\tilde{\boldsymbol{\theta}})(\boldsymbol{\theta} - \boldsymbol{\theta}_h, \boldsymbol{\theta} - \boldsymbol{\theta}_h) \;=\;& \sum_{T \in \mathcal{T}_h} \int_T \big(\mathbf{a}(\cdot, \boldsymbol{\theta}) - \mathbf{a}(\cdot, \boldsymbol{\theta}_h)\big) \cdot \nabla (p - \Pi_0 p) \, dx \\ &+ \int_\Omega \nabla_h (u - u_h) \cdot \operatorname{curl} \psi \, dx \;+\; \int_\Omega (\nabla_h u_h - \boldsymbol{\theta}_h) \cdot \operatorname{curl} \psi \, dx \,. \end{aligned}$$

The rest of the proof proceeds as in Theorem 4.3.3 and hence we omit the details. $\square$

THEOREM 4.4.3 *There exists $C_{\texttt{efl}} > 0$, independent of the meshsize, such that*

$$||\boldsymbol{\sigma} - \boldsymbol{\sigma}_h||^2_{[L^2(\Omega)]^2} \,\leq\, C_{\texttt{efl}}\,\widehat{\boldsymbol{\vartheta}}^2 \,:=\, C_{\texttt{efl}} \sum_{T\in\mathcal{T}_h} \widehat{\vartheta}_T^2\,,$$

where, for each $T \in \mathcal{T}_h$, we define

$$\widehat{\vartheta}_T^2 \,:=\, \tilde{\vartheta}_T^2 \,+\, ||\mathbf{a}(\cdot,\boldsymbol{\theta}_h) - \boldsymbol{\sigma}_h||^2_{[L^2(T)]^2}\,.$$

PROOF. It suffices to consider the relation $(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h) \,=\, \mathbf{a}(\cdot,\boldsymbol{\theta}) - \mathbf{a}(\cdot,\boldsymbol{\theta}_h) + \mathbf{a}(\cdot,\boldsymbol{\theta}_h) - \boldsymbol{\sigma}_h$, and then apply triangle inequality, the Lipschitz-continuity of the nonlinear operator N (see [32]), and the a-posteriori estimate provided by Theorem 4.4.2. □

Differently to the case of $|||u - u_h|||_h$, we observe here that when $\mathbf{a}(\cdot,\boldsymbol{\zeta}) = \boldsymbol{\kappa}\,\boldsymbol{\zeta}$, the a-posteriori error estimates given in Theorems 4.4.2 and 4.4.3 become the corresponding ones obtained in Section 4.3 for the associated linear problem (see Theorems 4.3.3 and 4.3.4), that is $\tilde{\boldsymbol{\eta}} = \tilde{\boldsymbol{\vartheta}}$ and $\widehat{\boldsymbol{\eta}} = \widehat{\boldsymbol{\vartheta}}$.

Our global a-posteriori error estimate is summarized as follows.

THEOREM 4.4.4 *Assume the hypotheses of* Lemma 4.3.4. *Then, there exists a constant $C > 0$, independent of the meshsize, such that*

$$|||u - u_h|||^2_h \,+\, ||\boldsymbol{\theta} - \boldsymbol{\theta}_h||^2_{[L^2(\Omega)]^2} \,+\, ||\boldsymbol{\sigma} - \boldsymbol{\sigma}_h||^2_{[L^2(\Omega)]^2} \,\leq\, C\,\boldsymbol{\vartheta}^2 \,:=\, C \sum_{T\in\mathcal{T}_h} \vartheta_T^2\,, \tag{4.4.21}$$

where, for each $T \in \mathcal{T}_h$, we define

$$\begin{aligned}\vartheta_T^2 \,:=\, & h_T^2\,||f + \operatorname{div}\mathbf{a}(\cdot,\boldsymbol{\theta}_h)||^2_{L^2(T)} \,+\, h_T\,||\widehat{\boldsymbol{\sigma}}\cdot\boldsymbol{\nu}_T - \mathbf{a}(\cdot,\boldsymbol{\theta}_h)\cdot\boldsymbol{\nu}_T||^2_{L^2(\partial T\setminus\Gamma_D)} \\ & + ||\mathbf{a}(\cdot,\boldsymbol{\theta}_h) - \boldsymbol{\sigma}_h||^2_{[L^2(T)]^2} \,+\, ||\nabla u_h - \boldsymbol{\theta}_h||^2_{[L^2(T)]^2} \,+ ||\nabla\tilde{\varphi}_h - \nabla u_h||^2_{[L^2(T)]^2} \\ & + ||\alpha^{1/2}(g_D - u_h)||^2_{L^2(\partial T\cap\Gamma_D)} \,+\, ||\alpha^{1/2}[\![u_h]\!]||^2_{[L^2(\partial T\cap\Gamma_I)]^2} \\ & + \log(1+\kappa_T) \sum_{e\subseteq\partial T\cap\Gamma_D} h_e \left\|\frac{\partial}{\partial s}(g_D - \tilde{\varphi}_h)\right\|^2_{L^2(e)}\,. \end{aligned} \tag{4.4.22}$$

PROOF. It follows from Theorems 4.4.1, 4.4.2 and 4.4.3, and Lemma 4.3.4. □

It is important to remark here that, due to the difference with the estimates for $|||u{-}u_h|||_h$ and the coincidence with the ones for $||\boldsymbol{\theta}{-}\boldsymbol{\theta}_h||_{[L^2(\Omega)]^2}$ and $||\boldsymbol{\sigma}{-}\boldsymbol{\sigma}_h||_{[L^2(\Omega)]^2}$, the restriction of the global a-posteriori error estimate $\boldsymbol{\vartheta}$ (cf. Theorem 4.4.4) to the linear case, contains two less terms than $\boldsymbol{\eta}$ (cf. Theorem 4.3.5), and hence it becomes cheaper. Therefore, if the adaptive algorithm is going to be based on a global a-posteriori error estimate, one should use the one induced by $\boldsymbol{\vartheta}$ with $\mathbf{a}(\cdot, \boldsymbol{\zeta}) = \boldsymbol{\kappa}\,\boldsymbol{\zeta}$, instead of $\boldsymbol{\eta}$. On the other hand, if this algorithm is going to be based on the a-posteriori estimate for the individual error $|||u - u_h|||_h$, then one should just use $\bar{\boldsymbol{\eta}}$ instead of $\bar{\boldsymbol{\vartheta}}$.

It remains to address the choice of an appropriate auxiliary function $\tilde{\varphi}_h$. As established by the residual-type terms involving $\tilde{\varphi}_h$ in the definitions of $\boldsymbol{\eta}$ and $\boldsymbol{\vartheta}$ (cf. Theorems 4.3.5 and 4.4.4), we observe that $\tilde{\varphi}_h|_T$ should be as close as possible to the exact solution u. Therefore, we propose next an heuristic procedure to choose it, which depends on the polynomial degree of our approximated solution u_h.

Let $k \in \mathbf{N}$ and assume that $u_h|_T \in \mathbf{P}_k(T)$ for each $T \in \mathcal{T}_h$. Then, in the case of regular triangulations, we define $\tilde{\varphi}_h$ as the $\mathbf{P}_k$*−continuous average* of the functions $u_h|_T$. In other words, $\tilde{\varphi}_h$ is the unique function in $C(\bar{\Omega})$ such that

1. $\tilde{\varphi}_h|_T \in \mathbf{P}_k(T)$ for each $T \in \mathcal{T}_h$.

2. $\tilde{\varphi}_h(\bar{\mathbf{x}}) := g_D(\bar{\mathbf{x}})$ for each node $\bar{\mathbf{x}}$ of $\mathcal{T}_h$ lying on Γ_D.

3. For each node $\bar{\mathbf{x}}$ of $\mathcal{T}_h$ not lying on the boundary Γ_D, $\tilde{\varphi}_h(\bar{\mathbf{x}})$ is the average of the values of $u_h(\bar{\mathbf{x}})$ on all the triangles $T \in \mathcal{T}_h$ to which $\bar{\mathbf{x}}$ belongs.

Since we also allow non-regular triangulations, the above procedure needs to be suitably modified for the case of meshes with *hanging nodes*.

4.5 Numerical results

In this section we provide several numerical examples illustrating the performance of the LDG method and the a-posteriori error estimates $\boldsymbol{\eta}$ and $\boldsymbol{\vartheta}$ with the above described choice of the auxiliary function $\tilde{\varphi}_h$. Hereafter, N is the number of degrees of freedom defining the subspaces V_h and $\boldsymbol{\Sigma}_h$, that is $N := 15$ (number of triangles of $\mathcal{T}_h$) for the $\mathbf{P}_1 - \mathbf{P}_1 - \mathbf{P}_1$ approximation, and $N := 18$ (number of triangles of $\mathcal{T}_h$) for the $\mathbf{P}_2 - \mathbf{P}_1 - \mathbf{P}_1$ approximation. Further, the individual and global errors are defined as follows

$$\mathbf{e}(\boldsymbol{\theta}) := ||\boldsymbol{\theta} - \boldsymbol{\theta}_h||_{[L^2(\Omega)]^2}\,, \quad \mathbf{e}(\boldsymbol{\sigma}) := ||\boldsymbol{\sigma} - \boldsymbol{\sigma}_h||_{[L^2(\Omega)]^2}\,,$$

$$\mathbf{e}_h(u) := |||u - u_h|||_h\,, \quad \mathbf{e}_0(u) := ||u - u_h||_{L^2(\Omega)}\,,$$

and

$$\mathbf{e} := \left\{ [\mathbf{e}(\boldsymbol{\theta})]^2 \,+\, [\mathbf{e}(\boldsymbol{\sigma})]^2 \,+\, [\mathbf{e}_h(u)]^2 \right\}^{1/2}\,,$$

where $(\boldsymbol{\theta}, \boldsymbol{\sigma}, u)$ and $(\boldsymbol{\theta}_h, \boldsymbol{\sigma}_h, u_h)$ are the unique solutions of the continuous and discrete mixed formulations, respectively. In addition, if $\mathbf{e}$ and $\tilde{\mathbf{e}}$ stand for the errors at two consecutive triangulations with N and $\tilde{N}$ degrees of freedom, respectively, then the experimental rate of convergence is given by $\gamma := -2\,\dfrac{\log(\mathbf{e}/\tilde{\mathbf{e}})}{\log(N/\tilde{N})}$. Similar definitions hold for $\gamma(\boldsymbol{\theta})$, $\gamma(\boldsymbol{\sigma})$, $\gamma_h(u)$, and $\gamma_0(u)$.

The adaptive algorithm based on the estimate $\boldsymbol{\eta}$, without *hanging nodes*, is the following (see [96]):

1. Start with a coarse mesh $\mathcal{T}_h$.

2. Solve the discrete problem (4.2.4) for the actual mesh $\mathcal{T}_h$.

3. Compute the auxiliary function $\tilde{\varphi}_h$.

4. Compute the local indicators η_T for each triangle $T \in \mathcal{T}_h$.

5. Evaluate stopping criterion and decide to finish or go to next step.

6. Use *red-blue-green* procedure to refine each $T' \in \mathcal{T}_h$ whose indicator $\eta_{T'}$ satisfies

$$\eta_{T'} \,\geq\, \frac{1}{2}\,\max\{\,\eta_T : \quad T \in \mathcal{T}_h\,\}\,.$$

7. Define resulting mesh as actual mesh $\mathcal{T}_h$ and go to step 2.

The adaptive algorithms based on other a-posteriori error estimates, such as $\bar{\boldsymbol{\eta}}$ or $\boldsymbol{\vartheta}$, are analogue to the one described above. Also, we observe that in the case of meshes with *hanging nodes*, the *red-blue-green* procedure in step 6 is replaced by a simple *red* refinement.

In what follows we present four examples. The first two refer to the linear case (4.2.1), and the other two consider the nonlinear boundary value problem (4.4.1) with $\mathbf{a}(\cdot,\boldsymbol{\zeta}) := \left(2 + \frac{1}{1+||\boldsymbol{\zeta}||}\right)\boldsymbol{\zeta}$ for all $\boldsymbol{\zeta} \in \mathbf{R}^2$. It is not difficult to see that this nonlinearity satisfies the hypotheses (H.1)-(H.4) specified in [32]. In Examples 1 and 3 we take the L-shaped domain $\Omega := (-1,1)^2 \backslash ([0,1] \times [-1,0])$ with Dirichlet boundary conditions on $\partial\Omega$, while Examples 2 and 4 use the convex domain $\Omega := (0,1)^2$ with mixed boundary conditions on $\Gamma_D := ([0,1] \times \{0\}) \cup (\{1\} \times [0,1])$ and $\Gamma_N := \partial\Omega \setminus \Gamma_D$. The data f, g_D and g_N are chosen so that the exact solutions u are the ones shown below in Table 4.5.0. The solutions of Examples 1 and 3, which coincide and are given in polar coordinates, have a singularity at $(0,0)$, while the solutions of Examples 2 and 4, which also coincide, present an inner layer around the origin, as well.

Table 4.5.0: Summary of data for the 4 examples.

Example	Mapping $\mathbf{a}$	Domain Ω	Conditions on $\partial\Omega$	Solution u
1	$\boldsymbol{\zeta}$	L-shaped	Dirichlet	$r^{1/3}\sin\left(\frac{2}{3}\theta\right)$
2	$\boldsymbol{\zeta}$	Square	Mixed	$\sqrt{1000}\,e^{-\sqrt{1000}\,(x_1+x_2)}$
3	$\left(2+\frac{1}{1+\|\|\boldsymbol{\zeta}\|\|}\right)\boldsymbol{\zeta}$	L-shaped	Dirichlet	$r^{1/3}\sin\left(\frac{2}{3}\theta\right)$
4	$\left(2+\frac{1}{1+\|\|\boldsymbol{\zeta}\|\|}\right)\boldsymbol{\zeta}$	Square	Mixed	$\sqrt{1000}\,e^{-\sqrt{1000}\,(x_1+x_2)}$

The numerical results presented below were obtained in a *Compaq Alpha ES40 Parallel Computer* using a MATLAB code. We remark that in the nonlinear case, the corresponding LDG scheme (see (3.7) in [32]), which becomes a nonlinear algebraic system with N unknowns, is solved by Newton's method with the initial guess given by the solution of the associated Poisson problem, and a tolerance of 10^{-10} for the residual. In all cases we take the parameters $\widehat{\alpha} = 1$ and $\boldsymbol{\beta} = (1,1)^t$ in the primal formulation.

In Tables 4.5.1-4.5.6 we give the individual and global errors, the effectivity index $\mathbf{e}_h(u)/\bar{\boldsymbol{\eta}}$, and the corresponding experimental rates of convergence for the uniform, red-blue-green, and red refinements (based on $\bar{\boldsymbol{\eta}}$) as applied to Examples 1 and 2 with $\mathbf{P}_1 - \mathbf{P}_1 - \mathbf{P}_1$ and $\mathbf{P}_2 - \mathbf{P}_1 - \mathbf{P}_1$ approximations. The errors are computed on each triangle using a 7 points Gaussian quadrature rule. We observe that the effectivity indexes are bounded above and below, which confirms the reliability of $\bar{\boldsymbol{\eta}}$ and provides numerical evidences for its efficiency. Also, Figures 4.5.1, 4.5.2, 4.5.7, and 4.5.8 show the global errors $\mathbf{e}$, $\mathbf{e}^{\mathtt{rbg}}$, and $\mathbf{e}^{\mathtt{r}}$, corresponding to the uniform, red-blue-green, and red refinements, respectively, versus the degrees of freedom N. In all cases the errors of the adaptive methods decrease much faster than that of the uniform one. This is also emphasized by the experimental rates of convergence provided in the tables, which show that the adaptive algorithms recover $O(h)$ and $O(h^2)$ for $\mathbf{P}_1 - \mathbf{P}_1 - \mathbf{P}_1$ and $\mathbf{P}_2 - \mathbf{P}_1 - \mathbf{P}_1$, respectively. Next, Figures 4.5.3-4.5.6, 4.5.9, and 4.5.10 display some intermediate meshes obtained with the different refinements. We remark, as expected, that the adaptive algorithms are able to recognize the singular point and the inner layer of Examples 1 and 2, respectively. In addition, we notice that the red refinement (with hanging nodes) is more localized around the singularities than the red-blue-green one (without hanging nodes). It is also important to observe that no much differences, except for the resulting orders of convergence, are observed between the $\mathbf{P}_1 - \mathbf{P}_1 - \mathbf{P}_1$ and $\mathbf{P}_2 - \mathbf{P}_1 - \mathbf{P}_1$ approximations.

Similarly as for the linear examples, we present in Tables 4.5.7-4.5.10 the individual and global errors, the effectivity index $\mathbf{e}/\boldsymbol{\vartheta}$, and the corresponding experimental rates of convergence for the uniform, red-blue-green, and red refinements (based on $\boldsymbol{\vartheta}$) as applied to the nonlinear problems given by Examples 3 and 4 with $\mathbf{P}_1 - \mathbf{P}_1 - \mathbf{P}_1$ approximation. In addition, Figures 4.5.11 and 4.5.13 show the global errors $\mathbf{e}$, $\mathbf{e}^{\mathtt{rbg}}$, and $\mathbf{e}^{\mathtt{r}}$ versus the degrees of freedom N, while Figures 4.5.12 and 4.5.14 display some intermediate meshes obtained with the red refinement (with hanging nodes). The remarks and conclusions here are the same of the linear examples. In particular, the effectivity indexes confirm now the reliability of $\boldsymbol{\vartheta}$ and provide experimental evidences of an eventual efficiency. Also, the adaptive refinements lead again to the quasi-optimal rates of convergence, and are able to identify the singularities of each problem.

Table 4.5.1: Example 1 with $\mathbf{P}_1 - \mathbf{P}_1 - \mathbf{P}_1$ approximation: uniform, red-blue-green and red refinements.

N	$\mathbf{e}_h(u)$	$\gamma_h(u)$	$\mathbf{e}_h(u)/\bar{\boldsymbol{\eta}}$	$\mathbf{e}(\boldsymbol{\theta})$	$\gamma(\boldsymbol{\theta})$	$\mathbf{e}$	γ
90	0.2963	——	0.2867	0.1608	——	0.3371	——
360	0.2182	0.4414	0.2705	0.1257	0.3548	0.2518	0.4208
1440	0.1468	0.5717	0.2695	0.0832	0.5966	0.1687	0.5778
5760	0.0960	0.6131	0.2695	0.0537	0.6299	0.1100	0.6171
23040	0.0618	0.6349	0.2696	0.0344	0.6453	0.0707	0.6374
92160	0.0395	0.6474	0.2697	0.0218	0.6537	0.0451	0.6489
360	0.2182	0.4414	0.2705	0.1257	0.3548	0.2518	0.4208
900	0.1550	0.7458	0.2751	0.0851	0.8519	0.1769	0.7713
1440	0.1204	1.0748	0.2817	0.0617	1.3705	0.1353	1.1397
1980	0.1037	0.9421	0.2884	0.0493	1.4050	0.1148	1.0329
3390	0.0807	0.9275	0.2821	0.0390	0.8725	0.0896	0.9172
5505	0.0631	1.0129	0.2812	0.0307	0.9744	0.0702	1.0056
8415	0.0505	1.0525	0.2779	0.0251	0.9622	0.0564	1.0349
13770	0.0401	0.9358	0.2784	0.0195	1.0174	0.0446	0.9517
21585	0.0322	0.9790	0.2785	0.0154	1.0512	0.0357	0.9926
34215	0.0254	1.0238	0.2763	0.0123	0.9581	0.0282	1.0114
55710	0.0199	0.9892	0.2773	0.0095	1.0708	0.0221	1.0045
85650	0.0160	1.0072	0.2766	0.0076	1.0070	0.0178	1.0071
136215	0.0127	1.0054	0.2764	0.0060	1.0027	0.0141	1.0049
360	0.2182	0.4414	0.2705	0.1257	0.3548	0.2518	0.4208
630	0.1683	0.9275	0.2758	0.0881	1.2726	0.1900	1.0075
990	0.1391	0.8408	0.2797	0.0670	1.2086	0.1544	0.9149
1620	0.1147	0.7836	0.2840	0.0540	0.8724	0.1268	0.8001
3015	0.0851	0.9616	0.2778	0.0399	0.9722	0.0940	0.9635
4770	0.0688	0.9220	0.2778	0.0328	0.8499	0.0763	0.9088
7650	0.0544	0.9941	0.2799	0.0249	1.1716	0.0599	1.0259
11790	0.0435	1.0353	0.2786	0.0200	1.0128	0.0479	1.0314
18720	0.0346	0.9853	0.2769	0.0157	1.0333	0.0380	0.9936
30735	0.0272	0.9739	0.2774	0.0124	0.9553	0.0299	0.9707
47925	0.0215	1.0587	0.2767	0.0097	1.0812	0.0236	1.0626
74205	0.0174	0.9680	0.2758	0.0079	0.9723	0.0191	0.9687
123750	0.0136	0.9686	0.2759	0.0062	0.9556	0.0149	0.9664

Table 4.5.2: Example 1 with $\mathbf{P}_2 - \mathbf{P}_1 - \mathbf{P}_1$ approximation: uniform and red-blue-green refinements.

N	$\mathbf{e}_h(u)$	$\gamma_h(u)$	$\mathbf{e}_h(u)/\bar{\boldsymbol{\eta}}$	$\mathbf{e}(\boldsymbol{\theta})$	$\gamma(\boldsymbol{\theta})$	$\mathbf{e}$	γ
108	0.1440	——	0.2149	0.1637	——	0.2180	——
432	0.0966	0.5765	0.1988	0.1052	0.6380	0.1428	0.6105
1728	0.0612	0.6590	0.1971	0.0666	0.6600	0.0904	0.6595
6912	0.0385	0.6674	0.1966	0.0420	0.6647	0.0570	0.6659
27648	0.0243	0.6673	0.1964	0.0265	0.6660	0.0359	0.6666
110592	0.0153	0.6670	0.1964	0.0167	0.6664	0.0226	0.6667
108	0.1440	——	0.2149	0.1637	——	0.2180	——
432	0.0966	0.5765	0.1988	0.1052	0.6380	0.1428	0.6105
1080	0.0618	0.9753	0.1971	0.0668	0.9913	0.0910	0.9839
1728	0.0401	1.8316	0.1967	0.0426	1.9105	0.0585	1.8738
2376	0.0272	2.4328	0.1970	0.0276	2.7172	0.0388	2.5802
3024	0.0199	2.5814	0.1977	0.0186	3.2803	0.0273	2.9211
3672	0.0162	2.1583	0.1985	0.0134	3.3559	0.0210	2.6808
4860	0.0128	1.6744	0.1974	0.0099	2.1808	0.0161	1.8723
7200	0.0085	2.0794	0.1993	0.0064	2.2068	0.0106	2.1263
9432	0.0062	2.2522	0.1963	0.0044	2.6848	0.0077	2.4032
12348	0.0048	1.9563	0.1945	0.0032	2.2666	0.0058	2.0576
17316	0.0034	1.9413	0.1949	0.0023	2.0327	0.0041	1.9700
23220	0.0025	2.0061	0.1954	0.0017	2.1164	0.0031	2.0400
31662	0.0018	2.0227	0.1966	0.0012	2.1132	0.0022	2.0499
41994	0.0014	2.0251	0.1961	0.0009	2.1873	0.0016	2.0727
56970	0.0010	1.9964	0.1957	0.0006	2.0630	0.0012	2.0155
79146	0.0007	1.9013	0.1967	0.0004	1.8839	0.0009	1.8963

Table 4.5.3: Example 1 with $\mathbf{P}_2 - \mathbf{P}_1 - \mathbf{P}_1$ approximation: uniform and red refinements.

N	$\mathbf{e}_h(u)$	$\gamma_h(u)$	$\mathbf{e}_h(u)/\bar{\boldsymbol{\eta}}$	$\mathbf{e}(\boldsymbol{\theta})$	$\gamma(\boldsymbol{\theta})$	$\mathbf{e}$	γ
108	0.1440	——	0.2149	0.1637	——	0.2180	——
432	0.0966	0.5765	0.1988	0.1052	0.6380	0.1428	0.6105
1728	0.0612	0.6590	0.1971	0.0666	0.6600	0.0904	0.6595
6912	0.0385	0.6674	0.1966	0.0420	0.6647	0.0570	0.6659
27648	0.0243	0.6673	0.1964	0.0265	0.6660	0.0359	0.6666
110592	0.0153	0.6670	0.1964	0.0167	0.6664	0.0226	0.6667
108	0.1440	——	0.2149	0.1637	——	0.2180	——
432	0.0966	0.5765	0.1988	0.1052	0.6380	0.1428	0.6105
756	0.0642	1.4556	0.1954	0.0676	1.5754	0.0933	1.5196
1080	0.0454	1.9423	0.1926	0.0446	2.3382	0.0636	2.1435
1512	0.0343	1.6573	0.1914	0.0305	2.2457	0.0460	1.9314
2106	0.0260	1.6854	0.1915	0.0218	2.0183	0.0339	1.8278
3132	0.0194	1.4761	0.1897	0.0148	1.9544	0.0244	1.6634
3834	0.0170	1.3096	0.2121	0.0125	1.6982	0.0211	1.4495
4374	0.0146	2.3023	0.1939	0.0102	2.9453	0.0178	2.5219
5022	0.0131	1.5577	0.1922	0.0090	1.9305	0.0159	1.6793
5400	0.0113	3.9653	0.1984	0.0082	2.4313	0.0140	3.4545
7830	0.0085	1.5464	0.1816	0.0056	2.0768	0.0102	1.7180
8964	0.0073	2.2397	0.1964	0.0048	2.0604	0.0088	2.1850
11124	0.0058	2.1394	0.1913	0.0039	2.0510	0.0070	2.1120
16470	0.0039	3.6324	0.1984	0.0026	2.6305	0.0047	3.3308
20520	0.0032	1.9331	0.1917	0.0020	2.0545	0.0038	1.9700
23166	0.0027	2.4026	0.1974	0.0018	2.2768	0.0033	2.3645
25164	0.0025	2.2936	0.1971	0.0016	1.9401	0.0030	2.1851
28350	0.0022	2.1926	0.1999	0.0014	2.1323	0.0026	2.1742
32832	0.0019	1.7977	0.1966	0.0012	1.9815	0.0023	1.8534
39420	0.0016	1.6092	0.1953	0.0010	1.7383	0.0019	1.6481
52812	0.0012	1.6052	0.1919	0.0007	2.0298	0.0014	1.7310
56160	0.0011	3.3277	0.2009	0.0007	1.9836	0.0013	2.9247
66582	0.0009	1.6213	0.1949	0.0006	1.9821	0.0011	1.7300

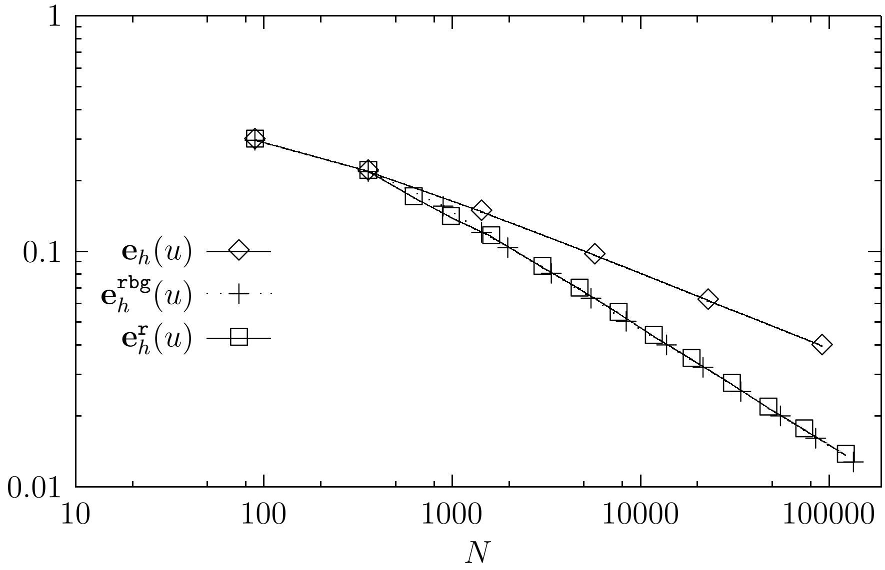

Figure 4.5.1: Example 1 with $\mathbf{P}_1 - \mathbf{P}_1 - \mathbf{P}_1$ approximation: energy-norm error $\mathbf{e}_h$ for the uniform and adaptive refinements.

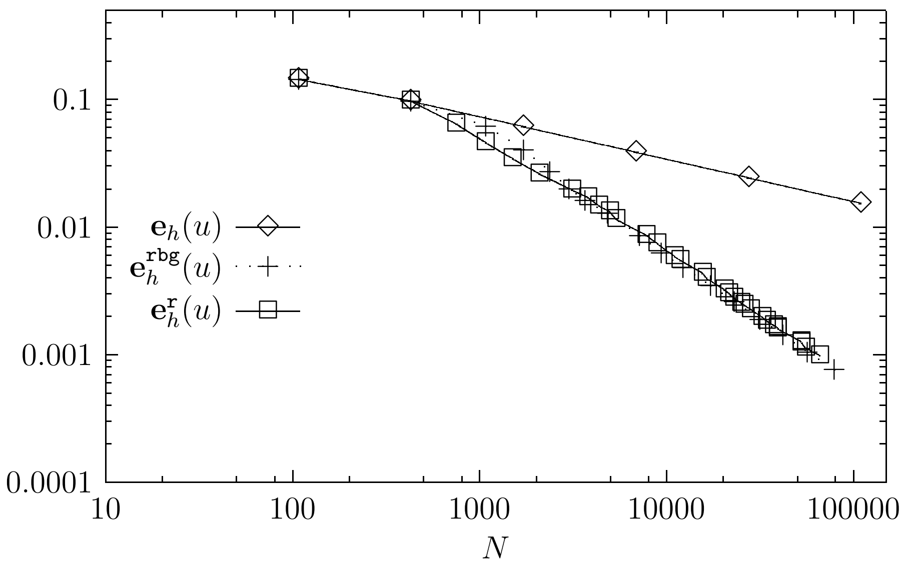

Figure 4.5.2: Example 1 with $\mathbf{P}_2 - \mathbf{P}_1 - \mathbf{P}_1$ approximation: energy-norm error $\mathbf{e}_h$ for the uniform and adaptive refinements.

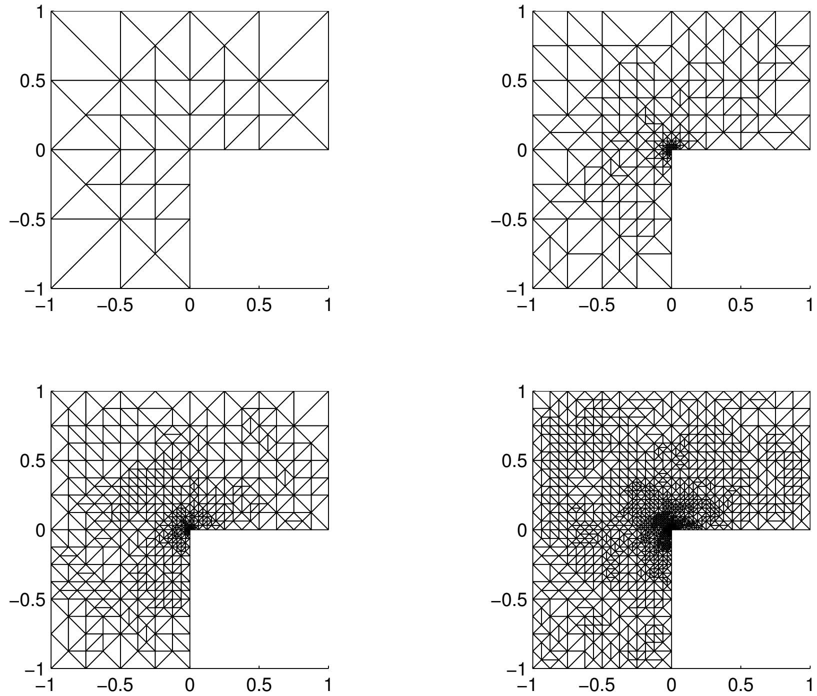

Figure 4.5.3: Example 1 with $\mathbf{P}_1 - \mathbf{P}_1 - \mathbf{P}_1$ approximation, without hanging nodes: adapted intermediate meshes with 900, 5505, 13770 and 34215 degrees of freedom.

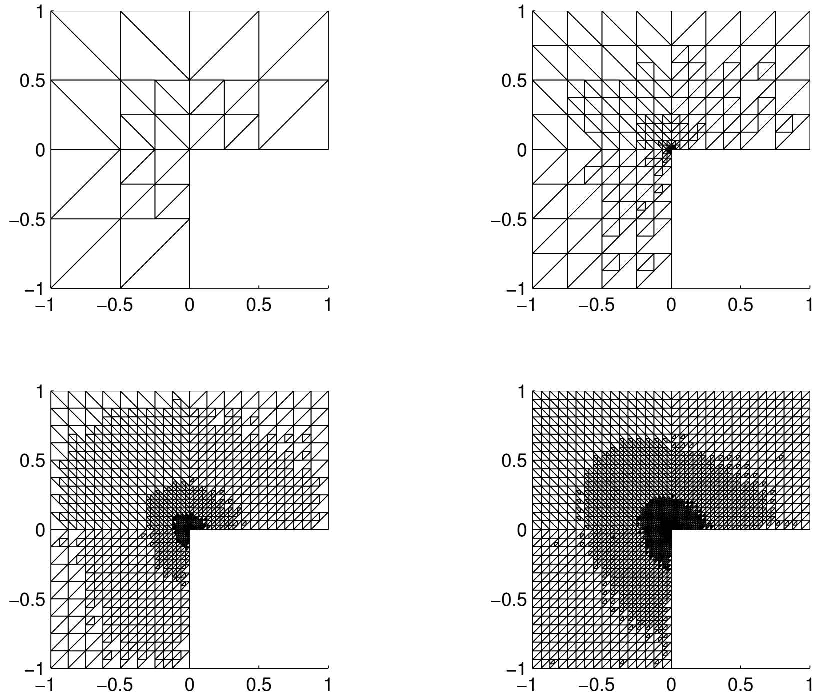

Figure 4.5.4: Example 1 with $\mathbf{P}_1 - \mathbf{P}_1 - \mathbf{P}_1$ approximation, with hanging nodes: adapted intermediate meshes with 630, 4770, 30735 and 74205 degrees of freedom.

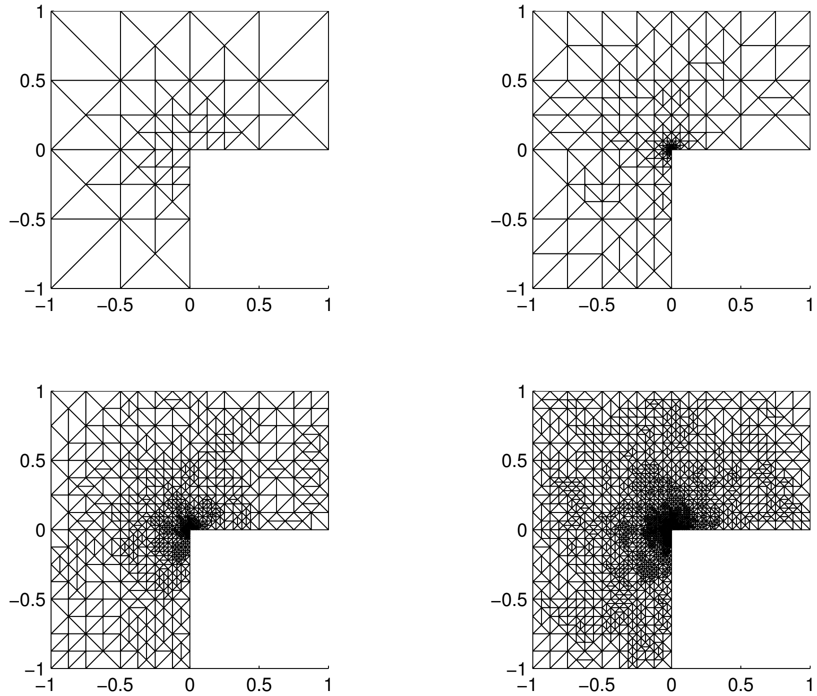

Figure 4.5.5: Example 1 with $\mathbf{P}_2 - \mathbf{P}_1 - \mathbf{P}_1$ approximation, without hanging nodes: adapted intermediate meshes with 1728, 7200, 31662 and 79146 degrees of freedom.

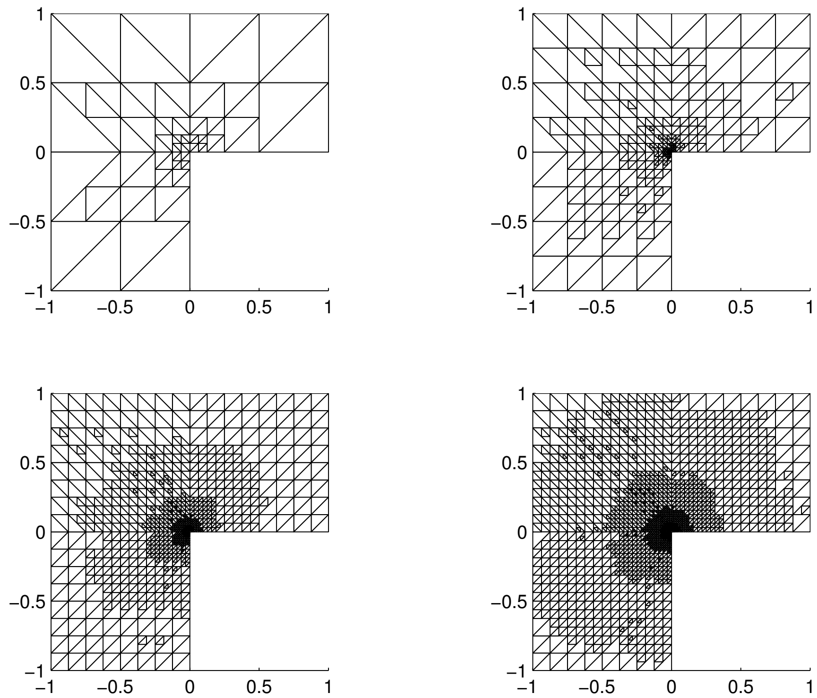

Figure 4.5.6: Example 1 with $\mathbf{P}_2 - \mathbf{P}_1 - \mathbf{P}_1$ approximation, with hanging nodes: adapted intermediate meshes with 1512, 8964, 32832 and 66582 degrees of freedom.

Table 4.5.4: Example 2 with $\mathbf{P}_1 - \mathbf{P}_1 - \mathbf{P}_1$ approximation: uniform, red-blue-green and red refinements.

N	$\mathbf{e}_h(u)$	$\gamma_h(u)$	$\mathbf{e}_h(u)/\bar{\boldsymbol{\eta}}$	$\mathbf{e}(\boldsymbol{\theta})$	$\gamma(\boldsymbol{\theta})$	$\mathbf{e}$	γ
60	5.2477	——	0.0328	3.7619	——	6.4568	——
240	11.0117	——	0.0555	11.9835	——	16.2745	——
960	19.7917	——	0.0916	17.2889	——	26.2796	——
3840	18.4130	0.1042	0.1384	13.4611	0.3611	22.8087	0.2044
15360	11.3188	0.7020	0.1616	6.7656	0.9925	13.1867	0.7905
61440	5.9466	0.9286	0.1649	2.6061	1.3763	6.4926	1.0222
210	11.0311	——	0.0556	11.9879	——	16.2909	——
390	19.7921	——	0.0916	17.2891	——	26.2800	——
570	18.4130	0.3806	0.1384	13.4611	1.3190	22.8088	0.7466
750	11.3248	3.5422	0.1616	6.7735	5.0050	13.1959	3.9881
930	6.3064	5.4431	0.1659	2.9868	7.6131	6.9779	5.9240
1200	4.2514	3.0938	0.1701	1.9646	3.2867	4.6834	3.1284
2730	2.5141	1.2782	0.1782	1.0662	1.4869	2.7308	1.3124
5430	1.6283	1.2633	0.1771	0.7153	1.1609	1.7785	1.2472
10500	1.2300	0.8506	0.1805	0.5483	0.8060	1.3467	0.8433
21000	0.8430	1.0900	0.1819	0.3659	1.1672	0.9190	1.1025
39870	0.6178	0.9695	0.1826	0.2610	1.0538	0.6707	0.9826
85080	0.4186	1.0269	0.1836	0.1767	1.0283	0.4544	1.0271
150	11.0423	——	0.0557	11.9912	——	16.3010	——
240	19.8073	——	0.0916	17.2943	——	26.2949	——
330	18.4164	0.4573	0.1385	13.4641	1.5723	22.8133	0.8920
420	11.3411	4.0206	0.1612	6.7564	5.7185	13.2011	4.5367
510	6.6767	5.4576	0.1609	2.6741	9.5476	7.1923	6.2557
780	4.6356	1.7174	0.1650	1.9498	1.4869	5.0289	1.6841
1770	2.7292	1.2929	0.1669	0.9041	1.8757	2.8751	1.3646
3390	1.9659	1.0095	0.1680	0.5876	1.3259	2.0519	1.0380
7080	1.2890	1.1463	0.1680	0.3064	1.7685	1.3249	1.1878
13740	0.9174	1.0257	0.1681	0.2252	0.9284	0.9447	1.0203
28320	0.6305	1.0368	0.1683	0.1452	1.2136	0.6470	1.0463
58470	0.4417	0.9820	0.1682	0.0945	1.1830	0.4517	0.9914
118680	0.3064	1.0331	0.1681	0.0636	1.1193	0.3129	1.0368

Table 4.5.5: Example 2 with $\mathbf{P}_2 - \mathbf{P}_1 - \mathbf{P}_1$ approximation: uniform and red-blue-green refinements.

N	$\mathbf{e}_h(u)$	$\gamma_h(u)$	$\mathbf{e}_h(u)/\bar{\boldsymbol{\eta}}$	$\mathbf{e}(\boldsymbol{\theta})$	$\gamma(\boldsymbol{\theta})$	$\mathbf{e}$	γ
72	9.7756	——	0.0633	3.6649	——	10.4401	——
288	47.9771	——	0.1249	14.2572	——	50.0508	——
1152	62.1958	——	0.1388	19.8255	——	65.2792	——
4608	33.8042	0.8796	0.1557	11.3388	0.8060	35.6552	0.8725
18432	9.6176	1.8134	0.1532	3.5060	1.6933	10.2368	1.8125
73728	2.6232	1.8743	0.1383	0.8519	2.0410	2.7581	1.8920
72	9.7757	——	0.0713	3.6649	——	10.4401	——
252	47.9613	——	0.2539	14.2599	——	50.0363	——
468	62.2383	——	0.3055	19.8285	——	65.3206	——
684	33.9208	3.1987	0.3348	11.3418	2.9441	35.7666	3.1742
900	9.7454	9.0894	0.3185	3.5169	8.5333	10.3605	9.0295
1116	3.8240	8.6977	0.3824	1.0457	11.2764	3.9644	8.9316
2304	1.6853	2.2605	0.1536	0.3826	2.7735	1.7282	2.2906
3600	1.0180	2.2590	0.1377	0.1756	3.4901	1.0330	2.3060
8280	0.4462	1.9804	0.1483	0.0871	1.6823	0.4546	1.9707
12312	0.2905	2.1634	0.1393	0.0492	2.8768	0.2946	2.1864
22860	0.1532	2.0669	0.1401	0.0260	2.0607	0.1554	2.0668
32904	0.1103	1.8056	0.1398	0.0177	2.1163	0.1117	1.8139
54144	0.0677	1.9585	0.1386	0.0113	1.7762	0.0686	1.9537
92844	0.0398	1.9643	0.1376	0.0063	2.1699	0.0403	1.9696
128448	0.0289	1.9775	0.1393	0.0046	1.8936	0.0293	1.9754

Table 4.5.6: Example 2 with $\mathbf{P}_2 - \mathbf{P}_1 - \mathbf{P}_1$ approximation: uniform and red refinements.

N	$\mathbf{e}_h(u)$	$\gamma_h(u)$	$\mathbf{e}_h(u)/\bar{\boldsymbol{\eta}}$	$\mathbf{e}(\boldsymbol{\theta})$	$\gamma(\boldsymbol{\theta})$	$\mathbf{e}$	γ
72	9.7756	——	0.0633	3.6649	——	10.4401	——
288	47.9771	——	0.1249	14.2572	——	50.0508	——
1152	62.1958	——	0.1388	19.8255	——	65.2792	——
4608	33.8042	0.8796	0.1557	11.3388	0.8060	35.6552	0.8725
18432	9.6176	1.8134	0.1532	3.5060	1.6933	10.2368	1.8125
73728	2.6232	1.8743	0.1383	0.8519	2.0410	2.7581	1.8920
72	9.7757	——	0.0633	3.6649	——	10.4401	——
180	47.9129	——	0.1249	14.2630	——	49.9908	——
288	62.1729	——	0.1389	19.8261	——	65.2575	——
396	33.7837	3.8306	0.1559	11.3368	3.5104	35.6351	3.7997
504	9.6875	10.3594	0.1491	3.5241	9.6899	10.3086	10.2865
612	6.1475	4.6848	0.1411	1.2990	10.2805	6.2833	5.0999
1368	1.9383	2.8698	0.1456	0.5163	2.2938	2.0059	2.8388
2988	1.1475	1.3420	0.1549	0.2627	1.7297	1.1772	1.3644
3744	0.7638	3.6087	0.1531	0.1907	2.8372	0.7873	3.5669
5796	0.5243	1.7219	0.1564	0.1232	2.0005	0.5386	1.7373
10116	0.3100	1.8861	0.1559	0.0653	2.2751	0.3169	1.9045
12492	0.2275	2.9336	0.1534	0.0531	1.9718	0.2336	2.8884
18540	0.1629	1.6919	0.1544	0.0370	1.8257	0.1671	1.6987
24048	0.1180	2.4810	0.1550	0.0275	2.2789	0.1211	2.4708
39600	0.0758	1.7716	0.1553	0.0160	2.1602	0.0775	1.7899
70056	0.0433	1.9609	0.1553	0.0093	1.9155	0.0443	1.9589
72216	0.0415	2.8852	0.1546	0.0082	8.2896	0.0423	3.1054
124812	0.0250	1.8402	0.1544	0.0048	1.9026	0.0255	1.8425

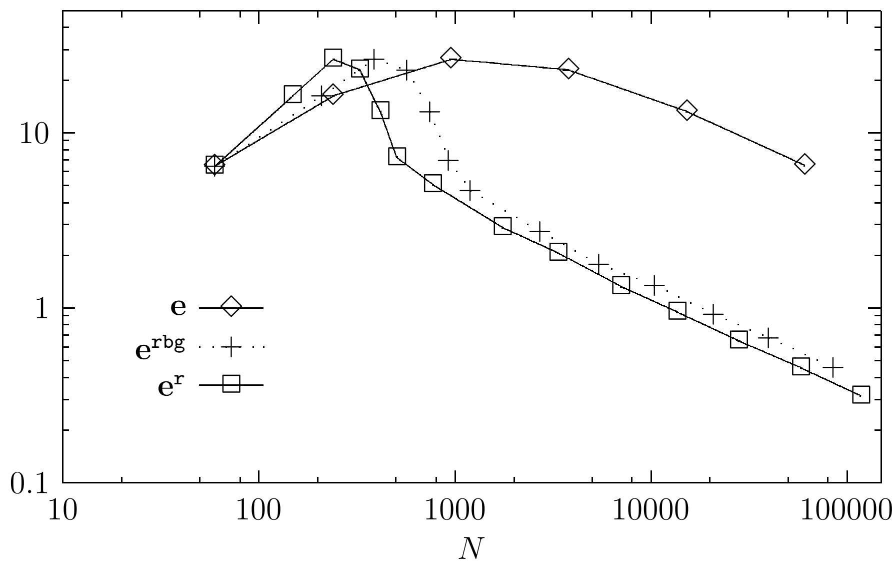

Figure 4.5.7: Example 2 with $\mathbf{P}_1 - \mathbf{P}_1 - \mathbf{P}_1$ approximation: global error $\mathbf{e}$ for the uniform and adaptive refinements.

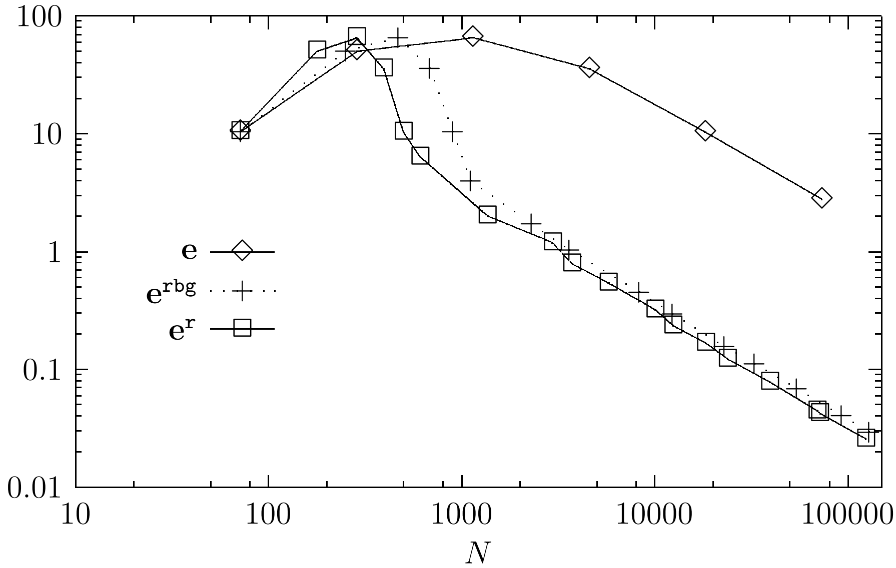

Figure 4.5.8: Example 2 with $\mathbf{P}_2 - \mathbf{P}_1 - \mathbf{P}_1$ approximation: global error $\mathbf{e}$ for the uniform and adaptive refinements.

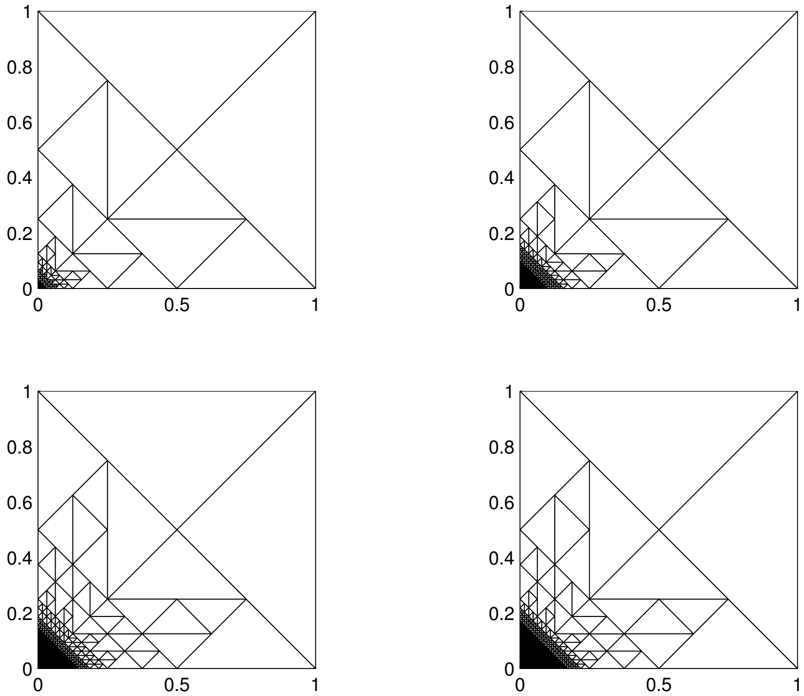

Figure 4.5.9: Example 2 with $\mathbf{P}_1 - \mathbf{P}_1 - \mathbf{P}_1$ approximation, with hanging nodes: adapted intermediate meshes with 1770, 13740, 58470 and 118680 degrees of freedom.

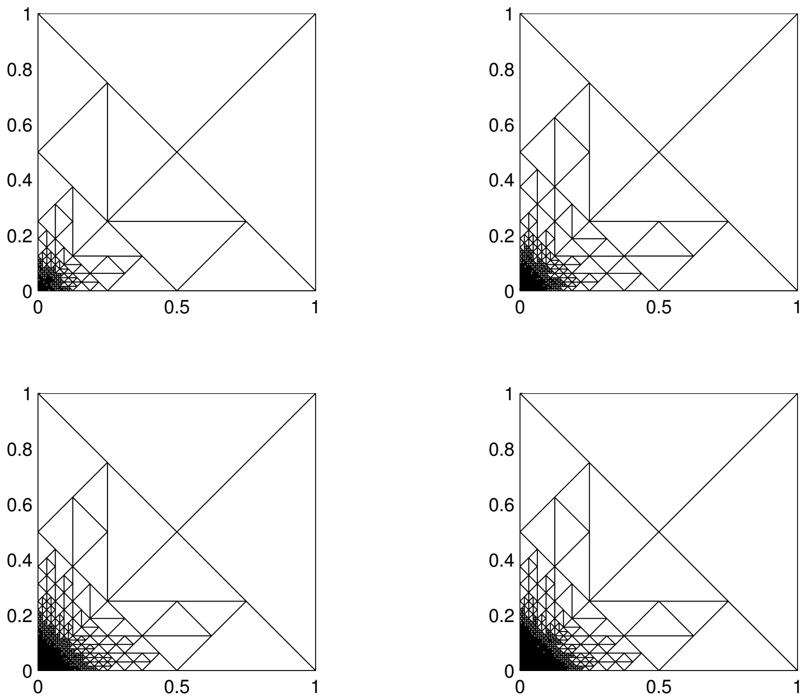

Figure 4.5.10: Example 2 with $\mathbf{P}_2 - \mathbf{P}_1 - \mathbf{P}_1$ approximation, with hanging nodes: adapted intermediate meshes with 5796, 18540, 39600 and 70056 degrees of freedom.

Table 4.5.7: Example 3 with $\mathbf{P}_1 - \mathbf{P}_1 - \mathbf{P}_1$ approximation: uniform and red-blue-green refinements.

N	$\mathbf{e}_h(u)$	$\gamma_h(u)$	$\mathbf{e}(\boldsymbol{\theta})$	$\gamma(\boldsymbol{\theta})$	$\mathbf{e}(\boldsymbol{\sigma})$	$\gamma(\boldsymbol{\sigma})$	γ	$\mathbf{e}/\boldsymbol{\vartheta}$
90	0.3024	——	0.1618	——	0.3815	——	——	0.4959
360	0.2241	0.4324	0.1267	0.3530	0.2924	0.3840	0.3972	0.2453
1440	0.1506	0.5728	0.0837	0.5971	0.1899	0.6227	0.6031	0.2329
5760	0.0983	0.6160	0.0541	0.6302	0.1206	0.6551	0.6387	0.2279
23040	0.0632	0.6368	0.0346	0.6452	0.0758	0.6687	0.6546	0.2264
92160	0.0403	0.6487	0.0220	0.6533	0.0475	0.6749	0.6628	0.2263
90	0.3023	——	0.1618	——	0.3815	——	——	0.4959
240	0.2631	0.2829	0.1375	0.3310	0.3198	0.3596	0.3295	0.3504
705	0.1968	0.5385	0.1075	0.4579	0.2475	0.4755	0.4961	0.2543
1635	0.1364	0.8716	0.0743	0.8761	0.1687	0.9105	0.8933	0.2380
2535	0.1046	1.2114	0.0543	1.4274	0.1230	1.4423	1.3564	0.1996
4605	0.0791	0.9345	0.0384	1.1630	0.0866	1.1732	1.0782	0.1925
9975	0.0508	1.1715	0.0234	1.2455	0.0537	1.2100	1.1968	0.1697
17700	0.0385	0.9674	0.0174	1.0374	0.0399	1.0330	1.0049	0.1670
25005	0.0322	1.0231	0.0143	1.1097	0.0332	1.0649	1.0504	0.1641
33810	0.0275	0.9612	0.0120	1.1289	0.0279	1.0983	1.0398	0.1635
59520	0.0206	1.4937	0.0091	1.2652	0.0213	1.2578	1.3626	0.1619
84120	0.0171	1.0660	0.0078	0.9220	0.0182	0.9140	0.9807	0.1593
97725	0.0157	1.1333	0.0072	1.0241	0.0168	1.0442	1.0804	0.1601
106890	0.0150	1.0616	0.0069	0.9835	0.0161	0.9728	1.0113	0.1594
144210	0.0130	0.9227	0.0059	1.0151	0.0138	1.0211	0.9787	0.1598
156315	0.0125	1.1213	0.0057	0.9702	0.0133	0.9755	1.0374	0.1596

Table 4.5.8: Example 3 with $\mathbf{P}_1 - \mathbf{P}_1 - \mathbf{P}_1$ approximation: uniform and red refinements.

N	$\mathbf{e}_h(u)$	$\gamma_h(u)$	$\mathbf{e}(\boldsymbol{\theta})$	$\gamma(\boldsymbol{\theta})$	$\mathbf{e}(\boldsymbol{\sigma})$	$\gamma(\boldsymbol{\sigma})$	γ	$\mathbf{e}/\boldsymbol{\vartheta}$
90	0.3024	——	0.1618	——	0.3815	——	——	0.4959
360	0.2241	0.4324	0.1267	0.3530	0.2924	0.3840	0.3972	0.2453
1440	0.1506	0.5728	0.0837	0.5971	0.1899	0.6227	0.6031	0.2329
5760	0.0983	0.6160	0.0541	0.6302	0.1206	0.6551	0.6387	0.2279
23040	0.0632	0.6368	0.0346	0.6452	0.0758	0.6687	0.6546	0.2264
92160	0.0403	0.6487	0.0220	0.6533	0.0475	0.6749	0.6628	0.2263
90	0.3024	——	0.1618	——	0.3815	——	——	0.4959
225	0.2820	0.1519	0.1350	0.3944	0.3153	0.4155	0.3146	0.3418
360	0.2786	0.0517	0.1403	——	0.3273	——	——	0.3702
855	0.1894	3.6092	0.1009	4.4728	0.2307	4.6473	4.2674	0.2844
1395	0.1484	0.9982	0.0717	1.3926	0.1632	1.4146	1.2521	0.2457
2340	0.1157	0.9605	0.0509	1.3265	0.1154	1.3398	1.1743	0.2354
3825	0.0934	0.8719	0.0379	1.1946	0.0865	1.1729	1.0316	0.2211
5805	0.0758	0.9966	0.0309	0.9865	0.0712	0.9291	0.9669	0.2035
7020	0.0636	1.8464	0.0270	1.4028	0.0621	1.4505	1.6361	0.1956
7740	0.0601	1.1621	0.0252	1.4318	0.0579	1.4140	1.2965	0.1918
10935	0.0516	0.8819	0.0215	0.9043	0.0494	0.9194	0.9003	0.1925
17055	0.0420	0.8491	0.0169	1.0962	0.0390	1.0270	0.9458	0.1882
21960	0.0378	0.8394	0.0147	1.0736	0.0342	1.0495	0.9463	0.1860
24165	0.0361	0.9534	0.0139	1.7702	0.0324	1.5732	1.2731	0.1874
33615	0.0305	1.0096	0.0121	0.8432	0.0283	0.8322	0.9223	0.1831
37170	0.0277	1.2146	0.0114	0.5658	0.0265	0.5143	0.8554	0.1806
67410	0.0212	0.8893	0.0087	0.8890	0.0203	0.8924	0.8907	0.1783
74160	0.0199	1.0892	0.0082	0.9718	0.0191	0.9677	1.0265	0.1811
85410	0.0184	1.1001	0.0076	1.0009	0.0178	0.9690	1.0342	0.1788
87885	0.0181	1.3083	0.0075	1.2781	0.0175	1.2479	1.2791	0.1797
105210	0.0168	0.8591	0.0068	1.0034	0.0160	1.0030	0.9342	0.1796

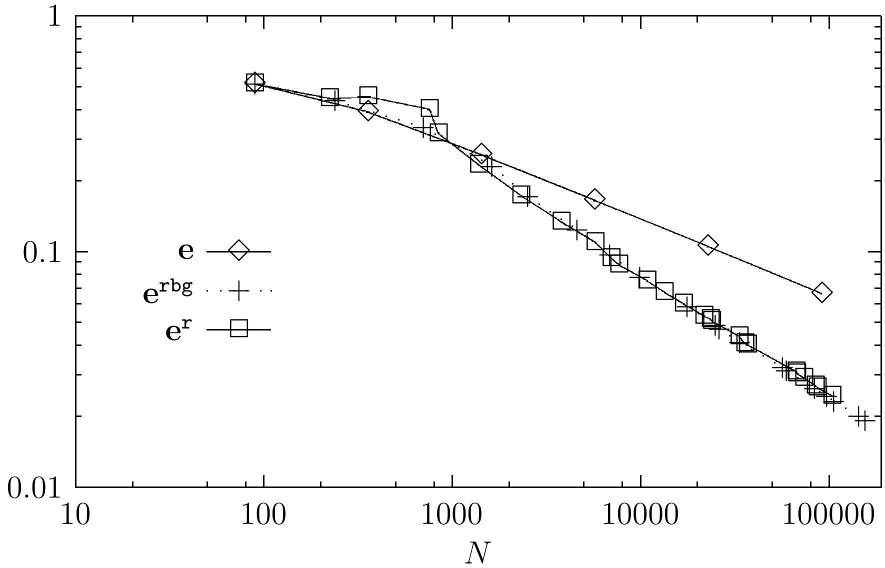

Figure 4.5.11: Example 3 with $\mathbf{P}_1 - \mathbf{P}_1 - \mathbf{P}_1$ approximation: global error $\mathbf{e}$ for the uniform and adaptive refinements.

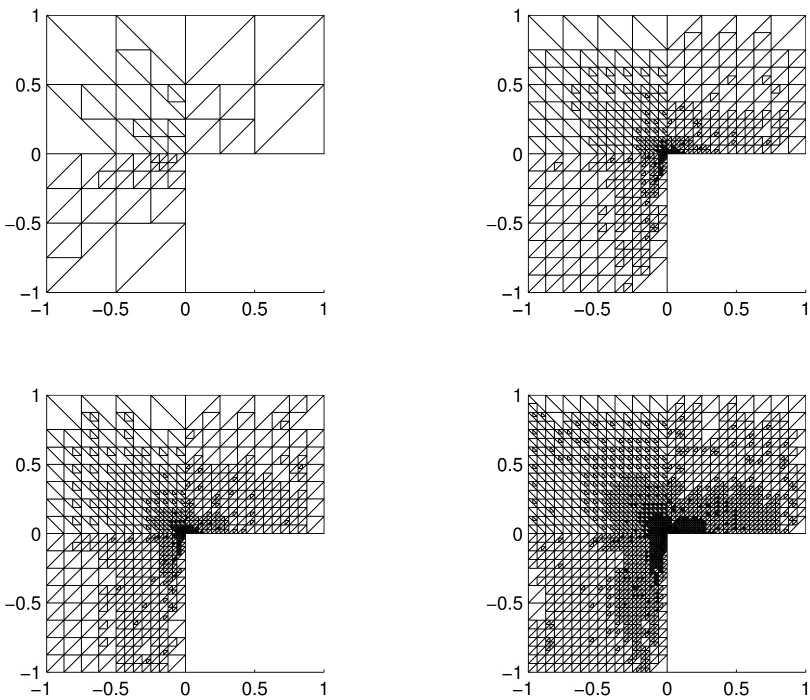

Figure 4.5.12: Example 3 with $\mathbf{P}_1 - \mathbf{P}_1 - \mathbf{P}_1$ approximation, with hanging nodes: adapted intermediate meshes with 1395, 17055, 24165 and 67410 degrees of freedom.

Table 4.5.9: Example 4 with $\mathbf{P}_1 - \mathbf{P}_1 - \mathbf{P}_1$ approximation: uniform and red-blue-green refinements.

N	$\mathbf{e}_h(u)$	$\gamma_h(u)$	$\mathbf{e}(\boldsymbol{\theta})$	$\gamma(\boldsymbol{\theta})$	$\mathbf{e}(\boldsymbol{\sigma})$	$\gamma(\boldsymbol{\sigma})$	γ	$\mathbf{e}/\boldsymbol{\vartheta}$
60	5.2179	——	3.8378	——	8.2157	——	——	0.0571
240	11.1938	——	12.4913	——	25.3062	——	——	0.0544
960	20.0539	——	17.8717	——	35.8747	——	——	0.0665
3840	18.5286	0.1141	13.7141	0.3820	27.4762	0.3847	0.3214	0.0952
15360	11.3263	0.7100	6.8160	1.0086	13.6477	1.0095	0.9165	0.1145
61440	5.9360	0.9321	2.6093	1.3852	5.2221	1.3859	1.1903	0.1129
60	5.2179	——	3.8378	——	8.2157	——	——	0.0571
210	11.2165	——	12.4972	——	25.3185	——	——	0.0544
390	20.0543	——	17.8718	——	35.8750	——	——	0.0665
570	18.5286	0.4170	13.7144	1.3954	27.4767	1.4055	1.1742	0.0952
750	11.3327	3.5827	6.8240	5.0867	13.6658	5.0900	4.6226	0.1145
930	6.2994	5.4598	2.9939	7.6596	6.0031	7.6482	6.7501	0.1130
1110	4.6206	3.5034	2.3143	2.9106	4.6473	2.8936	3.1731	0.1141
2850	2.4550	1.3413	1.0053	1.7685	2.0201	1.7670	1.5577	0.1251
4740	1.7037	1.4361	0.6650	1.6244	1.3397	1.6145	1.5166	0.1151
12420	1.0738	0.9584	0.4336	0.8876	0.8727	0.8899	0.9278	0.1217
19980	0.8349	1.0582	0.3337	1.1021	0.6735	1.0900	1.0736	0.1234
42540	0.5682	1.0184	0.2293	0.9931	0.4624	0.9947	1.0076	0.1246
81780	0.4089	1.0064	0.1583	1.1337	0.3194	1.1320	1.0621	0.1235
158940	0.2944	0.9883	0.1141	0.9840	0.2304	0.9834	0.9863	0.1247

Table 4.5.10: Example 4 with $\mathbf{P}_1 - \mathbf{P}_1 - \mathbf{P}_1$ approximation: uniform and red refinements.

N	$\mathbf{e}_h(u)$	$\gamma_h(u)$	$\mathbf{e}(\boldsymbol{\theta})$	$\gamma(\boldsymbol{\theta})$	$\mathbf{e}(\boldsymbol{\sigma})$	$\gamma(\boldsymbol{\sigma})$	γ	$\mathbf{e}/\boldsymbol{\vartheta}$
60	5.2179	——	3.8378	——	8.2157	——	——	0.0571
240	11.1938	——	12.4913	——	25.3062	——	——	0.0544
960	20.0539	——	17.8717	——	35.8747	——	——	0.0665
3840	18.5286	0.1141	13.7141	0.3820	27.4762	0.3847	0.3214	0.0952
15360	11.3263	0.7100	6.8160	1.0086	13.6477	1.0095	0.9165	0.1145
61440	5.9360	0.9321	2.6093	1.3852	5.2221	1.3859	1.1903	0.1129
60	5.2179	——	3.8378	——	8.2157	——	——	0.0571
150	11.2312	——	12.5029	——	25.3278	——	——	0.0545
240	20.0733	——	17.8799	——	35.8897	——	——	0.0666
330	18.5336	0.5012	13.7184	1.6639	27.4846	1.6757	1.4008	0.0952
420	11.3494	4.0671	6.8065	5.8122	13.6306	5.8160	5.2721	0.1140
510	6.6772	5.4642	2.6870	9.5742	5.3880	9.5607	7.7064	0.1053
690	5.0636	1.8301	1.7992	2.6537	3.6197	2.6318	2.1675	0.1056
1320	3.3137	1.3072	0.9221	2.0609	1.8620	2.0494	1.5561	0.1048
2760	2.1691	1.1489	0.5100	1.6054	1.0279	1.6109	1.2639	0.1034
5370	1.5411	1.0271	0.4266	0.5367	0.8572	0.5454	0.9073	0.1085
8610	1.1610	1.1998	0.2889	1.6505	0.5813	1.6455	1.3151	0.1065
16260	0.8398	1.0186	0.1616	1.8263	0.3269	1.8100	1.1746	0.1039
32910	0.5897	1.0029	0.1167	0.9248	0.2366	0.9170	0.9892	0.1048
55590	0.4539	0.9979	0.0856	1.1797	0.1737	1.1798	1.0271	0.1045
106710	0.3241	1.0330	0.0565	1.2726	0.1141	1.2880	1.0692	0.1038

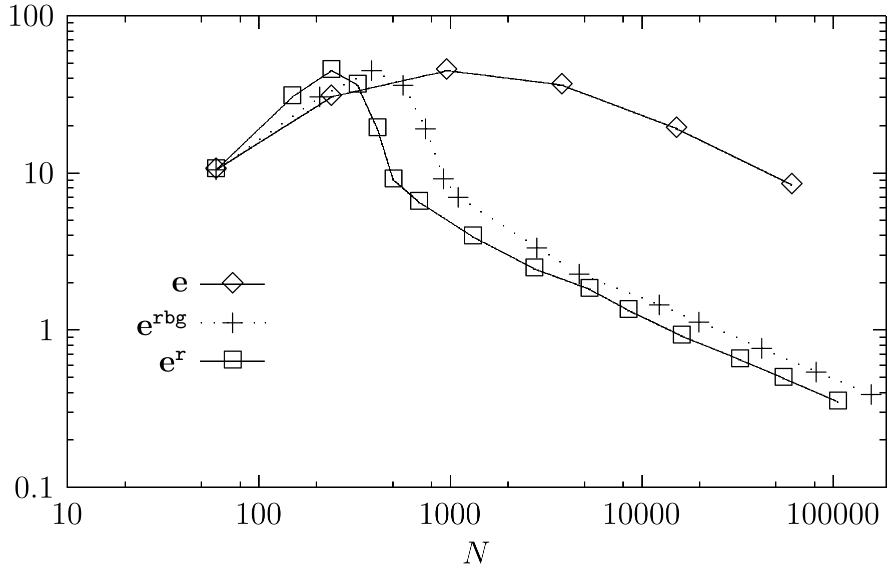

Figure 4.5.13: Example 4 with $\mathbf{P}_1 - \mathbf{P}_1 - \mathbf{P}_1$ approximation: global error $\mathbf{e}$ for the uniform and adaptive refinements.

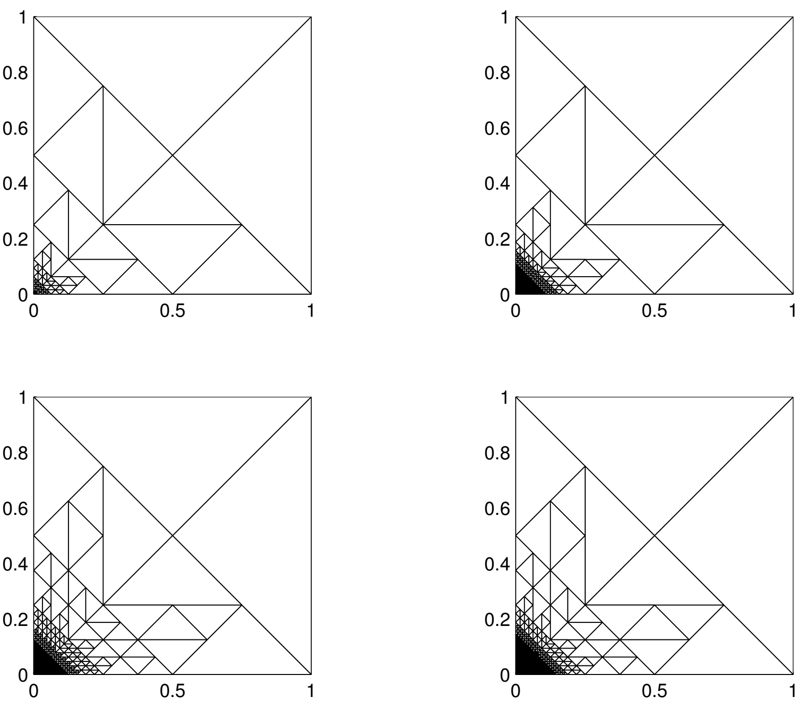

Figure 4.5.14: Example 4 with $\mathbf{P}_1 - \mathbf{P}_1 - \mathbf{P}_1$ approximation, with hanging nodes: adapted intermediate meshes with 1320, 16260, 32910 and 55590 degrees of freedom.

Chapter 5

A mixed local discontinuous finite element method for a class of quasi-Newtonian Stokes flows

In this chapter we present and analyze a new mixed local discontinuous Galerkin (LDG) method for a class of nonlinear Stokes model that appears in quasi-Newtonian Stokes fluids. The approach is based on the introduction of the flux and the tensor gradient of the velocity as further unknowns. Moreover, a suitable Lagrange multiplier is needed, in order to ensure that the corresponding discrete variational formulation is well posed. This yields a two-fold saddle point operator equation as the resulting LDG mixed formulation, which is reduced to a dual mixed formulation. Then, applying a slight generalization of the well known Babuška-Brezzi theory, we prove that the discrete formulation is well posed, and derive the associated a priori error analysis. We also develop an a-posteriori error estimate and propose a reliable adaptive algorithm to compute the finite element solutions. Finally, several numerical results illustrate the performance of the method and its capability to localize boundary and inner layers, as well as singularities.

5.1 Introduction

Nowadays, the discontinuous Galerkin (DG) methods (see, for e.g. [8], and the references therein for an overview) are widely used to solve diverse problems in physics and engineering sciences. This is mainly due to the fact that no interelement continuity is required, which is attractive to be analized in the frame of the *hp*-version. There are many applications of these approaches to different kind of linear elliptic problems, such as the Stokes, Maxwell and Oseen equations (see, for e.g., [41], [42], [43], [85]). Recently, we developed in [32] the extension of one of these methods, the local discontinuous Galerkin (LDG) method, to a class of nonlinear diffusion problems. Then, in [30] we derived an explicit reliable a-posteriori error estimate for the nonlinear model presented in [32], respect to the energy norm. There, we first analyze the corresponding linear elliptic model, via a suitable Helmholtz decomposition of the gradient of the error. Then, we extend the idea to the nonlinear case.

In the present chapter we are interested in certain type of nonlinear Stokes models, whose kinematic viscosities are nonlinear monotone functions of the gradient of the velocity. In order to define it explicitly, we first let Ω be a bounded open (polygon) subset of $\mathbf{R}^2$ with Lipschitz continuous (polygonal) boundary Γ. Then, given $\mathbf{f} \in [L^2(\Omega)]^2$ and $\mathbf{g} \in [H^{1/2}(\Gamma)]^2$, we look for the velocity $\mathbf{u} := (u_1, u_2)^{\mathbf{T}}$ and the pressure p of a fluid occupying the region Ω, such that

$$\begin{aligned} -\,\mathbf{div}\,(\psi(|\nabla \mathbf{u}|)\,\nabla \mathbf{u} \,-\, p\,\mathbf{I}) = \mathbf{f} \quad \text{in} \quad \Omega\,, \\ \operatorname{div}\mathbf{u} = 0 \quad \text{in} \quad \Omega \quad \text{and} \quad \mathbf{u} = \mathbf{g} \quad \text{on} \quad \Gamma\,, \end{aligned} \tag{5.1.1}$$

where $\mathbf{div}$ and div are the usual vector and scalar divergence operators, $\nabla\mathbf{u}$ is the tensor gradient of $\mathbf{u}$, $|\cdot|$ is the euclidean norm of $\mathbf{R}^2$, $\mathbf{I}$ is the identity matrix of $\mathbf{R}^{2\times 2}$, and $\psi : \mathbf{R}^+ \to \mathbf{R}^+$ is the nonlinear kinematic viscosity function of the fluid. We remark that, as a consequence of the incompressibility of the fluid, the Dirichlet datum $\mathbf{g}$ must satisfy the compatibility condition $\int_\Gamma \mathbf{g}\cdot\mathbf{n} = 0$, where $\mathbf{n}$ is the unit outward normal to Γ. Hereafter, given any Hilbert space S, we denote by S^2 and $S^{2\times 2}$ the spaces of vectors and tensors of order 2, respectively, with entries in S, provided with the product norms induced by the norm of S.

We now let $\psi_{ij} : \mathbf{R}^{2\times 2} \to \mathbf{R}$ be the mapping given by $\psi_{ij}(\mathbf{r}) := \psi(|\mathbf{r}|) r_{ij}$ for all $\mathbf{r} := (r_{ij}) \in \mathbf{R}^{2\times 2}$, for all $i, j \in \{1, 2\}$, and define the tensor $\boldsymbol{\psi} : \mathbf{R}^{2\times 2} \to \mathbf{R}^{2\times 2}$ by $\boldsymbol{\psi}(\mathbf{r}) := (\psi_{ij}(\mathbf{r}))$ for all $\mathbf{r} \in \mathbf{R}^{2\times 2}$. For vectors $\mathbf{v}, \mathbf{w} \in \mathbf{R}^2$, and tensors $\mathbf{r}, \mathbf{s} \in \mathbf{R}^{2\times 2}$, we use the standard notation $\mathbf{r} : \mathbf{s} := \sum_{i,j=1}^{2} r_{ij} s_{ij}$, and denote by $\mathbf{v} \otimes \mathbf{w}$ the tensor of order 2 whose ij-th entry is $v_i\, w_j$. Note that the following identity holds: $\mathbf{v} \cdot (\mathbf{r}\,\mathbf{w}) = \mathbf{r} : (\mathbf{v} \otimes \mathbf{w})$.

Then, throughout this chapter we assume that ψ is of class C^1 and that there exist C_1, $C_2 > 0$ such that for all $\mathbf{r} := (r_{ij})$, $\mathbf{s} := (s_{ij}) \in \mathbf{R}^{2\times 2}$, there hold

$$|\psi_{ij}(\mathbf{r})| \,\leq\, C_1||\mathbf{r}||_{\mathbf{R}^{2\times 2}}, \qquad \left|\frac{\partial}{\partial r_{kl}}\psi_{ij}\right| \,\leq\, C_1 \qquad \forall\, i,j,k,l \in \{1,2\}\,, \tag{5.1.2}$$

and

$$\sum_{i,j,k,l=1}^{2} \frac{\partial}{\partial r_{kl}}\psi_{ij}(\mathbf{r}) s_{ij} s_{kl} \,\geq\, C_2\,||\mathbf{s}||^2_{\mathbf{R}^{2\times 2}}\,. \tag{5.1.3}$$

In [53, 54], a dual-mixed formulation of (5.1.1) is proposed, which is based on low-order finite element subspaces (Raviart-Thomas spaces of order zero to approximate the flux, and piecewise constants to approximate the other unknowns). There, the variables $\mathbf{t} := \nabla\mathbf{u}$ and $\boldsymbol{\sigma} := \boldsymbol{\psi}(\mathbf{t}) - p\mathbf{I}$ are introduced as auxiliary unknowns, as well as the Lagrange multiplier ξ, yielding the continuous formulation: *Find* $(\mathbf{t},\boldsymbol{\sigma},p,\mathbf{u},\xi) \in [L^2(\Omega)]^{2\times 2} \times H(\mathrm{div}\,;\Omega) \times L^2(\Omega) \times [L^2(\Omega)]^2 \times \mathbf{R}$ *such that*

$$\begin{aligned}
&\int_\Omega \boldsymbol{\psi}(\mathbf{t}) : \mathbf{s} \quad - \int_\Omega \boldsymbol{\sigma} : \mathbf{s} - \int_\Omega p\,\mathrm{tr}\,(\mathbf{s}) && = 0\,,\\
&- \int_\Omega \boldsymbol{\tau} : \mathbf{t} - \int_\Omega q\,\mathrm{tr}\,(\mathbf{t}) \quad - \int_\Omega \mathbf{u}\cdot\mathbf{div}\,(\boldsymbol{\tau}) + \xi \int_\Omega \mathrm{tr}\,(\boldsymbol{\tau}) && = -\langle \boldsymbol{\tau\nu}, \mathbf{g}\rangle_\Gamma\,,\\
&\qquad - \int_\Omega \mathbf{v}\cdot\mathbf{div}\,(\boldsymbol{\sigma}) + \eta \int_\Omega \mathrm{tr}\,(\boldsymbol{\sigma}) && = \int_\Omega \mathbf{f}\cdot\mathbf{v}\,,
\end{aligned} \tag{5.1.4}$$

for all $(\mathbf{s},\boldsymbol{\tau},q,\mathbf{v},\eta) \in [L^2(\Omega)]^{2\times 2} \times H(\mathrm{div}\,;\Omega) \times L^2(\Omega) \times [L^2(\Omega)]^2 \times \mathbf{R}$.

On the other hand, it is important to mention that in [43], the LDG method is applied to solve the Stokes problem by polinomial aproximations for the velocity and the pressure. Besides, in [93], Schötzau et al. present several mixed DG methods for the Stokes problem, in the classical velocity-pressure formulation, and propose an abstract framework for their analysis. Also, they derived a priori error estimates for hp-approximations on tensor product meshes.

Our plan now is to apply the mixed LDG approach to solve (5.1.1), extending in this way the application of the analysis developed in [32]. We consider regular and conform meshes made of straight triangles, and avoid the zero mean value condition on the pressure by introducing a suitable Lagrange multiplier. The rest of the chapter is organized as follows. In Section 5.2 we introduce the mixed local discontinuous Galerkin scheme, including the definition of the corresponding numerical fluxes and the reduced mixed formulation, whose unique solvability and stability is shown in Section 5.3. In contrast to the analysis presented in [53], we only need piecewise discontinuous polynomials to approximate the unknowns. In Section 5.4 we derive the usual a-priori error estimates in energy and L^2 norms. Then, in Section 5.5 we present an a-posteriori error analysis, following the approach given in [70]. Finally, some numerical experiments that validate the well behaviour of our a-posteriori estimator are reported in Section 5.6, even considering meshes with hanging nodes, whose analysis is not covered yet.

5.2 The mixed LDG formulation

We follow [53] and introduce the tensor gradient $\mathbf{t} := \nabla \mathbf{u}$ in Ω, and the flux $\boldsymbol{\sigma} := \boldsymbol{\psi}(\mathbf{t}) - p\mathbf{I}$ in Ω as additional unknowns. Since $\operatorname{div} \mathbf{u} = \operatorname{tr}(\nabla \mathbf{u})$, the incompressibility condition can be rewritten as $\operatorname{tr}(\mathbf{t}) = 0$ in Ω. In this way, (5.1.1) can be reformulated as the following problem in $\bar{\Omega}$: *Find* $(\mathbf{t}, \boldsymbol{\sigma}, \mathbf{u}, p)$ *in appropriate spaces such that, in the distributional sense,*

$$\begin{gathered}
\mathbf{t} = \nabla \mathbf{u} \quad \text{in} \quad \Omega\,, \qquad \boldsymbol{\sigma} = \boldsymbol{\psi}(\mathbf{t}) - p\mathbf{I} \quad \text{in} \quad \Omega\,, \qquad -\mathbf{div}\,\boldsymbol{\sigma} \;=\; \mathbf{f} \quad \text{in} \quad \Omega\,, \\
\operatorname{tr}(\mathbf{t}) = 0 \quad \text{in} \quad \Omega\,, \quad \text{and} \quad \mathbf{u} = \mathbf{g} \quad \text{on} \quad \Gamma\,.
\end{gathered} \tag{5.2.1}$$

As in [32], we now let $\mathcal{T}_h$ be a shape-regular triangulation of $\bar{\Omega}$ made up of straight triangles T with diameter h_T and unit outward normal to ∂T given by $\boldsymbol{\nu}_T$. As usual, the index h also denotes $h := \max_{T \in \mathcal{T}_h} h_T$. In addition, we define the edges of $\mathcal{T}_h$ as follows. An *interior edge of* $\mathcal{T}_h$ *is the* (non-empty) interior of $\partial T \cap \partial T'$, where T and T' are two adjacent elements of $\mathcal{T}_h . h_e$ represents its diameter. Also, in what follows we assume that $\mathcal{T}_h$ is of *bounded variation*, that is there exists a constant $l > 1$, independent of h, such that $l^{-1} \;\leq\; \dfrac{h_T}{h_{T'}} \;\leq\; l$, for each pair T, $T' \in \mathcal{T}_h$ sharing an interior edge.

The LDG variational formulation is described next. We first multiply the first fourth equations of (5.2.1) by smooth test functions $\boldsymbol{\tau}$, $\mathbf{s}$, $\mathbf{v}$ and q, respectively, integrate by parts over each $T \in \mathcal{T}_h$, and obtain

$$
\begin{aligned}
&\int_T \boldsymbol{\psi}(\mathbf{t}) : \mathbf{s} \,-\, \int_T \boldsymbol{\sigma} : \mathbf{s} \,-\, \int_{\partial T} p \,\mathrm{tr}\,(\mathbf{s}) \;=\; 0\,,\\
&\int_T \mathbf{t} : \boldsymbol{\tau} \,+\, \int_T \mathbf{u} \cdot \mathrm{div}\,\boldsymbol{\tau} \,-\, \int_{\partial T} \boldsymbol{\tau} : \mathbf{u} \otimes \boldsymbol{\nu}_T \;=\; 0\,,\\
&\int_T q \,\mathrm{tr}\,(\mathbf{t}) \;=\; 0\,,\\
&\int_T \boldsymbol{\sigma} : \nabla \mathbf{v} \,-\, \int_{\partial T} \boldsymbol{\sigma} : \mathbf{v} \otimes \boldsymbol{\nu}_T \;=\; \int_T \mathbf{f} \cdot \mathbf{v}\,.
\end{aligned}
\tag{5.2.2}
$$

Then, we want to approximate the exact solution $(\mathbf{t}, \boldsymbol{\sigma}, \mathbf{u}, p)$ by discrete functions $(\mathbf{t}_h, \boldsymbol{\sigma}_h, \mathbf{u}_h, p_h)$ in the finite element space $\boldsymbol{\Sigma}_h \times \boldsymbol{\Sigma}_h \times V_h \times W_h$, where

$$
\boldsymbol{\Sigma}_h := \left\{ \mathbf{t}_h \in [L^2(\Omega)]^{2\times 2} : \quad \mathbf{t}_h|_T \in [\mathbf{P}_r(T)]^{2\times 2} \quad \forall\, T \in \mathcal{T}_h \right\} \,,
$$

$$
\mathbf{V}_h := \left\{ \mathbf{v}_h \in [L^2(\Omega)]^{2} : \quad \mathbf{v}_h|_T \in [\mathbf{P}_k(T)]^{2} \quad \forall\, T \in \mathcal{T}_h \right\} \,,
$$

$$
W_h := \left\{ q_h \in L^2(\Omega) : \quad q_h|_T \in \mathbf{P}_{k-1}(T) \quad \forall\, T \in \mathcal{T}_h \right\} \,,
$$

with $k \geq 1$ and $r = k$ or $r = k - 1$. Hereafter, given an integer $\kappa \geq 0$ we denote by $\mathbf{P}_\kappa(T)$ the space of polynomials of total degree at most κ on T.

From now on, the spaces $\boldsymbol{\Sigma}_h$ and W_h are provided with the usual product norms of $[L^2(\Omega)]^{2\times 2}$ and $L^2(\Omega)$, which are denoted by $||\cdot||_{[L^2(\Omega)]^{2\times 2}}$ and $||\cdot||_{L^2(\Omega)}$, respectively. The norm for $\mathbf{V}_h$ will be defined later on in Section 5.3.

At this point we recall that the idea of the LDG method is to enforce the conservation laws given in (5.2.2) with the traces of $\boldsymbol{\sigma}$ and $\mathbf{u}$ on the boundary of each $T \in \mathcal{T}_h$ being replaced by suitable numerical approximations of them. In other words, we consider the following formulation: *Find* $(\mathbf{t}_h, \boldsymbol{\sigma}_h, \mathbf{u}_h, p_h) \in \boldsymbol{\Sigma}_h \times \boldsymbol{\Sigma}_h \times \mathbf{V}_h \times W_h$ *such that on each* $T \in \mathcal{T}_h$ *there hold*

$$
\begin{aligned}
&\int_T \boldsymbol{\psi}(\mathbf{t}_h) : \mathbf{s}_h \,-\, \int_T \boldsymbol{\sigma}_h : \mathbf{s}_h \,-\, \int_T p_h \operatorname{tr}(\mathbf{s}_h) \,=\, 0\,,\\
&\int_T \mathbf{t}_h : \boldsymbol{\tau}_h \,+\, \int_T \mathbf{u}_h \cdot \operatorname{div} \boldsymbol{\tau}_h \,-\, \int_{\partial T} \boldsymbol{\tau}_h : \hat{\mathbf{u}} \otimes \boldsymbol{\nu}_T \,=\, 0\,,\\
&\int_T q_h \operatorname{tr}(\mathbf{t}_h) \,=\, 0\,,\\
&\int_T \boldsymbol{\sigma}_h : \nabla \mathbf{v}_h \,-\, \int_{\partial T} \widehat{\boldsymbol{\sigma}} : \mathbf{v}_h \otimes \boldsymbol{\nu}_T \,=\, \int_T \mathbf{f} \cdot \mathbf{v}_h\,,
\end{aligned}
\tag{5.2.3}
$$

for all $(\mathbf{s}_h, \boldsymbol{\tau}_h, \mathbf{v}_h, q_h) \in \boldsymbol{\Sigma}_h \times \boldsymbol{\Sigma}_h \times \mathbf{V}_h \times W_h$, we have on each $T \in \mathcal{T}_h$, where the *numerical fluxes* $\hat{\mathbf{u}}$ and $\widehat{\boldsymbol{\sigma}}$, which usually depend on $\mathbf{u}_h$, $\boldsymbol{\sigma}_h$, and the boundary conditions, are chosen so that some compatibility condition are satisfied.

Then, we define the *average* and the *jump* of $q := (q_T)_{T\in\mathcal{T}_h} \in \prod_{T\in\mathcal{T}_h} L^2(\partial T)$ across $e \subseteq \mathcal{E}_I$ by

$$
\{q\} := \frac{1}{2}(q_{T,e} + q_{T',e}) \quad \text{and} \quad [\![q]\!] := q_{T,e}\,\boldsymbol{\nu}_T + q_{T',e}\,\boldsymbol{\nu}_{T'}\,. \tag{5.2.4}
$$

Analogously, the corresponding *average* and *jump* of $\boldsymbol{\zeta} := (\boldsymbol{\zeta}_T)_{T\in\mathcal{T}_h} \in \prod_{T\in\mathcal{T}_h} [L^2(\partial T)]^{2\times 2}$ across $e \subseteq \mathcal{E}_I$ are defined by

$$
\{\boldsymbol{\zeta}\} := \frac{1}{2}(\boldsymbol{\zeta}_{T,e} + \boldsymbol{\zeta}_{T',e}) \quad \text{and} \quad [\![\boldsymbol{\zeta}]\!] := \boldsymbol{\zeta}_{T,e}\boldsymbol{\nu}_T + \boldsymbol{\zeta}_{T',e}\boldsymbol{\nu}_{T'}\,. \tag{5.2.5}
$$

Finally, for any $\mathbf{v} := (\mathbf{v}_T)_{T\in\mathcal{T}_h} \in \prod_{T\in\mathcal{T}_h} [L^2(\partial T)]^2$, we let its average and jump across $e \subseteq \mathcal{E}_I$ by

$$
\{\mathbf{v}\} := \frac{1}{2}(\mathbf{v}_{T,e} + \mathbf{v}_{T',e}) \quad \text{and} \quad [\![\mathbf{v}]\!] := \mathbf{v}_{T,e} \cdot \boldsymbol{\nu}_T + \mathbf{v}_{T',e} \cdot \boldsymbol{\nu}_{T'}\,, \tag{5.2.6}
$$

and introduce its *tensorial* jump by

$$
\underline{[\![\mathbf{v}]\!]} := \mathbf{v}_{T,e} \otimes \boldsymbol{\nu}_T + \mathbf{v}_{T',e} \otimes \boldsymbol{\nu}_{T'}\,, \tag{5.2.7}
$$

We notice that for any $e \subseteq \mathcal{E}_D$, the traces on e of every scalar, vector and tensor functions $q \in \prod_{T\in\mathcal{T}_h} L^2(\partial T)$, $\mathbf{v} \in \prod_{T\in\mathcal{T}_h} [L^2(\partial T)]^2$, and $\boldsymbol{\zeta} \in \prod_{T\in\mathcal{T}_h} [L^2(\partial T)]^{2\times 2}$, respectively, are uniquely defined, and hence we set

$$
\{q\} := q\,, \qquad \{\mathbf{v}\} := \mathbf{v}\,, \quad \text{and} \quad \{\boldsymbol{\zeta}\} := \boldsymbol{\zeta}\,,
$$

as well as

$$
[\![q]\!] := q\,\boldsymbol{\nu}_T\,, \qquad [\![\mathbf{v}]\!] := \mathbf{v} \cdot \boldsymbol{\nu}_T\,, \qquad \underline{[\![\mathbf{v}]\!]} := \mathbf{v} \otimes \boldsymbol{\nu}_T\,, \quad \text{and} \quad [\![\boldsymbol{\tau}]\!] := \boldsymbol{\tau}\,\boldsymbol{\nu}_T\,.
$$

We are now ready to complete the mixed LDG formulation (5.2.3). Indeed, using the approach from [32] (see also [41], [43], [93]), we define the numerical fluxes $\widehat{\mathbf{u}}$ and $\widehat{\boldsymbol{\sigma}}$ for each $T \in \mathcal{T}_h$, as follows:

$$\widehat{\mathbf{u}}_{T,e} := \begin{cases} \{\mathbf{u}_h\} + \underline{[\![\mathbf{u}_h]\!]}\,\boldsymbol{\beta} & \text{if } e \subseteq \mathcal{E}_I\,, \\ \mathbf{g} & \text{if } e \subseteq \mathcal{E}_D\,, \end{cases} \tag{5.2.8}$$

and

$$\widehat{\boldsymbol{\sigma}}_{T,e} := \begin{cases} \{\boldsymbol{\sigma}_h\} - [\![\boldsymbol{\sigma}_h]\!] \otimes \boldsymbol{\beta} - \alpha \underline{[\![\mathbf{u}_h]\!]} & \text{if } e \subseteq \mathcal{E}_I\,, \\ \boldsymbol{\sigma}_h - \alpha(\mathbf{u}_h - \mathbf{g}) \otimes \boldsymbol{\nu} & \text{if } e \subseteq \mathcal{E}_D\,, \end{cases} \tag{5.2.9}$$

where the auxiliary functions α (scalar) and $\boldsymbol{\beta}$ (vector), to be chosen appropriately, are single-valued on each edge $e \subseteq \mathcal{E}$. As in [32], these numerical fluxes are *consistent* and *conservative*.

Now, summing up in (5.2.3) over all the elements $T \in \mathcal{T}_h$, integrating by parts appropriately, using the definitions of the numerical fluxes, and applying some algebraic identities, we arrive to the formulation: *Find* $(\mathbf{t}_h, \mathbf{u}_h, \boldsymbol{\sigma}_h, p_h) \in \boldsymbol{\Sigma}_h \times \mathbf{V}_h \times \boldsymbol{\Sigma}_h \times W_h$ *such that*

$$\int_\Omega \boldsymbol{\psi}(\mathbf{t}_h) : \mathbf{s}_h \;-\; \int_\Omega \boldsymbol{\sigma}_h : \mathbf{s}_h \;-\; \int_\Omega p_h \,\mathrm{tr}\,(\mathbf{s}_h) \;=\; 0\,, \tag{5.2.10}$$

$$\begin{aligned} \int_\Omega \mathbf{t}_h : \boldsymbol{\tau}_h - \int_\Omega \nabla_h \mathbf{u}_h : \boldsymbol{\tau}_h + \int_{\mathcal{E}_I} (\{\boldsymbol{\tau}_h\} - [\![\boldsymbol{\tau}_h]\!] \otimes \boldsymbol{\beta}) : \underline{[\![\mathbf{u}_h]\!]} \\ + \int_{\mathcal{E}_D} \mathbf{u}_h \cdot \boldsymbol{\tau}_h \boldsymbol{\nu} = \int_{\mathcal{E}_D} \mathbf{g} \cdot \boldsymbol{\tau}_h \boldsymbol{\nu}\,, \end{aligned} \tag{5.2.11}$$

$$\begin{aligned} \int_\Omega \boldsymbol{\sigma}_h : \nabla_h \mathbf{v}_h - \int_{\mathcal{E}_I} \underline{[\![\mathbf{v}_h]\!]} : (\{\boldsymbol{\sigma}_h\} - [\![\boldsymbol{\sigma}_h]\!] \otimes \boldsymbol{\beta}) - \int_{\mathcal{E}_D} \mathbf{v}_h \cdot \boldsymbol{\sigma}_h \boldsymbol{\nu} \\ + \int_{\mathcal{E}_I} \alpha\, \underline{[\![\mathbf{u}_h]\!]} : \underline{[\![\mathbf{v}_h]\!]} + \int_{\mathcal{E}_D} \alpha\,(\mathbf{u}_h \otimes \boldsymbol{\nu}) : (\mathbf{v}_h \otimes \boldsymbol{\nu}) \\ = \int_\Omega \mathbf{f} \cdot \mathbf{v}_h + \int_{\mathcal{E}_D} \alpha\,(\mathbf{g} \otimes \boldsymbol{\nu}) : (\mathbf{v}_h \otimes \boldsymbol{\nu})\,, \end{aligned} \tag{5.2.12}$$

and

$$\int_\Omega q_h \,\mathrm{tr}\,(\mathbf{t}_h) \;=\; 0\,, \tag{5.2.13}$$

for all $(\mathbf{s}_h, \mathbf{v}_h, \boldsymbol{\tau}_h, q_h) \in \boldsymbol{\Sigma}_h \times \mathbf{V}_h \times \boldsymbol{\Sigma}_h \times W_h$.

Consequently, adding (5.2.10) and (5.2.12), and also (5.2.11) together with (5.2.13), our formulation becomes: *Find* $((\mathbf{t}_h,\mathbf{u}_h),(\boldsymbol{\sigma}_h,p_h)) \in (\boldsymbol{\Sigma}_h\times\mathbf{V}_h)\times(\boldsymbol{\Sigma}_h\times W_h)$ *such that*

$$\begin{array}{lcl} A((\mathbf{t}_h,\mathbf{u}_h),(\mathbf{s}_h,\mathbf{v}_h)) \;+\; B((\mathbf{s}_h,\mathbf{v}_h),(\boldsymbol{\sigma}_h,p_h)) & = & F(\mathbf{s}_h,\mathbf{v}_h)\,, \\ B((\mathbf{t}_h,\mathbf{u}_h),(\boldsymbol{\tau}_h,q_h)) & = & G(\boldsymbol{\tau}_h,q_h)\,, \end{array} \tag{5.2.14}$$

for all $((\mathbf{s}_h,\mathbf{v}_h),(\boldsymbol{\tau}_h,q_h)) \in (\boldsymbol{\Sigma}_h\times\mathbf{V}_h)\times(\boldsymbol{\Sigma}_h\times W_h)$, where the semilinear form $A:(\boldsymbol{\Sigma}_h\times\mathbf{V}_h)\times(\boldsymbol{\Sigma}_h\times\mathbf{V}_h)\to\mathbf{R}$, the bilinear form $B:(\boldsymbol{\Sigma}_h\times\mathbf{V}_h)\times(\boldsymbol{\Sigma}_h\times W_h)\to\mathbf{R}$, and the functionals $F:(\boldsymbol{\Sigma}_h\times\mathbf{V}_h)\to\mathbf{R}$, $G:(\boldsymbol{\Sigma}_h\times W_h)\to\mathbf{R}$, are defined by

$$A((\mathbf{t}_h,\mathbf{u}_h),(\mathbf{s}_h,\mathbf{v}_h)) := \int_\Omega \boldsymbol{\psi}(\mathbf{t}_h):\mathbf{s}_h + \int_{\mathcal{E}_I}\alpha\,\underline{[\![\mathbf{u}_h]\!]}:\underline{[\![\mathbf{v}_h]\!]}$$

$$+\int_{\mathcal{E}_D}\alpha\,(\mathbf{u}_h\otimes\boldsymbol{\nu}):(\mathbf{v}_h\otimes\boldsymbol{\nu})\,,$$

$$B((\mathbf{s}_h,\mathbf{v}_h),(\boldsymbol{\tau}_h,q_h)) := -\int_\Omega \mathbf{s}_h:\boldsymbol{\tau}_h + \int_\Omega \nabla_h\mathbf{v}_h:\boldsymbol{\tau}_h \;-\; \int_\Omega q_h\,\mathrm{tr}\,(\mathbf{s}_h)$$

$$-\int_{\mathcal{E}_I}\underline{[\![\mathbf{v}_h]\!]}:(\{\boldsymbol{\tau}_h\}-[\![\boldsymbol{\tau}_h]\!]\otimes\boldsymbol{\beta}) \;-\; \int_{\mathcal{E}_D}\mathbf{v}_h\cdot\boldsymbol{\tau}_h\boldsymbol{\nu}\,,$$

$$F(\mathbf{s}_h,\mathbf{v}_h) := \int_\Omega \mathbf{f}\cdot\mathbf{v}_h + \int_{\mathcal{E}_D}\alpha\,(\mathbf{g}\otimes\boldsymbol{\nu}):(\mathbf{v}_h\otimes\boldsymbol{\nu})\,,$$

and

$$G(\boldsymbol{\tau}_h,q_h) := -\int_{\mathcal{E}_D}\mathbf{g}\cdot\boldsymbol{\tau}_h\boldsymbol{\nu}\,,$$

for all $(\mathbf{t}_h,\mathbf{u}_h,\boldsymbol{\sigma}_h,p_h),(\mathbf{s}_h,\mathbf{v}_h,\boldsymbol{\tau}_h,q_h)\in\boldsymbol{\Sigma}_h\times\mathbf{V}_h\times\boldsymbol{\Sigma}_h\times W_h$.

However, we notice that (5.2.14) is not uniquely solvable since adding $(\mathbf{0},\mathbf{0},-c\mathbf{I},c)$ to $(\mathbf{t}_h,\mathbf{u}_h,\boldsymbol{\sigma}_h,p_h)$, for any $c\in\mathbf{R}$, yields further solutions of this problem. Therefore, in order to guarantee uniqueness, we proceed as in [53] and require additionally that $\int_\Omega \mathrm{tr}\,(\boldsymbol{\sigma}_h)=0$, which leads the introduction of the Lagrange multiplier $\xi_h\in\mathbf{R}$ as a further unknown. Then, our formulation can be written as the dual-dual system: *Find* $((\mathbf{t}_h,\mathbf{u}_h),(\boldsymbol{\sigma}_h,p_h),\xi_h)\in(\boldsymbol{\Sigma}_h\times\mathbf{V}_h)\times(\boldsymbol{\Sigma}_h\times W_h)\times\mathbf{R}$ *such that*

$$\begin{array}{lcl} A((\mathbf{t}_h,\mathbf{u}_h),(\mathbf{s}_h,\mathbf{v}_h)) \;+\; B((\mathbf{s}_h,\mathbf{v}_h),(\boldsymbol{\sigma}_h,p_h)) & = & F(\mathbf{s}_h,\mathbf{v}_h)\,, \\ B((\mathbf{t}_h,\mathbf{u}_h),(\boldsymbol{\tau}_h,q_h)) \;+\; C((\boldsymbol{\tau}_h,q_h),\xi_h) & = & G(\boldsymbol{\tau}_h,q_h)\,, \\ C((\boldsymbol{\sigma}_h,p_h),\lambda_h) & = & H(\lambda_h)\,, \end{array} \tag{5.2.15}$$

for all $((\mathbf{s}_h, \mathbf{v}_h), (\boldsymbol{\tau}_h, q_h), \lambda_h) \in (\boldsymbol{\Sigma}_h \times \mathbf{V}_h) \times (\boldsymbol{\Sigma}_h \times W_h) \times \mathbf{R}$, where the bilinear form $C : (\boldsymbol{\Sigma}_h \times W_h) \times \mathbf{R} \to \mathbf{R}$ is defined by

$$C((\boldsymbol{\tau}_h, q_h), \lambda_h) \;:=\; \lambda_h \int_\Omega \mathrm{tr}\,(\boldsymbol{\tau}_h)\,,$$

and H is the null functional.

We point out here that one knows in advance that $\xi_h = 0$. In fact, it suffices to take $\boldsymbol{\tau}_h = \mathbf{I}$ and $q_h = -1$ in the second equation of (5.2.15), and use the compatibility condition for the Dirichlet datum $\mathbf{g}$. However, we do keep this artifitial unknown since it is needed to insure the symmetry of the whole formulation.

The stability and unique solvability of (5.2.15) will be established next by means of an equivalent mixed formulation that arises after expressing the unknowns $\boldsymbol{\sigma}_h$ and $\mathbf{t}_h$ in terms of $\mathbf{u}_h$ and the Lagrange multiplier ξ_h. Then we will apply a slight generalization of the classical Babuška-Brezzi theory to the resulting mixed formulation.

At this point we introduce the norm associated to $\mathbf{V}_h$. First, according to [32], we define the function $\mathtt{h}$ in $L^\infty(\mathcal{E})$, related to the local meshsize, as

$$\mathtt{h}(x) := \begin{cases} \min\{h_T, h_{T'}\} & \text{if } x \in \mathrm{int}(\partial T \cap \partial T') \\ h_T & \text{if } x \in \mathrm{int}(\partial T \cap \partial \Omega). \end{cases} \tag{5.2.16}$$

Then, we define $\alpha \in L^\infty(\mathcal{E})$ as

$$\alpha := \frac{\widehat{\alpha}}{\mathtt{h}}\,, \tag{5.2.17}$$

and consider $\boldsymbol{\beta} \in [L^\infty(\mathcal{E}_I)]^2$ such that

$$\|\boldsymbol{\beta}\|_{[L^\infty(\mathcal{E}_I)]^2} \le \widehat{\beta}\,, \tag{5.2.18}$$

where $\widehat{\alpha} > 0$ and $\widehat{\beta}$ are independent of the meshsize.

Next, we introduce the space $\mathbf{V}(h) := \mathbf{V}_h + [H^1(\Omega)]^2$, and define the seminorm $|\cdot| : \mathbf{V}(h) \to \mathbf{R}$ and the energy norm $|||\cdot|||_h : \mathbf{V}(h) \to \mathbf{R}$, respectively, by

$$|\mathbf{v}|_h^2 \;:=\; \|\alpha^{1/2}\, \underline{[\![\mathbf{v}]\!]}\|^2_{[L^2(\mathcal{E}_I)]^{2\times 2}} \;+\; \|\alpha^{1/2}\,(\mathbf{v} \otimes \boldsymbol{\nu})\|^2_{[L^2(\mathcal{E}_D)]^{2\times 2}} \quad \forall\, \mathbf{v} \in \mathbf{V}(h)\,, \tag{5.2.19}$$

and

$$|||\mathbf{v}|||_h^2 := \|\nabla_h \mathbf{v}\|^2_{[L^2(\Omega_2)]^2} \;+\; |\mathbf{v}|_h^2 \quad \forall\, \mathbf{v} \in \mathbf{V}(h)\,. \tag{5.2.20}$$

Now, we let $S : \mathbf{V}(h) \times \mathbf{\Sigma}_h \to \mathbf{R}$ be the bilinear form

$$S(\mathbf{v}, \boldsymbol{\tau}_h) \, := \, \int_{\mathcal{E}_I} \Big(\{\boldsymbol{\tau}_h\} - [\![\boldsymbol{\tau}_h]\!] \otimes \boldsymbol{\beta} \Big) : \underline{[\![\mathbf{v}]\!]} \, ds + \int_{\mathcal{E}_D} \mathbf{v} \cdot \boldsymbol{\tau}_h \boldsymbol{\nu} \quad \forall \, (\mathbf{v}, \boldsymbol{\tau}_h) \in \mathbf{V}(h) \times \mathbf{\Sigma}_h \,,$$

and let $\mathbf{G} : \mathbf{\Sigma}_h \to \mathbf{R}$ be the linear functional defined by

$$\mathbf{G}(\boldsymbol{\tau}_h) \, := \, \int_{\mathcal{E}_D} \mathbf{g} \cdot \boldsymbol{\tau}_h \boldsymbol{\nu} \quad \forall \, \boldsymbol{\tau}_h \in \mathbf{\Sigma}_h \,.$$

Following the ideas given in [32], it is easy to check that S and $\mathbf{G}$ are bounded. Then, we introduce $\mathbf{S} : \mathbf{V}(h) \to \mathbf{\Sigma}_h$ as the linear and bounded operator induced by the bilinear form S, that is, given $\mathbf{v} \in \mathbf{V}(h)$, $\mathbf{S}(\mathbf{v})$ is the unique element in $\mathbf{\Sigma}_h$ such that

$$\int_\Omega \mathbf{S}(\mathbf{v}) : \boldsymbol{\tau}_h \, = \, S(\mathbf{v}, \boldsymbol{\tau}_h) \quad \forall \, \boldsymbol{\tau}_h \in \mathbf{\Sigma}_h \,. \tag{5.2.21}$$

Similarly, in virtue of the Riesz Theorem, we let $\mathcal{G}$ be the unique element in $\mathbf{\Sigma}_h$ such that

$$\int_\Omega \mathcal{G} : \boldsymbol{\tau}_h \, = \, \mathbf{G}(\boldsymbol{\tau}_h) \quad \forall \, \boldsymbol{\tau}_h \in \mathbf{\Sigma}_h \,.$$

As in [32], we point out that $\mathbf{S}(\mathbf{u}) = \mathcal{G}$, with $\mathbf{u}$ being the exact solution of (5.1.1). Moreover, the following result holds

Lemma 5.2.1 *There exists $C_{\mathbf{S}} > 0$, independent of the meshsize, such that*

$$||\mathbf{S}(\mathbf{v})||_{[L^2(\Omega)]^{2\times 2}} \, \leq \, C_{\mathbf{S}} \, |\mathbf{v}|_h \quad \forall \, \mathbf{v} \in \mathbf{V}(h) \,.$$

Now, it follows from the second equation of (5.2.15) that

$$\mathbf{t}_h = \Pi_{\mathbf{\Sigma}_h} \Big(\nabla_h \mathbf{u}_h - \mathbf{S}(\mathbf{u}_h) + \mathcal{G} + \xi_h \mathbf{I} \Big) = \nabla_h \mathbf{u}_h - \mathbf{S}(\mathbf{u}_h) + \mathcal{G} + \xi_h \mathbf{I} \,, \tag{5.2.22}$$

whereas (5.2.10) yields

$$\boldsymbol{\sigma}_h = \Pi_{\mathbf{\Sigma}_h} \Big(\boldsymbol{\psi}(\mathbf{t}_h) - p_h \mathbf{I} \Big) = \Pi_{\mathbf{\Sigma}_h} \Big(\boldsymbol{\psi}(\nabla_h \mathbf{u}_h - \mathbf{S}(\mathbf{u}_h) + \mathcal{G} + \xi_h \mathbf{I}) \, - \, p_h \, \mathbf{I} \Big) \,. \tag{5.2.23}$$

Letting $W := L^2(\Omega)$, we now define the operators $A_h : (\mathbf{V}(h) \times \mathbf{R}) \to (\mathbf{V}(h) \times \mathbf{R})'$, $B_h : (\mathbf{V}(h) \times \mathbf{R}) \to W'$, and the functionals $F_h : \mathbf{V}(h) \times \mathbf{R} \to \mathbf{R}$ and $G_h : W \to \mathbf{R}$, as follows:

$$\begin{aligned} [A_h(\mathbf{w}, \zeta), (\mathbf{v}, \lambda)] := & \int_\Omega \boldsymbol{\psi}(\nabla_h \mathbf{w} - \mathbf{S}(\mathbf{w}) + \mathcal{G} + \zeta \mathbf{I}) : (\nabla_h \mathbf{v} - \mathbf{S}(\mathbf{v}) + \lambda \mathbf{I}) \\ & + \int_{\mathcal{E}_I} \alpha \, \underline{[\![\mathbf{w}]\!]} : \underline{[\![\mathbf{v}]\!]} + \int_{\mathcal{E}_D} \alpha \, (\mathbf{w} \otimes \boldsymbol{\nu}) : (\mathbf{v} \otimes \boldsymbol{\nu}) \end{aligned} \tag{5.2.24}$$

$$[B_h(\mathbf{v},\lambda), q] := -\int_\Omega q\,\mathrm{div_h}\,\mathbf{v} + \int_\Omega (q\mathbf{I}) : (\mathbf{S}(\mathbf{v}) - \lambda\mathbf{I})\,, \tag{5.2.25}$$
$$[F_h, (\mathbf{v},\lambda)] := \int_\Omega \mathbf{f}\cdot\mathbf{v} + \int_{\mathcal{E}_D} \alpha\,(\mathbf{g}\otimes\boldsymbol{\nu}) : (\mathbf{v}\otimes\boldsymbol{\nu})\,,$$
$$[G_h, q] := \int_{\mathcal{E}_D} q\,\mathbf{g}\cdot\boldsymbol{\nu}\,,$$

for all $(\mathbf{w},\zeta), (\mathbf{v},\lambda) \in \mathbf{V}(h)\times\mathbf{R}$.

Therefore, according to (5.2.22) and (5.2.23), problem (5.2.15) can be written as: *Find* $((\mathbf{u}_h,\xi_h),p_h) \in (\mathbf{V}_h\times\mathbf{R})\times W_h$ *such that*

$$\begin{array}{rcl} [A_h(\mathbf{u}_h,\xi_h),(\mathbf{v}_h,\lambda_h)] \;+\; [B_h(\mathbf{v}_h,\lambda_h),p_h] & = & [F_h,(\mathbf{v}_h,\lambda_h)]\,, \\ [B_h(\mathbf{u}_h,\xi_h),q_h] \qquad\qquad\qquad & = & [G_h,q_h]\,, \end{array} \tag{5.2.26}$$

for all $((\mathbf{v}_h,\lambda_h),q_h) \in (\mathbf{V}_h\times\mathbf{R})\times W_h$.

5.3 Solvability and stability of the mixed LDG formulation

We now let $X := [L^2(\Omega)]^{2\times 2}$ and introduce the pure nonlinear operator $\mathcal{N} : X \to X'$ forming part of (5.1.4), that is

$$[\mathcal{N}(\mathbf{r}),(\mathbf{s})] \;:=\; \int_\Omega \boldsymbol{\psi}(\mathbf{r}) : \mathbf{s} \quad \forall\,\mathbf{r},\mathbf{s}\in X\,. \tag{5.3.1}$$

We observe that $\mathcal{N}$ is Gâteaux differentiable at each $\tilde{\mathbf{r}} \in X$. In fact, this derivative can be seen as the bounded bilinear form $D\mathcal{N}(\tilde{\mathbf{r}}) : X\times X \to \mathbf{R}$ given by

$$D\mathcal{N}(\tilde{\mathbf{r}})(\mathbf{r},\mathbf{s}) \;:=\; \int_\Omega \left\{ \sum_{i,j,k,l=1}^{2} \frac{\partial}{\partial \tilde{r}_{kl}} \psi_{ij}(\tilde{\mathbf{r}}) r_{kl}\, s_{ij} \right\} \quad \forall\,\mathbf{r},\mathbf{s}\in X\,, \tag{5.3.2}$$

which, according to (5.1.2) and (5.1.3), implies the existence of positive constants $\tilde{C}_1$ and $\tilde{C}_2$ such that

$$|D\mathcal{N}(\tilde{\mathbf{r}})(\mathbf{r},\mathbf{s})| \;\le\; \tilde{C}_1\,||\mathbf{r}||_X\,||\mathbf{s}||_X \quad \text{and} \quad D\mathcal{N}(\tilde{\mathbf{r}})(\mathbf{s},\mathbf{s}) \;\ge\; \tilde{C}_2\,||\mathbf{s}||_X^2 \tag{5.3.3}$$

for all $\tilde{\mathbf{r}},\mathbf{r},\mathbf{s} \in X$. Thus, the above properties yield the strong monotonicity and Lipschitz continuity of the nonlinear operator $\mathcal{N}$ on X.

Next, we introduce the applications $\boldsymbol{\varphi}\,,\tilde{\boldsymbol{\varphi}} : \mathbf{V}(h)\times\mathbf{R} \to X$ given by

$$\boldsymbol{\varphi}(\mathbf{v},\lambda) \;:=\; \nabla_h\mathbf{v} \,-\, \mathbf{S}(\mathbf{v}) \,-\, \lambda\mathbf{I} \qquad \forall\,(\mathbf{v},\lambda)\in\mathbf{V}(h)\times\mathbf{R}\,, \tag{5.3.4}$$

and

$$\tilde{\boldsymbol{\varphi}}(\mathbf{v},\lambda) \,:=\, \nabla_h \mathbf{v} \,-\, \mathbf{S}(\mathbf{v}) \,+\, \mathcal{G} \,-\, \lambda \mathbf{I} \qquad \forall\,(\mathbf{v},\lambda) \in \mathbf{V}(h) \times \mathbf{R}\,, \tag{5.3.5}$$

so that the corresponding non-linear part $\mathcal{N}_h : (\mathbf{V}(h) \times \mathbf{R}) \to (\mathbf{V}(h) \times \mathbf{R})'$ of A_h is defined by

$$[\mathcal{N}_h(\mathbf{u},\xi),(\mathbf{v},\lambda)] \,:=\, [\mathcal{N}(\tilde{\boldsymbol{\varphi}}(\mathbf{u},\xi)),\boldsymbol{\varphi}(\mathbf{v},\lambda)] \quad \forall\,(\mathbf{u},\xi),(\mathbf{v},\lambda) \in \mathbf{V}(h) \times \mathbf{R}\,. \tag{5.3.6}$$

It is easy to see that $\mathcal{N}_h$ admits a Gâteaux derivative at each $(\mathbf{z},\zeta) \in \mathbf{V}(h) \times \mathbf{R}$, which can be seen as the bounded bilinear form $D\mathcal{N}_h(\mathbf{z},\zeta) : (\mathbf{V}(h) \times \mathbf{R}) \times (\mathbf{V}(h) \times \mathbf{R}) \to \mathbf{R}$ given by

$$D\mathcal{N}_h(\mathbf{z},\zeta)((\mathbf{u},\xi),(\mathbf{v},\lambda)) \,:=\, D\mathcal{N}(\tilde{\boldsymbol{\varphi}}(\mathbf{z},\zeta))(\boldsymbol{\varphi}(\mathbf{u},\xi),\boldsymbol{\varphi}(\mathbf{v},\lambda)) \tag{5.3.7}$$

for all $(\mathbf{u},\xi),(\mathbf{v},\lambda) \in \mathbf{V}(h) \times \mathbf{R}$.

In this way, the Gâteaux derivative of A_h at $(\mathbf{z},\zeta) \in \mathbf{V}(h) \times \mathbf{R}$ reduces to the bounded bilinear form $DA_h(\mathbf{z},\zeta) : (\mathbf{V}(h) \times \mathbf{R}) \times (\mathbf{V}(h) \times \mathbf{R}) \to \mathbf{R}$ defined by

$$\begin{aligned} DA_h(\mathbf{z},\zeta)((\mathbf{w},\eta),(\mathbf{v},\lambda)) \,&:=\, D\mathcal{N}_h(\mathbf{z},\zeta)((\mathbf{w},\eta),(\mathbf{v},\lambda)) \\ + \int_{\mathcal{E}_I} \alpha\, \underline{[\![\mathbf{w}]\!]} : \underline{[\![\mathbf{v}]\!]} \,+\, \int_{\mathcal{E}_D} \alpha\,(\mathbf{w}\otimes\boldsymbol{\nu}) : (\mathbf{v}\otimes\boldsymbol{\nu}) &\qquad \forall\,(\mathbf{w},\eta),(\mathbf{v},\lambda) \in \mathbf{V}(h) \times \mathbf{R}\,. \end{aligned} \tag{5.3.8}$$

In what follows, we introduce the norm $||(\cdot,\cdot)||_{\mathbf{V}(h)\times\mathbf{R}} : \mathbf{V}(h) \times \mathbf{R} \to \mathbf{R}_0^+$ as

$$||(\mathbf{w},\zeta)||^2_{\mathbf{V}(h)\times\mathbf{R}} \,=\, |||\mathbf{w}|||_h^2 \,+\, |\zeta|^2 \quad \forall\,(\mathbf{w},\zeta) \in \mathbf{V}(h) \times \mathbf{R}\,.$$

Then, taking into account that

$$\int_\Omega \mathbf{I} : (\nabla_h \mathbf{w} - \mathbf{S}(\mathbf{w})) \,=\, 0 \quad \forall\,\mathbf{w} \in \mathbf{V}(h)\,, \tag{5.3.9}$$

we deduce that

$$||\boldsymbol{\varphi}(\mathbf{v},\lambda)||_X^2 \,=\, ||\nabla_h \mathbf{v} - \mathbf{S}(\mathbf{v})||^2_{[L^2(\Omega)]^{2\times2}} \,+\, 2\,|\Omega|\,|\lambda|^2 \qquad \forall\,(\mathbf{v},\lambda) \in \mathbf{V}(h) \times \mathbf{R}\,,$$

which allows us to show that the nonlinear operator A_h is also Lipschitz continuous and strongly monotone.

Lemma 5.3.1 *There exist $C_{\mathtt{LC}} > 0$ and $C_{\mathtt{SM}} > 0$, independent of the meshsize, such that*

$$||A_h(\mathbf{w},\zeta) - A_h(\mathbf{v},\lambda)||_{(\mathbf{V}(h)\times\mathbf{R})'} \leq C_{\mathtt{LC}}\,||(\mathbf{w}-\mathbf{v},\zeta-\lambda)||_{\mathbf{V}(h)\times\mathbf{R}}\,,$$

and

$$[A_h(\mathbf{w},\zeta) - A_h(\mathbf{v},\lambda),(\mathbf{w}-\mathbf{v},\zeta-\lambda)] \geq C_{\mathtt{SM}}\,||(\mathbf{w}-\mathbf{v},\zeta-\lambda)||^2_{\mathbf{V}(h)\times\mathbf{R}}\,,$$

for all $(\mathbf{w},\zeta),(\mathbf{v},\lambda) \in \mathbf{V}(h)\times\mathbf{R}$.

Our next goal is to proof the discrete inf-sup condition of the bilinear form B_h. To this end, we first let $L^2_0(\Omega)$ br the subspace of functions in $L^2(\Omega)$ with zero mean value, and note that $L^2(\Omega) = L^2_0(\Omega)\oplus\mathbf{R}$, i.e., each $q\in L^2(\Omega)$ can be uniquely decomposed as $q = \tilde{q}+\bar{q}$, with

$$\tilde{q} := \left(q - \frac{1}{|\Omega|}\int_\Omega q\right) \in L^2_0(\Omega) \quad \text{and} \quad \bar{q} := \frac{1}{|\Omega|}\int_\Omega q \in \mathbf{R}\,.$$

Moreover, it is easy to see that $||q||^2_{L^2(\Omega)} = ||\tilde{q}||^2_{L^2(\Omega)} + |\Omega|\,\bar{q}^2$. Then, we have the following result.

Lemma 5.3.2 *There exists a constant $C_{\mathtt{I}} > 0$, independent of the meshsize, such that,*

$$\sup_{\mathbf{v}_h\in\mathbf{V}_h\setminus\{\mathbf{0}\}} \frac{[B_h(\mathbf{v}_h,0),r_h]}{|||\mathbf{v}_h|||_h} \geq C_{\mathtt{I}}\,||r_h||_{L^2(\Omega)} \quad \forall\, r_h\in W_h\cap L^2_0(\Omega)\,.$$

Proof. It is very similar to the proof of Theorem 6.12 in [93]. First, given $\mathbf{w}\in[H^1(\Omega)]^2$ we introduce a global operator $\Pi : [H^1(\Omega)]^2 \to \mathbf{V}_h$ by $\Pi\mathbf{w}|_T := \Pi_{RT}\mathbf{w}\,, T\in\mathcal{T}_h$, where Π_{RT} is the Raviart-Thomas projector of degree $k-1$ on T. We realize that when $\mathbf{w}\in[H^1_0(\Omega)]^2$, the normal component of $\Pi\mathbf{w}$ is continuous across the inter-element boundaries and vanishes on $\partial\Omega$, i.e., $[\![\Pi\mathbf{w}]\!] = 0$ on $\mathcal{E}$. We point out here that this last property is no longer true if the mesh has hanging nodes.

By doing some algebraic computations, it is easy to check that

$$[B_h(\Pi\mathbf{w},0),r_h] = -\int_\Omega r_h\,\mathrm{div}\,\mathbf{w}\,,\quad \forall\, r_h\in W_h\cap L^2_0(\Omega)\,,$$

and

$$|||\Pi\mathbf{w}|||_h \leq C\,||\nabla\mathbf{w}||_{[L^2(\Omega)]^2}\,,$$

where $C>0$ is independent of the meshsize. The rest reduces to apply the well known Fortin property. We omit further details. □

We are now in position to show the discrete inf-sup condition of the operator B_h, which is established in the next lemma.

LEMMA 5.3.3 *There exists a constant $C_{\mathtt{INF}} > 0$, independent of the meshsize, such that,*

$$\sup_{(\mathbf{0},0)\neq(\mathbf{v}_h,\lambda_h)\in\mathbf{V}_h\times\mathbf{R}} \frac{[B_h(\mathbf{v}_h,\lambda_h),q_h]}{||(\mathbf{v}_h,\lambda_h)||_{\mathbf{V}(h)\times\mathbf{R}}} \,\geq\, C_{\mathtt{INF}}\,||q_h||_{L^2(\Omega)} \quad \forall\, q_h\in W_h\,.$$

PROOF. Let $q_h \in W_h \subseteq L^2(\Omega)$. Since $L^2(\Omega) = L^2_0(\Omega)\oplus\mathbf{R}$, there exists $\tilde{q}_h \in W_h \cap L^2_0(\Omega)$ and $\bar{q}_h \in \mathbf{R}$ such that $q_h = \tilde{q}_h + \bar{q}_h$. Then, applying the linearity of B_h (see (5.2.25) for its definition), together with (5.3.9), we have

$$[B_h(\mathbf{v}_h,\lambda_h),q_h] \,=\, [B_h(\mathbf{v}_h,\lambda_h),\tilde{q}_h] \,-\, 2\,\bar{q}_h\,\lambda_h\,|\Omega| \quad \forall\,(\mathbf{v}_h,\lambda_h)\in\mathbf{V}_h\times\mathbf{R}\,, \tag{5.3.10}$$

Therefore, using (5.3.10), we find that

$$\sup_{(\mathbf{0},0)\neq(\mathbf{v}_h,\lambda_h)\in\mathbf{V}_h\times\mathbf{R}} \frac{[B_h(\mathbf{v}_h,\lambda_h),q_h]}{||(\mathbf{v}_h,\lambda_h)||_{\mathbf{V}(h)\times\mathbf{R}}} \,\geq\, \sup_{\mathbf{0}\neq\mathbf{v}_h\in\mathbf{V}_h} \frac{[B_h(\mathbf{v}_h,0),\tilde{q}_h]}{|||\mathbf{v}_h|||_h}\,,$$

so, thanks to Lemma 5.3.2, we conclude that there exists a constant $C_{\mathtt{I}} > 0$, independent of the meshsize, such that

$$\sup_{(\mathbf{0},0)\neq(\mathbf{v}_h,\lambda_h)\in\mathbf{V}_h\times\mathbf{R}} \frac{[B_h(\mathbf{v}_h,\lambda_h),q_h]}{||(\mathbf{v}_h,\lambda_h)||_{\mathbf{V}(h)\times\mathbf{R}}} \,\geq\, C_{\mathtt{I}}\,||\tilde{q}_h||_{L^2(\Omega)}\,. \tag{5.3.11}$$

On the other hand, we also have that

$$\begin{aligned}\sup_{(\mathbf{0},0)\neq(\mathbf{v}_h,\lambda_h)\in\mathbf{V}_h\times\mathbf{R}} \frac{[B_h(\mathbf{v}_h,\lambda_h),q_h]}{||(\mathbf{v}_h,\lambda_h)||_{\mathbf{V}(h)\times\mathbf{R}}} \,&\geq\, \sup_{0\neq\lambda_h\in\mathbf{R}} \frac{[B_h(\mathbf{0},\lambda_h),q_h]}{|\lambda_h|} \\ &\geq\, \frac{[B_h(\mathbf{0},-\bar{q}_h),q_h]}{|\bar{q}_h|} \,=\, 2\,|\Omega|\,|\bar{q}_h|\,.\end{aligned} \tag{5.3.12}$$

Finally, the result follows easily from (5.3.11) and (5.3.12), with $C_{\mathtt{INF}} := \min\left\{C_{\mathtt{I}}, 2\,|\Omega|^{1/2}\right\}$. $\square$

Next, the boundedness of the functionals F_h and G_h is established in the following lemma.

LEMMA 5.3.4 *There exist $C_{\mathtt{F}}\,, C_{\mathtt{G}} > 0$, depending on $\widehat{\alpha}, l$ and k, but independent of the meshsize, such that*

$$|[F_h,(\mathbf{v}_h,\lambda_h)]| \leq C_{\mathtt{F}}\,\mathcal{B}(\mathbf{f},\mathbf{g})\,||(\mathbf{v}_h,\lambda_h)||_{\mathbf{V}(h)\times\mathbf{R}} \quad \forall\,(\mathbf{v}_h,\lambda_h)\in\mathbf{V}_h\times\mathbf{R}\,, \tag{5.3.13}$$

and

$$|[G_h, q_h]| \leq C_{\mathtt{G}} \, ||\alpha^{1/2} \mathbf{g} \cdot \boldsymbol{\nu}||_{L^2(\mathcal{E}_D)} \, ||q_h||_{L^2(\Omega)} \qquad \forall \, q_h \in W_h \,, \tag{5.3.14}$$

with

$$\mathcal{B}(\mathbf{f}, \mathbf{g}) := \Big(||\mathbf{f}||^2_{L^2(\Omega_2)} + ||\alpha^{1/2} \mathbf{g} \otimes \boldsymbol{\nu}||^2_{[L^2(\mathcal{E}_D)]^{2\times 2}} \Big)^{1/2} .$$

PROOF. It is similar to the proof of Lemma 4.4 in [32]. □

From now on, we introduce the norm $|| \cdot ||_{LDG}$ on $\mathbf{V}(h) \times L^2(\Omega)$, given by

$$||(\mathbf{v}, \lambda, q)||^2_{LDG} := |||\mathbf{v}|||^2_h \, + \, |\lambda|^2 \, + \, ||q||^2_{L^2(\Omega)} \quad \forall \, (\mathbf{v}, \lambda, q) \in \mathbf{V}(h) \times \mathbf{R} \times L^2(\Omega) \,.$$

THEOREM 5.3.1 *There exists a unique* $(\mathbf{u}_h, \xi_h, p_h) \in \mathbf{V}_h \times \mathbf{R} \times W_h$ *solution of* (5.2.26)*, for which there is a constant* $C > 0$*, independent of the meshsize, such that*

$$||(\mathbf{u}_h, \xi_h, p_h)||_{LDG} \, \leq \, C \, \Big(\mathcal{B}(\mathbf{f}, \mathbf{g}) + ||\alpha^{1/2} \mathbf{g} \cdot \boldsymbol{\nu}||_{L^2(\mathcal{E}_D)} \Big) \,. \tag{5.3.15}$$

Moreover, denoting by $C_{\mathtt{B}}$ *the boundedness constant associated to* B_h*, there hold the Strang-type estimates*

$$\begin{aligned} &||(\mathbf{u} - \mathbf{u}_h, \xi - \xi_h)||_{\mathbf{V}(h)\times\mathbf{R}} \, \leq \\ &\left(1 + \frac{C_{\mathtt{LC}}}{C_{\mathtt{SM}}}\right) \inf_{\mathbf{v}_h \in \mathbf{V}_h} |||\mathbf{u} - \mathbf{v}_h|||_h \, + \, \frac{C_{\mathtt{B}}}{C_{\mathtt{SM}}} \inf_{q_h \in W_h} ||p - q_h||_{L^2(\Omega)} \\ &+ C_{\mathtt{SM}}^{-1} \sup_{(\mathbf{0},0) \neq (\mathbf{v}_h, \lambda_h) \in \mathbf{V}_h \times \mathbf{R}} \frac{|[A_h(\mathbf{u}, \xi), (\mathbf{v}_h, \lambda_h)] + [B_h(\mathbf{v}_h, \lambda_h), p] - [F_h, (\mathbf{v}_h, \lambda_h)]|}{||(\mathbf{v}_h, \lambda_h)||_{\mathbf{V}(h)\times\mathbf{R}}} \,, \end{aligned} \tag{5.3.16}$$

and

$$\begin{aligned} ||p - p_h||_{L^2(\Omega)} \leq &\left(1 + \frac{C_{\mathtt{B}}}{C_{\mathtt{INF}}}\right) \inf_{q_h \in W_h} ||p - q_h||_{L^2(\Omega)} \, + \, \frac{C_{\mathtt{LC}}}{C_{\mathtt{INF}}} |||\mathbf{u} - \mathbf{u}_h|||_h \\ &+ C_{\mathtt{INF}}^{-1} \sup_{(\mathbf{0},0) \neq (\mathbf{v}_h, \lambda_h) \in \mathbf{V}_h \times \mathbf{R}} \frac{|[A_h(\mathbf{u}, \xi), (\mathbf{v}_h, \lambda_h)] + [B_h(\mathbf{v}_h, \lambda_h), p] - [F_h, (\mathbf{v}_h, \lambda_h)]|}{||(\mathbf{v}_h, \lambda_h)||_{\mathbf{V}(h)\times\mathbf{R}}} \,. \end{aligned} \tag{5.3.17}$$

PROOF. Thanks to Lemmas 5.3.1, 5.3.3, and 5.3.4, the unique solvability of (5.2.26), and its continuous dependence property, (5.3.15) follows from a slight version of Babuška-Brezzi theory (see, for e.g. Lemma 2.1 in [57]). Now, in order to show

(5.3.16), we apply the strong monotonicity of A_h (cf. Lemma 5.3.1). Then, given any $(\mathbf{w}_h, \zeta_h, q_h) \in \mathbf{V}_h \times \mathbf{R} \times W_h$, we have

$$C_{\texttt{SM}}\, ||(\mathbf{w}_h - \mathbf{u}_h, \zeta_h - \xi_h)||^2_{\mathbf{V}(h)\times\mathbf{R}} \,\leq\, [A_h(\mathbf{w}_h, \zeta_h) - A_h(\mathbf{u}_h, \xi_h), (\mathbf{w}_h - \mathbf{u}_h, \zeta_h - \xi_h)]$$

$$= \,[A_h(\mathbf{w}_h, \zeta_h) - A_h(\mathbf{u}, \xi), (\mathbf{w}_h - \mathbf{u}_h, \zeta_h - \xi_h)] \,+\, [B_h(\mathbf{w}_h - \mathbf{u}_h, \zeta_h - \xi_h), p - q_h]$$

$$+\, [B_h(\mathbf{w}_h, \zeta_h), p_h - q_h] \,+\, \Big([A_h(\mathbf{u}, \xi), (\mathbf{w}_h - \mathbf{u}_h, \zeta_h - \xi_h)]$$

$$+\, [B_h(\mathbf{w}_h - \mathbf{u}_h, \zeta_h - \xi_h), p] \,-\, [F_h, (\mathbf{w}_h - \mathbf{u}_h, \zeta_h - \xi_h)]\Big)$$

$$\leq\, ||(\mathbf{w}_h - \mathbf{u}_h, \zeta_h - \xi_h)||_{\mathbf{V}(h)\times\mathbf{R}} \left(C_{\texttt{LC}}\, ||(\mathbf{w}_h - \mathbf{u}, \zeta_h - \xi)||_{\mathbf{V}(h)\times\mathbf{R}} \,+\, C_{\texttt{B}}\, ||p - q_h||_{L^2(\Omega)}\right)$$

$$+\, C_{\texttt{B}}\, ||(\mathbf{w}_h - \mathbf{u}_h, \zeta_h - \xi_h)||_{\mathbf{V}(h)\times\mathbf{R}}\, ||p_h - q_h||_{L^2(\Omega)} \,+\, \Big([A_h(\mathbf{u}, \xi), (\mathbf{w}_h - \mathbf{u}_h, \zeta_h - \xi_h)]$$

$$+\, [B_h(\mathbf{w}_h - \mathbf{u}_h, \zeta_h - \xi_h), p] \,-\, [F_h, (\mathbf{w}_h - \mathbf{u}_h, \zeta_h - \xi_h)]\Big)\,,$$

which yields

$$C_{\texttt{SM}}\, ||(\mathbf{w}_h - \mathbf{u}_h, \zeta_h - \xi_h)||_{\mathbf{V}(h)\times\mathbf{R}} \,\leq$$

$$C_{\texttt{LC}}\, ||(\mathbf{w}_h - \mathbf{u}, \zeta_h - \xi)||_{\mathbf{V}(h)\times\mathbf{R}} \,+\, C_{\texttt{B}}\, ||p - q_h||_{L^2(\Omega)} \,+\, C_{\texttt{B}}\, ||p_h - q_h||_{L^2(\Omega)}$$

$$+ \sup_{\substack{(\mathbf{v}_h, \lambda_h)\in\mathbf{V}_h\times\mathbf{R}\\ (\mathbf{v}_h, \lambda_h)\neq(\mathbf{0},0)}} \frac{|[A_h(\mathbf{u}, \xi), (\mathbf{v}_h, \lambda_h)] \,+\, [B_h(\mathbf{v}_h, \lambda_h), p] \,-\, [F_h, (\mathbf{v}_h, \lambda_h)]|}{||(\mathbf{v}_h, \lambda_h)||_{\mathbf{V}(h)\times\mathbf{R}}}\,,$$

for all $(\mathbf{w}_h, \zeta_h, q_h) \in \mathbf{V}_h \times \mathbf{R} \times W_h$. Then, applying the triangle inequality, we deduce that

$$||(\mathbf{u} - \mathbf{u}_h, \xi - \xi_h)||_{\mathbf{V}(h)\times\mathbf{R}} \,\leq$$

$$\left(1 + \frac{C_{\texttt{LC}}}{C_{\texttt{SM}}}\right) \inf_{\mathbf{w}_h\in\mathbf{V}_h} |||\mathbf{w}_h - \mathbf{u}|||_h \,+\, \frac{C_{\texttt{B}}}{C_{\texttt{SM}}} \inf_{q_h\in W_h} ||p - q_h||_{L^2(\Omega)}$$

$$+\, \frac{1}{C_{\texttt{SM}}} \sup_{(\mathbf{0},0)\neq(\mathbf{v}_h, \lambda_h)\in\mathbf{V}_h\times\mathbf{R}} \frac{|[A_h(\mathbf{u}, \xi), (\mathbf{v}_h, \lambda_h)] + [B_h(\mathbf{v}_h, \lambda_h), p] - [F_h, (\mathbf{v}_h, \lambda_h)]|}{||(\mathbf{v}_h, \lambda_h)||_{\mathbf{V}(h)\times\mathbf{R}}}\,.$$

Similarly, using the idea of the proof of Proposition 4.3 in [93], we reach (5.3.17). □

5.4 A priori error estimates

From now on, we will assume additional and enough regularity on the exact solution $(\mathbf{u}, p)$ of (5.1.1), so that $[\![\mathbf{u}]\!] = 0$ on any interior edge of $\mathcal{T}_h$, and $\mathbf{S}(\mathbf{u}) = \mathcal{G}$. Besides, since $\boldsymbol{\sigma} = \boldsymbol{\psi}(\nabla\mathbf{u}) - p\mathbf{I} \in [L^2(\Omega_2)]^2$ and $-\mathbf{div}\,(\boldsymbol{\psi}(\nabla\mathbf{u}) - p\mathbf{I}) = \mathbf{f}$ in Ω, with $\mathbf{f} \in L^2(\Omega_2)$, we conclude that $\boldsymbol{\psi}(\nabla\mathbf{u}) - p\mathbf{I} \in H(\mathrm{div}\,;\Omega)$, whence $[\![\boldsymbol{\psi}(\nabla\mathbf{u}) - p\mathbf{I}]\!] = 0$ on each $e \in \mathcal{E}_I$. Moreover, $\xi = 0$, and $\boldsymbol{\sigma}$ satisfies $\displaystyle\int_\Omega \mathrm{tr}\,(\boldsymbol{\sigma}) = 0$, which due to the nonlinearity we are dealing with, is equivalent to the fact that $p \in L^2_0(\Omega)$. In addition, for simplicity we will consider convex domains, but we remark that the results we present below can be extended to non-convex domains, as in [32].

The following lemma establishes a well known approximation result (see [39] and [60] for more details), which will be used later on.

Lemma 5.4.1 *Let $\mathcal{T}_h$ be a regular triangulation and let $s \geq 0$ and $m \in \{0,1\}$. Given $T \in \mathcal{T}_h$, let $\Pi^k_T : H^{s+1}(T) \to \mathbf{P}_k(T)$ be the linear and bounded operator given by the $L^2(T)$-orthogonal projection, which satisfies $\Pi^k_T(p) = p$ for all $p \in \mathbf{P}_k(T)$. Then there exists $C_{\mathtt{ort}} > 0$, independent of the meshsize, such that*

$$|\,(\mathbf{I} - \Pi^k_T)(w)\,|_{H^m(T)} \;\leq\; C_{\mathtt{ort}}\, h_T^{\min\{s,k\}+1-m}\, |w|_{H^{s+1}(T)} \quad \forall\, w \in H^{s+1}(T)\,, \qquad (5.4.1)$$

and

$$\|\,(\mathbf{I} - \Pi^k_T)(w)\,\|_{L^2(\partial T)} \;\leq\; C_{\mathtt{ort}}\, h_T^{\min\{s,k\}+1/2}\, |w|_{H^{s+1}(T)} \quad \forall\, w \in H^{s+1}(T)\,. \qquad (5.4.2)$$

Our analysis now will focus in finding a bound for the consistency term in the Strang type error estimates (5.3.16) and (5.3.17) (cf. Theorem 5.3.1). To this end, we introduce the *discontinuous* global operator $\Pi^k_h : D \to D_h$, where, given $w \in D$, $\Pi^k_h(w)$ is the unique element in D_h such that $\Pi^k_h(w)|_T = \Pi^k_h(w|_T)$ for all $T \in \mathcal{T}_h$. Here, D (D_h, respectively) can be chosen as the space $[H^1(\Omega)]^2$ or $L^2(\Omega)$ ($\mathbf{V}_h$ or W_h respectively).

Lemma 5.4.2 *Let $s \in \mathbf{N} \cup \{0\}$ and $r \in \mathbf{N}$ such that $0 \leq s \leq k$ and $r \leq k$, and assume that $\boldsymbol{\psi}(\nabla u)|_T \in [H^{s+1}(T)]^{2\times 2}$ and $p|_T \in H^r(T)$ for all $T \in \mathcal{T}_h$. Then, there exists $C_{\mathtt{con}} > 0$, independent of the meshsize, but depending on $\widehat{\alpha}$, $\widehat{\beta}$, l, and the* *polynomial approximation degree k, such that for each $(\mathbf{w},\eta) \in \mathbf{V}(h) \times \mathbf{R}$*

$$|[A_h(\mathbf{u},\xi),(\mathbf{w},\eta)] + [B_h(\mathbf{w},\eta),p] - [F_h,(\mathbf{w},\eta)]| \leq$$

$$C_{\mathtt{con}} \left(\sum_{T\in\mathcal{T}_h} h_T^{2(s+1)} ||\boldsymbol{\psi}(\nabla\mathbf{u})||^2_{[H^{s+1}(T)]^{2\times 2}} + \sum_{T\in\mathcal{T}_h} h_T^{2r} ||p||^2_{H^r(T)} \right)^{1/2} |||\mathbf{w}|||_h \,.$$

Proof. Let $(\mathbf{w},\eta) \in \mathbf{V}(h) \times \mathbf{R}$. Since $\xi = 0$, $\mathbf{S}(\mathbf{u}) = \mathcal{G}$, $\underline{[\![\mathbf{u}]\!]} = 0$ on $\mathcal{E}_I$, $\mathbf{f} = -\mathbf{div}\,(\boldsymbol{\psi}(\nabla\mathbf{u}) - p\mathbf{I})$ in Ω, and $\mathbf{u} = \mathbf{g}$ on Γ, we find that

$$[A_h(\mathbf{u},\xi),(\mathbf{w},\eta)] + [B_h(\mathbf{w},\eta),p] - [F_h,(\mathbf{w},\eta)] = \int_\Omega \boldsymbol{\psi}(\nabla\mathbf{u}) : (\nabla_h\mathbf{w} - \mathbf{S}(\mathbf{w}) + \eta\mathbf{I})$$

$$+ \int_{\mathcal{E}_D} \alpha\,(\mathbf{u}\otimes\boldsymbol{\nu}) : (\mathbf{w}\otimes\boldsymbol{\nu}) - \int_\Omega p\mathbf{I} : (\nabla_h\mathbf{w} + \mathbf{S}(\mathbf{w}) + \eta\mathbf{I})$$

$$- \int_\Omega \mathbf{f}\cdot\mathbf{w} - \int_{\mathcal{E}_D} \alpha\,(\mathbf{g}\otimes\boldsymbol{\nu}) : (\mathbf{w}\otimes\boldsymbol{\nu})$$

$$= \int_\Omega \boldsymbol{\psi}(\nabla\mathbf{u}) : \Big(\nabla_h\mathbf{w} - \mathbf{S}(\mathbf{w})\Big) + \int_\Omega \mathbf{w}\cdot\mathbf{div}\,(\boldsymbol{\psi}(\nabla\mathbf{u}) - p\mathbf{I})$$

$$- \int_\Omega p\mathbf{I} : (\nabla_h\mathbf{w} + \mathbf{S}(\mathbf{w})) + \eta \int_\Omega \mathrm{tr}\,(\boldsymbol{\sigma})$$

$$= \int_\Omega \Big(\boldsymbol{\psi}(\nabla\mathbf{u}) - p\mathbf{I}\Big) : \Big(\nabla_h\mathbf{w} - \mathbf{S}(\mathbf{w})\Big) + \int_\Omega \mathbf{w}\cdot\mathbf{div}\,(\boldsymbol{\psi}(\nabla\mathbf{u}) - p\mathbf{I})\,. \tag{5.4.3}$$

Applying Gauss' formula on each element $T \in \mathcal{T}_h$, we obtain

$$\int_\Omega \mathbf{w}\cdot\mathbf{div}\,(\boldsymbol{\psi}(\nabla\mathbf{u}) - p\mathbf{I}) = \sum_{T\in\mathcal{T}_h} \int_T \mathbf{w}\cdot\mathbf{div}\,(\boldsymbol{\psi}(\nabla\mathbf{u}) - p\mathbf{I})$$
$$= \sum_{T\in\mathcal{T}_h} \left[- \int_T (\boldsymbol{\psi}(\nabla\mathbf{u}) - p\mathbf{I}) : \nabla\mathbf{w} + \int_{\partial T} \mathbf{w}\cdot(\boldsymbol{\psi}(\nabla\mathbf{u}) - p\mathbf{I})\,\boldsymbol{\nu} \right]$$

$$= - \int_\Omega (\boldsymbol{\psi}(\nabla\mathbf{u}) - p\mathbf{I}) : \nabla_h\mathbf{w} + \int_{\mathcal{E}_I} \{\boldsymbol{\psi}(\nabla\mathbf{u}) - p\mathbf{I}\} : \underline{[\![\mathbf{w}]\!]} + \int_{\mathcal{E}_D} \mathbf{w}\cdot(\boldsymbol{\psi}(\nabla\mathbf{u}) - p\mathbf{I})\,\boldsymbol{\nu}\,,$$

which, replaced back into (5.4.3), yields

$$[A_h(\mathbf{u},\xi),(\mathbf{w},\eta)] + [B_h(\mathbf{w},\eta),p] - [F_h,(\mathbf{w},\eta)] =$$

$$- \int_\Omega \mathbf{S}(\mathbf{w}) : \Big(\boldsymbol{\psi}(\nabla\mathbf{u}) - p\mathbf{I}\Big) + \int_{\mathcal{E}_I} \{\boldsymbol{\psi}(\nabla\mathbf{u}) - p\mathbf{I}\} : \underline{[\![\mathbf{w}]\!]} + \int_{\mathcal{E}_D} \mathbf{w}\cdot(\boldsymbol{\psi}(\nabla\mathbf{u}) - p\mathbf{I})\,\boldsymbol{\nu}\,.$$

Therefore, since

$$\int_\Omega \mathbf{S}(\mathbf{w}) : \Big(\boldsymbol{\psi}(\nabla\mathbf{u}) - p\mathbf{I}\Big) \;=\; \int_\Omega \mathbf{S}(\mathbf{w}) : \Big(\Pi_{\boldsymbol{\Sigma}_h}\boldsymbol{\psi}(\nabla\mathbf{u}) - \Pi_{W_h} p\mathbf{I}\Big)\,,$$

and applying the definition of $\mathbf{S}$ (cf. (5.2.21)), we deduce that

$$[A_h(\mathbf{u},\xi),(\mathbf{w},\eta)] + [B_h(\mathbf{w},\eta),p] - [F_h,(\mathbf{w},\eta)] \;=\; \int_{\mathcal{E}_D} (\mathbf{I}-\Pi_{\boldsymbol{\Sigma}_h})(\boldsymbol{\psi}(\nabla\mathbf{u})) : (\mathbf{w}\otimes\boldsymbol{\nu})$$

$$+ \int_{\mathcal{E}_I} \Big(\{(\mathbf{I}-\Pi_{\boldsymbol{\Sigma}_h})(\boldsymbol{\psi}(\nabla\mathbf{u})) - [\![(\mathbf{I}-\Pi_{\boldsymbol{\Sigma}_h})(\boldsymbol{\psi}(\nabla\mathbf{u}))]\!]\otimes\boldsymbol{\beta}\}\Big) : \underline{[\![\mathbf{w}]\!]}$$

$$- \int_{\mathcal{E}_I} \{(\mathbf{I}-\Pi_{W_h})(p)\}\,[\![\mathbf{w}]\!] - \int_{\mathcal{E}_D} (\mathbf{I}-\Pi_{W_h})(p)\,\mathbf{w}\cdot\boldsymbol{\nu}\,.$$

Finally, noting that $||[\![\mathbf{v}]\!]||_{L^2(\mathcal{E}_I)} \leq ||\underline{[\![\mathbf{v}]\!]}||_{[L^2(\mathcal{E}_I)]^{2\times 2}}$, the rest of the proof is analogue to the analysis shown in the proof of Lemma 5.2 in [32]. □

Next, applying an analogous version of Lemma 5.3 in [32], together with Lemma 5.4.1, we can establish the a-priori error estimates for $\mathbf{u}$ and p, in the energy norm $|||\cdot|||_h$ and the usual norm $||\cdot||_{L^2(\Omega)}$, respectively.

Theorem 5.4.1 *Let $(\mathbf{u}_h,\xi_h,p_h)$ and $(\mathbf{u},\xi,p)$ be the solutions of (5.2.26) and (5.1.1), respectively. Assume that $p|_T \in H^r(T)$, $\mathbf{u}|_T \in [H^{k+1}(T)]^2$ and $\boldsymbol{\psi}(\nabla\mathbf{u})|_T \in [H^{s+1}(T)]^{2\times 2}$, with $1 \leq s \leq k$, $r \leq k$, for all $T \in \mathcal{T}_h$. Then there exists $C_{\mathtt{err}} > 0$, independent of the meshsize, but depending on $\widehat{\alpha}$, $\widehat{\beta}$, l, and the polynomial approximation degree k, such that*

$$|||\mathbf{u}-\mathbf{u}_h|||_h^2 \;\leq$$

$$C_{\mathtt{err}} \sum_{T\in\mathcal{T}_h} \Big(h_T^{2k} ||\mathbf{u}||^2_{[H^{k+1}(T)]^2} \,+\, h_T^{2(s+1)} ||\boldsymbol{\psi}(\nabla\mathbf{u})||^2_{[H^{s+1}(T)]^{2\times 2}} \,+\, h_T^{2r} ||p||^2_{H^r(T)} \Big)\,,$$

and

$$||p-p_h||^2_{L^2(\Omega)} \;\leq$$

$$C_{\mathtt{err}} \sum_{T\in\mathcal{T}_h} \Big(h_T^{2k} ||\mathbf{u}||^2_{[H^{k+1}(T)]^2} \,+\, h_T^{2(s+1)} ||\boldsymbol{\psi}(\nabla\mathbf{u})||^2_{[H^{s+1}(T)]^{2\times 2}} \,+\, h_T^{2r} ||p||^2_{H^r(T)} \Big)\,.$$

We remark that when $k > 1$, and assuming that $p|_T \in H^k(T)$, $\mathbf{u}|_T \in [H^{k+1}(T)]^2$, $\psi(|\nabla\mathbf{u}|)|_T \in [H^k(T)]^{2\times 2}$, we obtain

$$|||\mathbf{u}-\mathbf{u}_h|||_h \;\leq$$

$$C_{\mathtt{err}}\, h^k \left(\sum_{T\in\mathcal{T}_h} ||\mathbf{u}||^2_{[H^{k+1}(T)]^2} \,+\, \sum_{T\in\mathcal{T}_h} ||\boldsymbol{\psi}(\nabla\mathbf{u})||^2_{[H^k(T)]^{2\times 2}} \,+\, \sum_{T\in\mathcal{T}_h} ||p||^2_{H^k(T)} \right)^{1/2}\,,$$

and

$$||p - p_h||_{L^2(\Omega)} \leq$$
$$C_{\texttt{err}}\, h^k \left(\sum_{T\in\mathcal{T}_h} ||\mathbf{u}||^2_{[H^{k+1}(T)]^2} + \sum_{T\in\mathcal{T}_h} ||\boldsymbol{\psi}(\nabla\mathbf{u})||^2_{[H^k(T)]^{2\times 2}} + \sum_{T\in\mathcal{T}_h} ||p||^2_{H^k(T)} \right)^{1/2} .$$

On the other hand, if $k = r = 1$, and assuming that $p|_T \in H^1(T)$, $\mathbf{u}|_T \in [H^2(T)]^2$, $\psi(|\nabla\mathbf{u}|)|_T \in [H^2(T)]^{2\times 2}$, we get

$$|||\mathbf{u} - \mathbf{u}_h|||_h \leq$$
$$C_{\texttt{err}}\, h \left(\sum_{T\in\mathcal{T}_h} ||\mathbf{u}||^2_{[H^2(T)]^2} + \sum_{T\in\mathcal{T}_h} ||\boldsymbol{\psi}(\nabla\mathbf{u})||^2_{[H^2(T)]^{2\times 2}} + \sum_{T\in\mathcal{T}_h} ||p||^2_{H^1(T)} \right)^{1/2} , \tag{5.4.4}$$

and

$$||p - p_h||_{L^2(\Omega)} \leq$$
$$C_{\texttt{err}}\, h \left(\sum_{T\in\mathcal{T}_h} ||\mathbf{u}||^2_{[H^2(T)]^2} + \sum_{T\in\mathcal{T}_h} ||\boldsymbol{\psi}(\nabla\mathbf{u})||^2_{[H^2(T)]^{2\times 2}} + \sum_{T\in\mathcal{T}_h} ||p||^2_{H^1(T)} \right)^{1/2} . \tag{5.4.5}$$

Analogously to Theorem 5.2 in [32], we establish the next result.

THEOREM 5.4.2 *Assume the same hypotheses of* Theorem 5.4.1. *Then there exists* $\tilde{C}_{\texttt{err}} > 0$*, independent of the meshsize, such that*

$$||\mathbf{t} - \mathbf{t}_h||^2_{[L^2(\Omega_2)]^2} \leq$$
$$\tilde{C}_{\texttt{err}} \left(\sum_{T\in\mathcal{T}_h} h_T^{2k} ||\mathbf{u}||^2_{[H^{k+1}(T)]^2} + \sum_{T\in\mathcal{T}_h} h_T^{2(s+1)} ||\boldsymbol{\psi}(\nabla\mathbf{u})||^2_{[H^{s+1}(T)]^{2\times 2}} + \sum_{T\in\mathcal{T}_h} h_T^{2r} ||p||^2_{H^r(T)} \right), \tag{5.4.6}$$

and

$$||\boldsymbol{\sigma} - \boldsymbol{\sigma}_h||^2_{[L^2(\Omega_2)]^2} \leq$$
$$\tilde{C}_{\texttt{err}} \left(\sum_{T\in\mathcal{T}_h} h_T^{2k} ||\mathbf{u}||^2_{[H^{k+1}(T)]^2} + \sum_{T\in\mathcal{T}_h} h_T^{2(s+1)} ||\boldsymbol{\psi}(\nabla\mathbf{u})||^2_{[H^{s+1}(T)]^{2\times 2}} + \sum_{T\in\mathcal{T}_h} h_T^{2r} ||p||^2_{H^r(T)} \right). \tag{5.4.7}$$

As before, we observe that when $k > 1$, and considering $p|_T \in H^k(T)$, $\mathbf{u}|_T \in [H^{k+1}(T)]^2$, $\psi(|\nabla\mathbf{u}|)|_T \in [H^k(T)]^{2\times 2}$, we have

$$||\mathbf{t} - \mathbf{t}_h||_{[L^2(\Omega_2)]^2} \leq$$
$$\tilde{C}_{\texttt{err}}\, h^k \left(\sum_{T\in\mathcal{T}_h} ||\mathbf{u}||^2_{[H^{k+1}(T)]^2} + \sum_{T\in\mathcal{T}_h} ||\boldsymbol{\psi}(\nabla\mathbf{u})||^2_{[H^k(T)]^{2\times 2}} + \sum_{T\in\mathcal{T}_h} ||p||^2_{H^k(T)} \right)^{1/2} ,$$

and

$$||\boldsymbol{\sigma} - \boldsymbol{\sigma}_h||_{[L^2(\Omega_2)]^2} \leq$$
$$\tilde{C}_{\mathbf{err}}\, h^k \left(\sum_{T\in\mathcal{T}_h} ||\mathbf{u}||^2_{[H^{k+1}(T)]^2} + \sum_{T\in\mathcal{T}_h} ||\boldsymbol{\psi}(\nabla \mathbf{u})||^2_{[H^k(T)]^{2\times 2}} + \sum_{T\in\mathcal{T}_h} ||p||^2_{H^k(T)} \right)^{1/2} .$$

On the other hand, if $k = 1$, and under the assumptions: $p|_T \in H^1(T)$, $\mathbf{u}|_T \in [H^2(T)]^2$, $\psi(|\nabla \mathbf{u}|)|_T \in [H^2(T)]^{2\times 2}$, we can conclude that

$$||\mathbf{t} - \mathbf{t}_h||_{[L^2(\Omega_2)]^2} \leq$$
$$\tilde{C}_{\mathbf{err}}\, h \left(\sum_{T\in\mathcal{T}_h} ||\mathbf{u}||^2_{[H^2(T)]^2} + \sum_{T\in\mathcal{T}_h} ||\boldsymbol{\psi}(\nabla \mathbf{u})||^2_{[H^2(T)]^{2\times 2}} + \sum_{T\in\mathcal{T}_h} ||p||^2_{H^1(T)} \right)^{1/2} ,$$

and

$$||\boldsymbol{\sigma} - \boldsymbol{\sigma}_h||_{[L^2(\Omega_2)]^2} \leq$$
$$\tilde{C}_{\mathbf{err}}\, h \left(\sum_{T\in\mathcal{T}_h} ||\mathbf{u}||^2_{[H^2(T)]^2} + \sum_{T\in\mathcal{T}_h} ||\boldsymbol{\psi}(\nabla \mathbf{u})||^2_{[H^2(T)]^{2\times 2}} + \sum_{T\in\mathcal{T}_h} ||p||^2_{H^1(T)} \right)^{1/2} .$$

5.4.1 L^2-norm error estimates

We now turn our attention to the L^2-norm for the error $(\mathbf{u} - \mathbf{u}_h)$. To this end, we notice from (5.3.7) that, given any $(\tilde{\mathbf{z}}, \tilde{\zeta}) \in \mathbf{V}(h) \times \mathbf{R}$, the Gâteaux derivative of $\mathcal{N}_h$ at $(\tilde{\mathbf{z}}, \tilde{\zeta})$ can be written as

$$D\mathcal{N}_h(\tilde{\mathbf{z}}, \tilde{\zeta})((\mathbf{w}, \eta), (\mathbf{v}, \lambda)) := \int_\Omega \Big(D\psi_{ij}(\tilde{\boldsymbol{\varphi}}(\tilde{\mathbf{z}}, \tilde{\zeta})) : \boldsymbol{\varphi}(\mathbf{w}, \eta) \Big)^2_{i,j=1} : \boldsymbol{\varphi}(\mathbf{v}, \lambda) \qquad (5.4.8)$$

for all $(\mathbf{w}, \eta), (\mathbf{v}, \lambda) \in \mathbf{V}(h) \times \mathbf{R}$, with $\tilde{\boldsymbol{\varphi}}$ and $\boldsymbol{\varphi}$ given before (cf. (5.3.5) and (5.3.4), respectively). Hereafter, $D\psi_{ij}$ denotes the derivative (jacobian) of ψ_{ij}, for $i, j = 1, 2$.

In what follows we assume that $\dfrac{\partial \psi_{ij}}{\partial \tilde{r}_{kl}}(\tilde{\mathbf{r}}) = \dfrac{\partial \psi_{kl}}{\partial \tilde{r}_{ij}}(\tilde{\mathbf{r}})$, for all $\tilde{\mathbf{r}} \in X$, and for all $i, j, k, l = 1, 2$, and that $D\mathcal{N}_h$ is *hemi-continuous*, that is for any $\mathbf{r}, \mathbf{s} \in X$, the mapping $\mathbf{R} \ni \mu \to D\mathcal{N}_h((\mathbf{w}, \eta) + \mu(\mathbf{v}, \lambda))((\mathbf{v}, \lambda), \cdot) \in (\mathbf{V}(h) \times \mathbf{R})'$ is continuous. Thus, applying the mean value theorem we deduce that there exists $(\tilde{\mathbf{u}}, \tilde{\xi}) \in \mathbf{V}(h) \times \mathbf{R}$, a convex combination of $(\mathbf{u}, \xi)$ and $(\mathbf{u}_h, \xi_h)$, such that

$$D\mathcal{N}_h(\tilde{\mathbf{u}}, \tilde{\xi})((\mathbf{u} - \mathbf{u}_h, \xi - \xi_h), (\mathbf{v}, \lambda)) = [\mathcal{N}_h(\mathbf{u}, \xi) - \mathcal{N}_h(\mathbf{u}_h, \xi_h), (\mathbf{v}, \lambda)] \qquad (5.4.9)$$

for all $(\mathbf{v}, \lambda) \in \mathbf{V}(h) \times \mathbf{R}$.

Further, it follows from (5.2.24), (5.3.8) and (5.4.9) that

$$DA_h(\tilde{\mathbf{u}},\tilde{\xi})((\mathbf{u}-\mathbf{u}_h,\xi-\xi_h),(\mathbf{v},\lambda)) \;=\; [A_h(\mathbf{u},\xi)-A_h(\mathbf{u}_h,\xi_h),(\mathbf{v},\lambda)] \tag{5.4.10}$$

for all $(\mathbf{v},\lambda)\in\mathbf{V}(h)\times\mathbf{R}$.

Next, setting $\tilde{\boldsymbol{\theta}} := \tilde{\boldsymbol{\varphi}}(\tilde{\mathbf{u}},\tilde{\xi})$, we let $(\mathbf{z},q)\in[H^1(\Omega)]^2\times L_0^2(\Omega)$ be the unique solution of the linear boundary value problem

$$\begin{gathered} -\mathbf{div}\,\left(\left(D\psi_{ij}(\tilde{\boldsymbol{\theta}}):\nabla\mathbf{z}\right)_{i,j=1}^{2}\right) + \nabla q \;=\; \mathbf{u}-\mathbf{u}_h \quad \text{in} \quad \Omega\,, \\ \operatorname{div}\mathbf{z} \;=\; 0 \quad \text{in} \quad \Omega\,, \quad \mathbf{z} \;=\; \mathbf{0} \quad \text{on} \quad \Gamma\,. \end{gathered} \tag{5.4.11}$$

We consider from now on enough elliptic regularity on the exact solution of (5.4.11), that is, we assume that $(\mathbf{z},q)\in([H^2(\Omega)]^2\cap[H_0^1(\Omega)]^2)\times\left(H^1(\Omega)\cap L_0^2(\Omega)\right)$, and $\left(D\psi_{ij}(\tilde{\boldsymbol{\theta}}):\nabla\mathbf{z}\right)_{i,j=1}^{2}\in[H^1(\Omega)]^{2\times 2}$, with

$$\left\|\left(D\psi_{ij}(\tilde{\boldsymbol{\theta}}):\nabla\mathbf{z}\right)_{i,j=1}^{2}\right\|_{[H^1(\Omega)]^{2\times 2}} + \,||\mathbf{z}||_{[H^2(\Omega)]^2} \,+\, ||q||_{H^1(\Omega)} \;\le\; C_{\mathtt{reg}}\,||\mathbf{u}-\mathbf{u}_h||_{L^2(\Omega_2)}\,, \tag{5.4.12}$$

and where the constant $C_{\mathtt{reg}}>0$ is independent of $\mathbf{u}$ and $\mathbf{u}_h$.

Then, we apply a similar result to Lemma 5.3 in [32] and the projection error estimates given in Lemma 5.4.1, with $s=k=1$ and $m=1$, to obtain

$$|||(\mathbf{I}-\Pi_h^1)(\mathbf{z})|||_h \;\le\; (2\,C_{\mathtt{upp}})^{1/2}C_{\mathtt{ort}}\,h\,||\mathbf{z}||_{[H^2(\Omega)]^2} \;\le\; \tilde{C}_{\mathtt{ort}}\,h\,||\mathbf{u}-\mathbf{u}_h||_{[L^2(\Omega)]^2}\,, \tag{5.4.13}$$

with $\tilde{C}_{\mathtt{ort}} := (2\,C_{\mathtt{upp}})^{1/2}C_{\mathtt{ort}}C_{\mathtt{reg}}$.

Now, using the method applied in Section 5.2, we deduce that the mixed LDG formulation of problem (5.4.11) reduces to: *Find* $(\mathbf{z}_h,\zeta_h,q_h)\in\mathbf{V}_h\times\mathbf{R}\times W_h$ *such that*

$$\begin{aligned} DA_h(\tilde{\mathbf{u}},\tilde{\xi})((\mathbf{z}_h,\zeta_h),(\mathbf{v}_h,\lambda_h)) \;+\; [B_h(\mathbf{v}_h,\lambda_h),q_h] \;&=\; [\tilde{F}_h,(\mathbf{v}_h,\lambda_h)]\,, \\ [B_h(\mathbf{z}_h,\zeta_h),r_h] \;&=\; [\tilde{G}_h,r_h]\,, \end{aligned} \tag{5.4.14}$$

for all $(\mathbf{v}_h,\lambda_h,q_h)\in\mathbf{V}_h\times\mathbf{R}\times W_h$, where $\tilde{F}_h:\mathbf{V}(h)\times\mathbf{R}\to\mathbf{R}$ and $\tilde{G}_h:W_h\to\mathbf{R}$ are defined by

$$[\tilde{F}_h,(\mathbf{v}_h,\lambda_h)] \;:=\; \int_\Omega(\mathbf{u}-\mathbf{u}_h)\cdot\mathbf{v}_h \quad \forall\,(\mathbf{v}_h,\lambda_h)\in\mathbf{V}(h)\times\mathbf{R}\,, \tag{5.4.15}$$

and

$$[\tilde{G}_h,r_h] \;:=\; 0 \quad \forall\,r_h\in W_h\,. \tag{5.4.16}$$

As a consequence of the assumption (5.1.3) on ψ, and proceeding as in the proof of Lemma 5.3.1, one can show that $DA_h(\tilde{\mathbf{u}}, \tilde{\xi})$ is $(\mathbf{V}(h) \times \mathbf{R})$-elliptic with respect to $||\cdot||_{\mathbf{V}(h)\times\mathbf{R}}$, and that B_h satisfies the discrete inf-sup condition (cf. Lemma 5.3.3), whence problem (5.4.14) has a unique solution $(\mathbf{z}_h, \zeta_h, q_h) \in \mathbf{V}_h \times \mathbf{R} \times W_h$. Furthermore, applying the linear version of the consistency estimate provided by Lemma 5.4.2, with $s = 0$, and $r = 1$, and using the estimate (5.4.12), we find that

$$|DA_h(\tilde{\mathbf{u}}, \tilde{\xi})((\mathbf{z}, \zeta), (\mathbf{w}, \eta)) + [B_h(\mathbf{w}, \eta), q] - [F_h, (\mathbf{w}, \eta)]|$$

$$\leq C_{\texttt{con}}\, h \left(\left\| \Big(D\psi_{ij}(\tilde{\boldsymbol{\theta}}) : \nabla \mathbf{z} \Big)_{i,j=1}^{2} \right\|^2_{[H^1(\Omega)]^{2\times 2}} + ||q||^2_{H^1(\Omega)} \right)^{1/2} |||\mathbf{w}|||_h \tag{5.4.17}$$

$$\leq \tilde{C}_{\texttt{con}}\, h \, ||\mathbf{u} - \mathbf{u}_h||_{[L^2(\Omega)]^2} \, |||\mathbf{w}|||_h \quad \forall\, (\mathbf{w}, \eta) \in \mathbf{V}(h) \times \mathbf{R}\,,$$

with $\tilde{C}_{\texttt{con}} := C_{\texttt{con}}\, C_{\texttt{reg}}$.

Therefore, extending the ideas of the analysis in [32] to the present case, we conclude the following main result.

THEOREM 5.4.3 *Assume the same hypotheses of* Theorem 5.4.1*, and that $D\psi_{ij}$ and the solution $(\mathbf{z}, q)$ of* (5.4.11) *satisfy the conditions stated in the present subsection. Then there exists $\bar{C}_{\texttt{err}} > 0$, independent of the meshsize, such that for $k > 1$ and $k = 1$, respectively, there hold*

$$\|\mathbf{u} - \mathbf{u}_h\|_{L^2(\Omega_2)} \leq$$

$$\bar{C}_{\texttt{err}}\, h^{k+1} \left(\sum_{T\in\mathcal{T}_h} \|\mathbf{u}\|^2_{[H^{k+1}(T)]^2} + \sum_{T\in\mathcal{T}_h} \|\boldsymbol{\psi}(\nabla\mathbf{u})\|^2_{[H^k(T)]^{2\times 2}} + \sum_{T\in\mathcal{T}_h} ||p||^2_{H^k(T)} \right)^{1/2},$$

and

$$\|\mathbf{u} - \mathbf{u}_h\|_{L^2(\Omega_2)} \leq$$

$$\bar{C}_{\texttt{err}}\, h^{2} \left(\sum_{T\in\mathcal{T}_h} \|\mathbf{u}\|^2_{[H^{2}(T)]^2} + \sum_{T\in\mathcal{T}_h} \|\boldsymbol{\psi}(\nabla\mathbf{u})\|^2_{[H^2(T)]^{2\times 2}} + \sum_{T\in\mathcal{T}_h} ||p||^2_{H^1(T)} \right)^{1/2}.$$

PROOF. Taking $(\mathbf{v}, \lambda) := (\mathbf{u} - \mathbf{u}_h, \xi - \xi_h) \in \mathbf{V}(h) \times \mathbf{R}$ in (5.4.15), we obtain

$$||\mathbf{u} - \mathbf{u}_h||^2_{[L^2(\Omega)]^2} = [\tilde{F}_h, (\mathbf{u} - \mathbf{u}_h, \xi - \xi_h)]$$

$$= DA_h(\tilde{\mathbf{u}}, \tilde{\xi})((\mathbf{z}, \zeta), (\mathbf{u} - \mathbf{u}_h, \xi - \xi_h)) + [B_h(\mathbf{u} - \mathbf{u}_h, \xi - \xi_h), q]$$

$$- \Big(DA_h(\tilde{\mathbf{u}}, \tilde{\xi})((\mathbf{z}, \zeta), (\mathbf{u} - \mathbf{u}_h, \xi - \xi_h)) + [B_h(\mathbf{u} - \mathbf{u}_h, \xi - \xi_h), q]$$

$$- [\tilde{F}_h, (\mathbf{u} - \mathbf{u}_h, \xi - \xi_h)] \Big)$$

which, according to (5.4.17), yields

$$||\mathbf{u}-\mathbf{u}_h||^2_{[L^2(\Omega)]^2} \,\leq\, |DA_h(\tilde{\mathbf{u}},\tilde{\xi})((\mathbf{z},\zeta),(\mathbf{u}-\mathbf{u}_h,\xi-\xi_h)) \,+\, [B_h(\mathbf{u}-\mathbf{u}_h,\xi-\xi_h),q]| \\ +\,\tilde{C}_{\mathtt{con}}\,h\,||\mathbf{u}-\mathbf{u}_h||_{[L^2(\Omega)]^2}\,|||\mathbf{u}-\mathbf{u}_h|||_h\,. \tag{5.4.18}$$

It is easy to check that $[B_h(\mathbf{u},\xi),r_h] \,=\, [B_h(\mathbf{u}_h,\xi_h),r_h]$, for all $r_h\in W_h$ (we remark that we know in advance that $\xi=\xi_h=0$). Therefore, we have

$$|[B_h(\mathbf{u}-\mathbf{u}_h,\xi-\xi_h),q]| \,=\, |[B_h(\mathbf{u}-\mathbf{u}_h,\xi-\xi_h),(\mathbf{I}-\Pi^1_h)(q)]| \\ \leq\, \bar{C}_{\mathtt{con}}\,h\,|||\mathbf{u}-\mathbf{u}_h|||_h\,||\mathbf{u}-\mathbf{u}_h||_{[L^2(\Omega)]^2}$$

Now, since $DA_h(\tilde{\mathbf{u}},\tilde{\xi})$ is symmetric, we can write

$$DA_h(\tilde{\mathbf{u}},\tilde{\xi})((\mathbf{z},\zeta),(\mathbf{u}-\mathbf{u}_h,\xi-\xi_h)) \,=\, DA_h(\tilde{\mathbf{u}},\tilde{\xi})((\mathbf{u}-\mathbf{u}_h,\xi-\xi_h),(\Pi^1_h(\mathbf{z}),\zeta_h)) \\ +\,DA_h(\tilde{\mathbf{u}},\tilde{\xi})((\mathbf{u}-\mathbf{u}_h,\xi-\xi_h),((\mathbf{I}-\Pi^1_h)(\mathbf{z}),\zeta-\zeta_h))$$

which, because of (5.4.10), reduces to

$$DA_h(\tilde{\mathbf{u}},\tilde{\xi})((\mathbf{z},\zeta),(\mathbf{u}-\mathbf{u}_h,\xi-\xi_h)) \,=\, [A_h(\mathbf{u},\xi)-A_h(\mathbf{u}_h,\xi_h),((\mathbf{I}-\Pi^1_h)(\mathbf{z}),\zeta-\zeta_h)] \\ +\,[A_h(\mathbf{u},\xi)-A_h(\mathbf{u}_h,\xi_h),(\Pi^1_h(\mathbf{z}),\zeta_h)]\,,$$

and noting that $\zeta=0=\zeta_h$, using the Lipschitz-continuity of A_h and the estimate (5.4.17), gives

$$|DA_h(\tilde{\mathbf{u}},\tilde{\xi})((\mathbf{z},\zeta),(\mathbf{u}-\mathbf{u}_h,\xi-\xi_h))| \,\leq\, C_{\mathtt{LC}}\,\tilde{C}_{\mathtt{ort}}\,h\,|||\mathbf{u}-\mathbf{u}_h|||_h\,||\mathbf{u}-\mathbf{u}_h||_{[L^2(\Omega)]^2} \\ +\,|[A_h(\mathbf{u},\xi)-A_h(\mathbf{u}_h,\xi_h),(\Pi^1_h(\mathbf{z}),\zeta_h)]|\,. \tag{5.4.19}$$

On the other hand, it is not difficult to see, using Gauss' formula, that $(\mathbf{z},\zeta,q)$ also satisfies $[A_h(\mathbf{u},\xi),(\mathbf{z},\zeta)] \,+\, [B_h(\mathbf{z},\zeta),p] \,=\, [F_h,(\mathbf{z},\zeta)]$, and from the mixed formulation (5.2.26) we have that $[A_h(\mathbf{u}_h,\xi_h),(\Pi^1_h(\mathbf{z}),\zeta_h)] \,+\, [B_h(\Pi^1_h(\mathbf{z}),\zeta_h),p_h] \,=\, [F_h,(\Pi^1_h(\mathbf{z}),\zeta_h)]$ and $[B_h(\mathbf{z},\zeta),p-p_h]=0$. In this way, it follows that

$$[A_h(\mathbf{u},\xi)-A_h(\mathbf{u}_h,\xi_h),(\Pi^1_h(\mathbf{z}),\zeta_h)] \,=\, -[B_h((\mathbf{I}-\Pi^1_h)(\mathbf{z}),\zeta-\zeta_h),p-p_h] \\ +\,[F_h,((\mathbf{I}-\Pi^1_h)(\mathbf{z}),\zeta-\zeta_h)] \,-\, [A_h(\mathbf{u},\xi),((\mathbf{I}-\Pi^1_h)(\mathbf{z}),\zeta-\zeta_h)] \\ -\,[B_h(((\mathbf{I}-\Pi^1_h)(\mathbf{z}),\zeta-\zeta_h),p]\,,$$

and hence, applying again Lemma 5.4.2 (with $s = k-1$ and $r = k$) and the estimate (5.4.17), we conclude that

$$
\begin{aligned}
&||[A_h(\mathbf{u},\xi) - A_h(\mathbf{u}_h,\xi_h), (\Pi_h^1(\mathbf{z}),\zeta_h)]|| \\
&\le\; C_{\mathtt{con}}\,\tilde{C}_{\mathtt{ort}}\,h^{k+1}\left(\sum_{T\in\mathcal{T}_h}||\boldsymbol{\psi}(\nabla\mathbf{u})||^2_{[H^k(T)]^{2\times2}} \;+\; \sum_{T\in\mathcal{T}_h}||p||^2_{H^k(T)}\right)^{1/2} ||\mathbf{u}-\mathbf{u}_h||_{[L^2(\Omega)]^2}\,.
\end{aligned}
\tag{5.4.20}
$$

Finally, (5.4.18), (5.4.19), and (5.4.20) imply that

$$
\begin{aligned}
||\mathbf{u}-\mathbf{u}_h||_{[L^2(\Omega)]^2} \;\le\; &\left(\tilde{C}_{\mathtt{con}} \;+\; C_{\mathtt{LC}}\,\tilde{C}_{\mathtt{ort}}\right) h\,|||\mathbf{u}-\mathbf{u}_h|||_h \\
&+ C_{\mathtt{con}}\,\tilde{C}_{\mathtt{ort}}\,h^{k+1}\left(\sum_{T\in\mathcal{T}_h}||\boldsymbol{\psi}(\nabla\mathbf{u})||^2_{[H^k(T)]^{2\times2}} \;+\; \sum_{T\in\mathcal{T}_h}||p||^2_{H^k(T)}\right)^{1/2},
\end{aligned}
$$

which, together with the estimate for $|||\mathbf{u}-\mathbf{u}_h|||_h$ (cf. Theorem 5.4.1), completes the proof. □

5.5 An a-posteriori error estimate for the energy norm

Hereafter we consider problem (5.1.1) with $\mathbf{g} = \mathbf{0}$, and then we re-define $\mathbf{V}(h)$ as $\mathbf{V}(h) := \mathbf{V}_h + [H_0^1(\Omega)]^2$. Next, we introduce first the semilinear global operator $\mathcal{A}_h : \Big(\mathbf{V}(h)\times\mathbf{R}\times L^2(\Omega)\Big) \to \Big(\mathbf{V}(h)\times\mathbf{R}\times L^2(\Omega)\Big)'$, given by

$$
[\mathcal{A}_h(\mathbf{w},\eta,r),(\mathbf{v},\lambda,q)] \;:=\; [A_h(\mathbf{w},\eta),(\mathbf{v},\lambda)] \;+\; [B_h(\mathbf{v},\lambda),r] \;+\; [B_h(\mathbf{w},\eta),q] \tag{5.5.1}
$$

for all $(\mathbf{w},\eta,r),(\mathbf{v},\lambda,q)\in\mathbf{V}(h)\times\mathbf{R}\times L^2(\Omega)$, and we consider its respective Gâteaux derivative at $(\mathbf{z},\zeta)\in\mathbf{V}(h)\times\mathbf{R}$ as the bounded bilinear form $D\mathcal{A}_h(\mathbf{z},\zeta) : \Big(\mathbf{V}(h)\times\mathbf{R}\times L^2(\Omega)\Big)\times\Big(\mathbf{V}(h)\times\mathbf{R}\times L^2(\Omega)\Big)\to\mathbf{R}$ defined by

$$
\begin{aligned}
D\mathcal{A}_h(\mathbf{z},\zeta)((\mathbf{w},\eta,r),(\mathbf{v},\lambda,q)) \;:=\; &DA_h(\mathbf{z},\zeta)((\mathbf{w},\eta),(\mathbf{v},\lambda)) \\
&+ [B_h(\mathbf{v},\lambda),r] \;+\; [B_h(\mathbf{w},\eta),q]\,,
\end{aligned}
\tag{5.5.2}
$$

for all $(\mathbf{w},\eta,r),(\mathbf{v},\lambda,q)\in\mathbf{V}(h)\times\mathbf{R}\times L^2(\Omega)$. Furthermore, we also define the linear operator $\mathcal{F}_h\in(\mathbf{V}(h)\times\mathbf{R}\times L^2(\Omega))'$ by

$$
[\mathcal{F}_h,(\mathbf{v},\lambda,q)] \;:=\; [F_h,(\mathbf{v},\lambda)] \;+\; [G_h,q] \quad \forall\,(\mathbf{v},\lambda,q)\in\mathbf{V}(h)\times\mathbf{R}\times L^2(\Omega)\,. \tag{5.5.3}
$$

In what follows we consider $(\tilde{\mathbf{u}},\tilde{\xi}) \in \mathbf{V}(h)\times\mathbf{R}$ such that (5.4.10) holds. Therefore, adapting the ideas of the proof of Lemma 4.3 in [70], we obtain the following result.

Lemma 5.5.1 *There exist $C\,,\tilde{C}>0$, independent of the meshsize (and $(\tilde{\mathbf{u}},\tilde{\xi})$), such that for any $(\mathbf{u},\xi,p) \in [H^1_0(\Omega)]^2\times\mathbf{R}\times L^2_0(\Omega)$ there is $(\mathbf{v},\lambda,q) \in [H^1_0(\Omega)]^2\times\mathbf{R}\times L^2_0(\Omega)$ with*

$$\begin{aligned} D\mathcal{A}_h(\tilde{\mathbf{u}},\tilde{\xi})((\mathbf{u},\xi,p),(\mathbf{v},\lambda,q)) \,&\geq\, C\,||(\mathbf{u},\xi,p)||^2_{LDG}\,,\\ ||(\mathbf{v},\lambda,q)||_{LDG} \,&\leq\, \tilde{C}\,||(\mathbf{u},\xi,p)||_{LDG}\,. \end{aligned} \tag{5.5.4}$$

Proof. Let $p\in L^2_0(\Omega)$. Then, there exists $\mathbf{w}\in[H^1_0(\Omega)]^2$ such that (cf. Corollary 2.4 in [64])

$$-\int_\Omega p\,\mathrm{div}\,\mathbf{w} \,\geq\, C_0\,||p||^2_{L^2(\Omega)}\,, \qquad |||\mathbf{w}|||_h \,\leq\, ||p||_{L^2(\Omega)}\,. \tag{5.5.5}$$

Now, we choose $\mathbf{v}:=\kappa_0\,\mathbf{u}+\kappa_1\,\mathbf{w}$, $q:=-\kappa_0\,p$, and $\lambda:=\kappa_0\,\xi$, where κ_0 and κ_1 are positive constants to be determined, in order to obtain (5.5.4). Since $\mathbf{u},\mathbf{v}\in[H^1_0(\Omega)]^2$, we have that $\mathbf{S}(\mathbf{v}) = \mathbf{S}(\mathbf{u}) = \mathbf{0}$, and $[\![\mathbf{v}]\!] = [\![\mathbf{u}]\!] = \mathbf{0}$ on $\mathcal{E}_I$, and $\mathbf{v}=\mathbf{0}$ on $\mathcal{E}_D$. Then, replacing back, we obtain from (5.5.2)

$$D\mathcal{A}_h(\tilde{\mathbf{u}},\tilde{\xi})((\mathbf{u},\xi,p),(\mathbf{v},\lambda,q)) \,=\, \kappa_1\int_\Omega(\nabla\mathbf{u}+\xi\mathbf{I}):\Big(D\psi_{ij}(\tilde{\boldsymbol{\varphi}}(\mathbf{z},\zeta)):\nabla\mathbf{w}\Big)_{i,j=1,2}$$

$$+\,\kappa_0\int_\Omega(\nabla\mathbf{u}+\xi\mathbf{I}):\Big(D\psi_{ij}(\tilde{\boldsymbol{\varphi}}(\mathbf{z},\zeta)):(\nabla\mathbf{u}+\xi\mathbf{I})\Big)_{i,j=1,2} \,-\, \kappa_1\int_\Omega p\,\mathrm{div}\,\mathbf{w}\,,$$

and, using the bounds in (5.1.2), (5.1.3), and (5.5.5), and applying then the arithmetic-geometric mean inequality, we have

$$D\mathcal{A}_h(\tilde{\mathbf{u}},\tilde{\xi})((\mathbf{u},\xi,p),(\mathbf{v},\lambda,q)) \,\geq$$

$$\left(\kappa_0\,\tilde{C}_2-\frac{\kappa_1\,\tilde{C}_1}{2\,\varepsilon}\right)||\nabla\mathbf{u}+\xi\mathbf{I}||^2_{[L^2(\Omega_2)]^2} \,+\, \kappa_1\left(C_0-\frac{\varepsilon\,\tilde{C}_1}{2}\right)||p||^2_{L^2(\Omega)}\,,$$

where $\varepsilon>0$ will be determined later, and such that $\kappa_0\,\tilde{C}_2-\dfrac{\kappa_1\,\tilde{C}_1}{2\,\varepsilon}>0$ and $C_0-\dfrac{\varepsilon\,\tilde{C}_1}{2}>0$. Hence, choosing $\varepsilon:=\dfrac{C_0}{\tilde{C}_1}$, $\kappa_0:=1$, and $\kappa_1:=\dfrac{\tilde{C}_2\,C_0}{\tilde{C}_1^2}$, and taking into account (5.3.9), the inequality becomes

$$D\mathcal{A}_h(\tilde{\mathbf{u}},\tilde{\xi})((\mathbf{u},\xi,p),(\mathbf{v},\lambda,q)) \,\geq\, \frac{\tilde{C}_2}{2}||\nabla\mathbf{u}||^2_{[L^2(\Omega_2)]^2} \,+\, \tilde{C}_2\,|\xi|^2 \,+\, \frac{\tilde{C}_2\,C_0^2}{2\,\tilde{C}_1^2}\,||p||^2_{L^2(\Omega)}\,. \tag{5.5.6}$$

and the proof follows straightforwardly. □

Next, we point out that given any $(\mathbf{v}, \lambda, q) \in [H_0^1(\Omega)]^2 \times \mathbf{R} \times L_0^2(\Omega)$, we have, by an approximation argument, that there exist $(\mathbf{v}_h, \lambda_h, q_h) \in \mathbf{V}_h \times \mathbf{R} \times (W_h \cap L_0^2(\Omega))$ with $\lambda_h = \lambda$, and $C > 0$, independent of the meshsize, such that

$$\sum_{T\in\mathcal{T}_h} \Big(h_T^{-2}\, ||\mathbf{v} - \mathbf{v}_h||^2_{[L^2(T)]^2} \,+\, ||\nabla(\mathbf{v} - \mathbf{v}_h)||^2_{[L^2(T)]^{2\times 2}} \,+\, h_T^{-1}\, ||\mathbf{v} - \mathbf{v}_h||^2_{[L^2(\partial T)]^2} \Big)$$
$$\leq\, C\, ||\nabla \mathbf{v}||^2_{[L^2(\Omega)]^{2\times 2}}\,, \tag{5.5.7}$$

as well as

$$||q - q_h||_{L^2(\Omega)} \,\leq\, C\, ||q||_{L^2(\Omega)}\,. \tag{5.5.8}$$

These assumptions are valid, for example, if $\mathbf{v}_h$ is chosen as the interpolant of $\mathbf{v}$ onto $\mathbf{V}_h$, and q_h as the L^2-projection of q onto W_h.

Then, we can establish the following result.

LEMMA 5.5.2 *Let $(\mathbf{v}, \lambda, q) \in [H_0^1(\Omega)]^2 \times \mathbf{R} \times L_0^2(\Omega)$, and $(\mathbf{v}_h, \lambda_h, q_h) \in \mathbf{V}_h \times \mathbf{R} \times (W_h \cap L_0^2(\Omega))$ such that* (5.5.7) *and* (5.5.8) *are satisfied. Then, there exists a constant $C_{\mathtt{con}} > 0$, independent of the meshsize, such that*

$$\Big|[\mathcal{F}_h, (\mathbf{v} - \mathbf{v}_h, \lambda - \lambda_h, q - q_h)] - [\mathcal{A}_h(\mathbf{u}_h, \xi_h, p_h), (\mathbf{v} - \mathbf{v}_h, \lambda - \lambda_h, q - q_h)]\Big|$$
$$\leq\, C_{\mathtt{con}} \left(\sum_{T\in\mathcal{T}_h} \eta_T^2 \right)^{1/2} \, ||(\mathbf{v}, \lambda, q)||_{LDG} \tag{5.5.9}$$

where, for each $T \in \mathcal{T}_h$, η_T is defined by

$$\begin{aligned} \eta_T^2 \,:=\, & h_T^2\, ||\mathbf{f} + \mathbf{div}\, \boldsymbol{\psi}(\mathbf{t}_h) - \nabla p_h||^2_{[L^2(T)]^2} \,+\, ||\mathrm{tr}\,(\mathbf{t}_h)||^2_{[L^2(T)]^2} \\ & + h_T\, ||[\![\boldsymbol{\psi}(\mathbf{t}_h) - p_h \mathbf{I}]\!]||^2_{[L^2(\partial T\setminus\Gamma)]^2} \,+\, ||\alpha^{1/2} \underline{[\![\mathbf{u}_h]\!]}||^2_{[L^2(\partial T\cap\mathcal{E}_I)]^{2\times 2}} \\ & + ||\alpha^{1/2} \mathbf{u}_h \otimes \boldsymbol{\nu}||^2_{[L^2(\partial T\cap\mathcal{E}_D)]^{2\times 2}} \,+\, h_T\, ||\boldsymbol{\sigma}_h - (\boldsymbol{\psi}(\mathbf{t}_h) - p_h \mathbf{I})||^2_{[L^2(\partial T\cap\mathcal{E}_D)]^{2\times 2}} \\ & + h_T\, ||\{\boldsymbol{\sigma}_h\} - [\![\boldsymbol{\sigma}_h]\!] \otimes \boldsymbol{\beta} - \{\boldsymbol{\psi}(\mathbf{t}_h) - p_h \mathbf{I}\}||^2_{[L^2(\partial T\cap\mathcal{E}_I)]^{2\times 2}}\,. \end{aligned} \tag{5.5.10}$$

Proof. Integrating by parts, we have that

$$[A_h(\mathbf{u}_h,\xi_h),(\mathbf{v}-\mathbf{v}_h,0)] \,+\, [B_h(\mathbf{v}-\mathbf{v}_h,0),p_h]$$

$$= \int_\Omega \Big(\boldsymbol{\psi}(\mathbf{t}_h)-p_h\mathbf{I}\Big) : \Big(\nabla_h(\mathbf{v}-\mathbf{v}_h)-\mathbf{S}(\mathbf{v}-\mathbf{v}_h)\Big)$$

$$+ \int_{\mathcal{E}_I}\alpha\,\underline{[\![\mathbf{u}_h]\!]} : \underline{[\![\mathbf{v}-\mathbf{v}_h]\!]} \,+\, \int_{\mathcal{E}_D}\alpha(\mathbf{u}_h\otimes\boldsymbol{\nu}) : ((\mathbf{v}-\mathbf{v}_h)\otimes\boldsymbol{\nu})$$

$$= \sum_{T\in\mathcal{T}_h}\Big(-\int_T \mathrm{div}\,(\boldsymbol{\psi}(\mathbf{t}_h)-p_h\mathbf{I})\cdot(\mathbf{v}-\mathbf{v}_h)$$

$$+ \int_{\partial T}(\boldsymbol{\psi}(\mathbf{t}_h)-p_h\mathbf{I}) : ((\mathbf{v}-\mathbf{v}_h)\otimes\boldsymbol{\nu})\Big) + \int_{\mathcal{E}_I}\alpha\,\underline{[\![\mathbf{u}_h]\!]} : \underline{[\![\mathbf{v}-\mathbf{v}_h]\!]}$$

$$+ \int_{\mathcal{E}_D}\alpha(\mathbf{u}_h\otimes\boldsymbol{\nu}) : ((\mathbf{v}-\mathbf{v}_h)\otimes\boldsymbol{\nu}) \,-\, \int_\Omega \boldsymbol{\sigma}_h : \mathbf{S}(\mathbf{v}-\mathbf{v}_h)\,,$$

where the last term was deduced from (5.2.23). Then, integrating by parts and applying the definition of $\mathbf{S}$ (cf. (5.2.21)), we obtain that

$$[A_h(\mathbf{u}_h,\xi_h),(\mathbf{v}-\mathbf{v}_h,0)] \,+\, [B_h(\mathbf{v}-\mathbf{v}_h,0),p_h] \,=\, -\int_\Omega \mathbf{div}_h\,(\boldsymbol{\psi}(\mathbf{t}_h)-p_h\mathbf{I})\cdot(\mathbf{v}-\mathbf{v}_h)$$

$$+ \int_{\mathcal{E}_I}\alpha\,\underline{[\![\mathbf{u}_h]\!]} : \underline{[\![\mathbf{v}-\mathbf{v}_h]\!]} + \int_{\mathcal{E}_D}\alpha(\mathbf{u}_h\otimes\boldsymbol{\nu}) : ((\mathbf{v}-\mathbf{v}_h)\otimes\boldsymbol{\nu})$$

$$- \int_{\mathcal{E}_I}\Big(\{\boldsymbol{\sigma}_h\} - [\![\boldsymbol{\sigma}_h]\!]\otimes\boldsymbol{\beta} - \{\boldsymbol{\psi}(\mathbf{t}_h)-p_h\mathbf{I}\}\Big) : \underline{[\![\mathbf{v}-\mathbf{v}_h]\!]}$$

$$- \int_{\mathcal{E}_D}(\boldsymbol{\sigma}_h-\boldsymbol{\psi}(\mathbf{t}_h)+p_h\mathbf{I}) : ((\mathbf{v}-\mathbf{v}_h)\otimes\boldsymbol{\nu}) \,+\, \int_{\mathcal{E}_I}[\![\boldsymbol{\psi}(\mathbf{t}_h)-p_h\mathbf{I}]\!]\cdot\{\mathbf{v}-\mathbf{v}_h\}\,. \tag{5.5.11}$$

Similarly, using the definition of $\mathbf{t}_h$ (cf. (5.2.22)), we have

$$[B_h(\mathbf{u}_h,\xi_h),q-q_h] \,=\, -\int_\Omega(q-q_h)\,\mathbf{I} : (\mathbf{t}_h-\mathcal{G}) \,=\, -\int_\Omega(q-q_h)\,\mathrm{tr}\,(\mathbf{t}_h)\,.$$

In this way, we get

$$[\mathcal{F}_h,(\mathbf{v}-\mathbf{v}_h,0,q-q_h)] - [\mathcal{A}_h(\mathbf{u}_h,\xi_h,p_h),(\mathbf{v}-\mathbf{v}_h,0,q-q_h)]$$

$$= \sum_{T\in\mathcal{T}_h}\int_T\Big(\mathbf{f}+\mathbf{div}\,(\boldsymbol{\psi}(\mathbf{t}_h)) - \nabla p_h\Big)\cdot(\mathbf{v}-\mathbf{v}_h) \,+\, \sum_{T\in\mathcal{T}_h}\int_T(q-q_h)\mathrm{tr}\,(\mathbf{t}_h)$$

$$- \int_{\mathcal{E}_I} [\![\boldsymbol{\psi}(\mathbf{t}_h) - p_h \mathbf{I}]\!] \cdot \{\mathbf{v} - \mathbf{v}_h\} \; - \; \int_{\mathcal{E}_I} \alpha \underline{[\![\mathbf{u}_h]\!]} : \underline{[\![\mathbf{v} - \mathbf{v}_h]\!]}$$

$$+ \int_{\mathcal{E}_I} \Big(\{\boldsymbol{\sigma}_h\} - [\![\boldsymbol{\sigma}_h]\!] \otimes \boldsymbol{\beta} - \{\boldsymbol{\psi}(\mathbf{t}_h) - p_h \mathbf{I}\} \Big) : \underline{[\![\mathbf{v} - \mathbf{v}_h]\!]}$$

$$+ \int_{\mathcal{E}_D} (\boldsymbol{\sigma}_h - \boldsymbol{\psi}(\mathbf{t}_h) + p_h \mathbf{I}) : ((\mathbf{v} - \mathbf{v}_h) \otimes \boldsymbol{\nu}) \; + \; \int_{\mathcal{E}_D} \alpha (\mathbf{u}_h \otimes \boldsymbol{\nu}) : ((\mathbf{v} - \mathbf{v}_h) \otimes \boldsymbol{\nu}) \,.$$

Then, applying the Cauchy-Schwarz inequality, we obtain

$$\Big| [\mathcal{F}_h, (\mathbf{v} - \mathbf{v}_h, \lambda - \lambda_h, q - q_h)] - [\mathcal{A}_h(\mathbf{u}_h, \xi_h, p_h), (\mathbf{v} - \mathbf{v}_h, \lambda - \lambda_h, q - q_h)] \Big|$$

$$\leq C \left(\sum_{T \in \mathcal{T}_h} \eta_T^2 \right)^{1/2} H(\mathbf{v}, q)^{1/2} \,,$$

with

$$H(\mathbf{v}, q) := \sum_{T \in \mathcal{T}_h} h_T^{-2} \, ||\mathbf{v} - \mathbf{v}_h||^2_{[L^2(T)]^2} + \sum_{T \in \mathcal{T}_h} ||q - q_h||^2_{L^2(T)} + ||\alpha^{1/2} \{\mathbf{v} - \mathbf{v}_h\}||^2_{[L^2(\mathcal{E}_I)]^2}$$

$$+ \, ||\alpha^{1/2} \underline{[\![\mathbf{v} - \mathbf{v}_h]\!]}||^2_{[L^2(\mathcal{E}_I)]^{2\times 2}} \; + \; ||\alpha^{1/2} (\mathbf{v} - \mathbf{v}_h) \otimes \boldsymbol{\nu}||^2_{[L^2(\mathcal{E}_D)]^{2\times 2}} \,.$$

Finally, applying similar relations to the ones stating in Lemma 3.1 in [32], together with the approximation results (5.5.7) and (5.5.8), $H(\mathbf{v}, q)$ is bounded above by a constant that is independent of the meshsize. $\square$

In what follows we let $\mathbf{V}_h^c := \mathbf{V}_h \cap [H_0^1(\Omega)]^2$, and $\mathbf{V}_h^\perp$ as the orthogonal complement in $\mathbf{V}_h$ of $\mathbf{V}_h^c$ with respect to the norm $|||\cdot|||_h$. Then, the following equivalence result, whose proof can be found in [74], holds.

LEMMA 5.5.3 *$|\cdot|_h$ is a norm on $\mathbf{V}_h^\perp$ that is equivalent to $|||\cdot|||_h$, with equivalence constants being independent of the meshsize.*

Therefore, the next result gives us an explicit and reliable a-posteriori estimate.

THEOREM 5.5.1 *Let $(\boldsymbol{\sigma}, \mathbf{t}, \mathbf{u}, \xi, p) \in H(\mathrm{div}\,; \Omega) \times [L^2(\Omega_2)]^2 \times [H_0^1(\Omega)]^2 \times \mathbf{R} \times L^2(\Omega)$ be the unique solution of problem (5.1.4), and let $(\boldsymbol{\sigma}_h, \mathbf{t}_h, \mathbf{u}_h, \xi_h, p_h) \in \boldsymbol{\Sigma}_h \times \boldsymbol{\Sigma}_h \times \mathbf{V}_h \times \mathbf{R} \times W_h$ be its mixed LDG approximation obtained by (5.2.26), (5.2.23) and (5.2.22). Then there exists a constant $C_{\mathtt{rel}} > 0$, independent of the meshsize, such that*

$$||(\mathbf{t} - \mathbf{t}_h, \boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \mathbf{u} - \mathbf{u}_h, \xi - \xi_h, p - p_h)||_{\mathbf{X}} \, \leq \, C_{\mathtt{rel}} \, \boldsymbol{\vartheta} \, := \left(\sum_{T \in \mathcal{T}_h} \vartheta_T^2 \right)^{1/2} , \quad (5.5.12)$$

where the local error estimator ϑ_T is given by

$$\vartheta_T^2 \;:=\; \eta_T^2 \;+\; |T|\,|\bar{p}_h|^2 \;+\; ||\boldsymbol{\sigma}_h - \boldsymbol{\psi}(\mathbf{t}_h) + p_h\mathbf{I}||^2_{[L^2(T)]^{2\times 2}}\,, \tag{5.5.13}$$

with $\bar{p}_h$, the mean value of p_h.

Proof. Taking into account that $\xi_h = 0$, we obtain from (5.2.22) that

$$||\mathbf{t} - \mathbf{t}_h||_{[L^2(\Omega)]^{2\times 2}} \;\leq\; (1 + C_{\mathtt{S}})\,|||\mathbf{u} - \mathbf{u}_h|||_h\,.$$

Next, applying the triangle inequality, we have that

$$||\boldsymbol{\sigma} - \boldsymbol{\sigma}_h||_{[L^2(\Omega)]^{2\times 2}}$$

$$\leq\; ||\boldsymbol{\psi}(\mathbf{t}) - \boldsymbol{\psi}(\mathbf{t}_h)||_{[L^2(\Omega)]^{2\times 2}} \;+\; ||p_h\mathbf{I} - p\mathbf{I}||_{[L^2(\Omega)]^{2\times 2}} \;+\; ||\boldsymbol{\sigma}_h - \boldsymbol{\psi}(\mathbf{t}_h) + p_h\mathbf{I}||_{[L^2(\Omega)]^{2\times 2}}$$

$$\leq\; C_{\mathtt{LC}}||\mathbf{t} - \mathbf{t}_h||_{[L^2(\Omega)]^{2\times 2}} \;+\; \sqrt{2}\,||p - p_h||_{L^2(\Omega)} \;+\; ||\boldsymbol{\sigma}_h - \boldsymbol{\psi}(\mathbf{t}_h) + p_h\mathbf{I}||_{[L^2(\Omega)]^{2\times 2}}\,.$$

Finally, for the rest of terms of the global error, we proceed as follows. Since $L^2(\Omega) = L^2_0(\Omega) \oplus \mathbf{R}$, p_h can be decomposed as $p_h = p_h^0 + \bar{p}_h \in L^2_0(\Omega) \oplus \mathbf{R}$. Then, noting that $p \in L^2_0(\Omega)$ and applying Lemma 5.5.3, we have

$$||(\mathbf{u} - \mathbf{u}_h, \xi - \xi_h, p - p_h)||^2_{LDG} \;\leq$$

$$\Big(||(\mathbf{u} - \mathbf{u}_h^c, \xi - \xi_h, p - p_h^0)||_{LDG} \;+\; ||(\mathbf{u}_h^{\perp}, 0, \bar{p}_h)||_{LDG}\Big)^2$$

$$\leq\; 2\,\Big(||(\mathbf{u} - \mathbf{u}_h^c, \xi - \xi_h, p - p_h^0)||^2_{LDG} + |||\mathbf{u}_h^{\perp}|||_h^2 \;+\; |\Omega|\,|\bar{p}_h|^2\Big)$$

$$\leq\; C_1\,\Big(||(\mathbf{u} - \mathbf{u}_h^c, \xi - \xi_h, p - p_h^0)||^2_{LDG} + |\mathbf{u}_h^{\perp}|_h^2 \;+\; |\Omega|\,|\bar{p}_h|^2\Big)\,,$$

and due to the fact that $\underline{[\![\mathbf{u}_h^{\perp}]\!]} = \underline{[\![\mathbf{u}_h]\!]}$ on $\mathcal{E}_I$, and $\mathbf{u}_h^{\perp} = \mathbf{u}_h$ on $\mathcal{E}_D$, it follows that

$$||(\mathbf{u}-\mathbf{u}_h, \xi-\xi_h, p-p_h)||^2_{LDG} \;\leq\; C_1\,\Big(||(\mathbf{u}-\mathbf{u}_h^c, \xi-\xi_h, p-p_h^0)||^2_{LDG} + |\mathbf{u}_h|_h^2 + |\Omega|\,|\bar{p}_h|^2\Big)\,. \tag{5.5.14}$$

Now, by Lemma 5.5.1, there exists $(\mathbf{v}, \lambda, q) \in [H^1_0(\Omega)]^2 \times \mathbf{R} \times L^2_0(\Omega)$ such that

$$C\,||(\mathbf{u} - \mathbf{u}_h^c, \xi - \xi_h, p - p_h^0)||^2_{LDG} \;\leq\; D\mathcal{A}_h(\tilde{\mathbf{u}}, \tilde{\xi})((\mathbf{u} - \mathbf{u}_h^c, \xi - \xi_h, p - p_h^0), (\mathbf{v}, \lambda, q))\,,$$

$$\text{and} \quad ||(\mathbf{v}, \lambda, q)||_{LDG} \;\leq\; \widetilde{C}\,||(\mathbf{u} - \mathbf{u}_h^c, \xi - \xi_h, p - p_h^0)||_{LDG}\,, \tag{5.5.15}$$

with $\widetilde{C} > 0$ independent of the meshsize. Then, setting $(\mathbf{v}_h, \lambda_h, q_h) \in \mathbf{V}_h \times \mathbf{R} \times (W_h \cap L_0^2(\Omega))$ such that $\lambda_h = \lambda$, and the approximation properties (5.5.7) and (5.5.8) hold, we have

$$
\begin{aligned}
& C\,||(\mathbf{u} - \mathbf{u}_h^c, \xi - \xi_h, p - p_h^0)||_{LDG}^2 \leq \\
& D\mathcal{A}_h(\tilde{\mathbf{u}}, \tilde{\xi})((\mathbf{u} - \mathbf{u}_h, \xi - \xi_h, p - p_h), (\mathbf{v}, \lambda, q)) \;+\; D\mathcal{A}_h(\tilde{\mathbf{u}}, \tilde{\xi})((\mathbf{u}_h^{\perp}, 0, \bar{p}_h), (\mathbf{v}, \lambda, q)) \\
& = \; D\mathcal{A}_h(\tilde{\mathbf{u}}, \tilde{\xi})((\mathbf{u} - \mathbf{u}_h, \xi - \xi_h, p - p_h), (\mathbf{v} - \mathbf{v}_h, 0, q - q_h)) \\
& + D\mathcal{A}_h(\tilde{\mathbf{u}}, \tilde{\xi})((\mathbf{u} - \mathbf{u}_h, \xi - \xi_h, p - p_h), (\mathbf{v}_h, \lambda_h, q_h)) \;+\; D\mathcal{A}_h(\tilde{\mathbf{u}}, \tilde{\xi})((\mathbf{u}_h^{\perp}, 0, \bar{p}_h), (\mathbf{v}, \lambda, q)) \\
& = \; [\mathcal{F}_h, (\mathbf{v} - \mathbf{v}_h, 0, q - q_h)] \;-\; [\mathcal{A}_h(\mathbf{u}_h, \xi_h, p_h), (\mathbf{v} - \mathbf{v}_h, 0, q - q_h)] \\
& + D\mathcal{A}_h(\tilde{\mathbf{u}}, \tilde{\xi})((\mathbf{u}_h^{\perp}, 0, \bar{p}_h), (\mathbf{v}, \lambda, q))\,,
\end{aligned}
$$

and using Lemmas 5.5.2 and 5.5.3, the uniform boundedness of $D\mathcal{A}_h(\tilde{\mathbf{u}}, \tilde{\xi})$, and the second relation in (5.5.15), we conclude the proof. □

5.6 Numerical results

In this section we provide several numerical examples illustrating the performance of the mixed LDG method and the fully explicit a-posteriori error estimate ϑ (cf. (5.5.13)) for the linear and nonlinear case, respectively.

Hereafter, N is the number of degrees of freedom defining the subspaces $\mathbf{\Sigma}_h$, $\mathbf{V}_h$ and $\mathbf{R}$, which implies that $N := 15 \times$ (number of triangles of $\mathcal{T}_h$) $+\ 1$ for the $\mathbf{P}_0 - \mathbf{P}_0 - \mathbf{P}_1 - \mathbf{P}_0$ approximation, while $N := 39 \times$ (number of triangles of $\mathcal{T}_h$) $+\ 1$ for the $\mathbf{P}_1 - \mathbf{P}_1 - \mathbf{P}_2 - \mathbf{P}_1$ approximation. Further, the individual and global errors are defined as follows

$$
\mathbf{e}(\mathbf{t}) := ||\mathbf{t} - \mathbf{t}_h||_{[L^2(\Omega)]^{2\times 2}}\,, \quad \mathbf{e}_h(\mathbf{u}) := |||\mathbf{u} - \mathbf{u}_h|||_h\,,
$$

$$
\mathbf{e}(p) := ||p - p_h||_{L^2(\Omega)}\,, \quad \mathbf{e}(\boldsymbol{\sigma}) := ||\boldsymbol{\sigma} - \boldsymbol{\sigma}_h||_{[L^2(\Omega)]^{2\times 2}}\,,
$$

$$
\text{and} \quad \mathbf{e} := \left\{ [\mathbf{e}_h(\mathbf{u})]^2 \,+\, [\mathbf{e}(\mathbf{t})]^2 \,+\, [\mathbf{e}(\boldsymbol{\sigma})]^2 \,+\, [\mathbf{e}(p)]^2 \right\}^{1/2}\,,
$$

where $(\mathbf{u}, \mathbf{t}, \boldsymbol{\sigma}, \xi, p)$ and $(\mathbf{u}_h, \mathbf{t}_h, \boldsymbol{\sigma}_h, \xi_h, p_h)$ are the unique solutions of the continuous and discrete mixed formulations, respectively. In particular, if $\mathbf{e}_h(\mathbf{u})$ and $\mathbf{e}_{\tilde{h}}(\mathbf{u})$ stand for the error at two consecutive triangulations with N and $\tilde{N}$ degrees of freedom, respectively, we set

$$r_h(\mathbf{u}) := -2\,\frac{\log(\mathbf{e}_h(\mathbf{u})/\mathbf{e}_{\tilde{h}}(\mathbf{u}))}{\log(N/\tilde{N})}\,,$$

as the experimental rate of convergence, and similarly for $r(\mathbf{t})$, $r(\boldsymbol{\sigma})$, $r(p)$, and r, which is related to $\mathbf{e}$.

On the other hand, the adaptive algorithm, used in the mesh refinement process, without *hanging nodes*, is the following (see [96]):

1. Start with a coarse mesh $\mathcal{T}_h$.

2. Solve the discrete problem (5.2.15) for the actual mesh $\mathcal{T}_h$.

3. Compute ϑ_T for each triangle $T \in \mathcal{T}_h$.

4. Evaluate stopping criterion and decide to finish or go to next step.

5. Use *red-blue-green* procedure to refine each $T' \in \mathcal{T}_h$ whose indicator $\vartheta_{T'}$ satisfies
$$\vartheta_{T'} \,\geq\, \frac{1}{2}\,\max\{\vartheta_T : \quad T \in \mathcal{T}_h\,\}\,.$$

6. Define resulting mesh as actual mesh $\mathcal{T}_h$ and go to step 2.

The numerical results presented below were obtained in a *Compaq Alpha ES40 Parallel Computer* using a MATLAB code. We remark that in the nonlinear case, the corresponding mixed LDG scheme (cf. (5.2.15)), which becomes a nonlinear algebraic system with N unknowns, is solved by Newton-Raphson's method with the initial guess given by the solution of the associated Stokes problem, and setting the tolerance in 10^{-3} for the relative error. In all cases we take the parameters $\widehat{\alpha} = 1$ and $\boldsymbol{\beta} = (1,1)^{\mathbf{T}}$ in the corresponding dual formulation. In addition, we test our results considering regular meshes and meshes with *hanging nodes*, fact that is not covered by the present analysis, giving us numerical evidences of its well behaviour. In this case, our refinement strategy is similar to the one described before, but instead of using the *red-blue-green* process in step 5, we apply only the *red* procedure.

We present four examples. The first two consider the linear version of the boundary value problem (5.1.1) and the other two deal with the nonlinear case, taking as the kinematic viscosity function, ψ, the one given by the Carreau law

$$\psi(t) \;=\; \kappa_0 \;+\; \kappa_1\,(1+t^2)^{(\beta-2)/2}\,,$$

which satifies (5.1.2) and (5.1.3) for all $\kappa_0 > 0$, and for all $\beta \in [1, 2]$. Note that when $\beta = 2$, we recover the usual linear Stokes model. In our case, we take $\kappa_0 = \kappa_1 = 1/2$ and $\beta = 3/2$, that is $\psi(t) := \frac{1}{2} + \frac{1}{2}(1 + t^2)^{-1/4}$. We remark here that since $\boldsymbol{\sigma}$ satisfies $\int_\Omega \mathrm{tr}\,(\boldsymbol{\sigma}) = 0$, the pressure p must be selected such that it belongs to $L_0^2(\Omega)$.

Example 1 is defined on the square domain $\Omega := (-1, 1)^2$, and we choose the data $\mathbf{f}$ and $\mathbf{g}$ so that the exact solution of (5.1.1), is

$$\begin{cases} \mathbf{u}_1(\mathbf{x}) := \Big(-(2.1 - x_1 - x_2)^{-1/3}, (2.1 - x_1 - x_2)^{-1/3} \Big)\,, \\[2ex] p_1(\mathbf{x}) := x_1 + x_2\,. \end{cases}$$

The L-shaped domain $\Omega := (-1, 1)^2 \backslash [0, 1]^2$ is consider in Example 2, and the data $\mathbf{f}$ and $\mathbf{g}$ are chosen so that the exact solution of (5.1.1), is

$$\begin{cases} \mathbf{u}_2(\mathbf{x}) := \Big(-\sqrt{1000}\,\mathrm{e}^{-\sqrt{1000}(x_1+x_2)}, \sqrt{1000}\,\mathrm{e}^{-\sqrt{1000}(x_1+x_2)} \Big)\,, \\[2ex] p_2(\mathbf{x}) := 2\,\mathrm{e}^{x_1}\,\sin(x_2) - \dfrac{2}{3}(\mathrm{e} - 1)(\cos(1) - 1)\,, \end{cases}$$

for all $\mathbf{x} := (x_1, x_2) \in \Omega$.

We notice that both $\mathbf{u}_1$ and $\mathbf{u}_2$ are divergence free in their respective domain Ω, and while $\mathbf{u}_1$ is singular in an exterior neighborhood of the point $(1, 1)$, $\mathbf{u}_2$ presents an inner layer around the origin.

Now, for the full nonlinear boundary value problem (5.1.1) we consider two L-shaped domains: $\Omega := (-1, 1)^2 - [0, 1] \times [-1, 0]$ for Example 3, and $\Omega := (-1, 1)^2 - [0, 1]^2$ for Example 4. For both of these two examples, we choose the data $\mathbf{f}$ and $\mathbf{g}$ so that the corresponding exact solutions of (5.1.1) are,

$$\begin{cases} \mathbf{u}_3(\mathbf{x}) := \Big[(x_1 - 0.1)^2 + (x_2 + 0.1)^2\Big]^{-1/2}\,(x_2 + 0.1, 0.1 - x_1)\,, \\[2ex] p_3(\mathbf{x}) := (2 - x_1 + x_2)^{1/2} - \dfrac{4}{45}\Big(34 - 12\sqrt{3}\Big)\,, \end{cases}$$

and

$$\begin{cases} \mathbf{u}_4(\mathbf{x}) := \Big[(x_1 - 0.01)^2 + (x_2 - 0.01)^2\Big]^{-1/2}\,(x_2 - 0.01, 0.01 - x_1)\,, \\[2ex] p_4(\mathbf{x}) := \dfrac{1}{1.1 - x_1} - \dfrac{1}{3}\ln\left(\dfrac{441}{11}\right)\,, \end{cases}$$

for all $\mathbf{x} := (x_1, x_2)$ belonging to its respective domain Ω. We note that $\mathbf{u}_3$ and $\mathbf{u}_4$ are divergence free in their corresponding domain, and singular in an exterior neighborhood of $(0,0)$. In addition, p_4 is singular in an exterior neighborhood of the segment $\{1\} \times [0,1]$.

In Tables 5.6.1, 5.6.2, 5.6.3, and 5.6.4 we present the individual errors (except $\mathbf{e}(\xi)$, which converges very rapidly to zero), the error estimate $\boldsymbol{\vartheta}$, the effectivity index $\mathbf{e}/\boldsymbol{\vartheta}$, and the corresponding experimental rates of convergence for the uniform, red-blue-green, and red refinements associated to Examples 1 and 2 with $\mathbf{P}_0 - \mathbf{P}_0 - \mathbf{P}_1 - \mathbf{P}_0$ and $\mathbf{P}_1 - \mathbf{P}_1 - \mathbf{P}_2 - \mathbf{P}_1$ approximations. The errors on each triangle were computed applying a 7 points Gaussian quadrature rule. We notice that the effectivity indexes are bounded above and below, which confirm the reliability of $\boldsymbol{\vartheta}$, and provide numerical evidences for their efficiency, even considering irregular meshes, which analysis is not covered here. In addition, Figures 5.6.1, 5.6.2, 5.6.7, and 5.6.8 show the global errors $\mathbf{e}$, $\mathbf{e}^{\mathrm{rbg}}$, and $\mathbf{e}^{\mathrm{r}}$, corresponding to the uniform, red-blue-green, and red refinements, respectively, versus the degrees of freedom N. In all cases the errors of the adaptive methods decrease much faster than those of the uniform ones, which is also emphasized by the experimental rates of convergence provided in the tables, showing that the adaptive algorithms recover $O(h)$ and $O(h^2)$ for $\mathbf{P}_0 - \mathbf{P}_0 - \mathbf{P}_1 - \mathbf{P}_0$ and $\mathbf{P}_1 - \mathbf{P}_1 - \mathbf{P}_2 - \mathbf{P}_1$, respectively. Next, Figures 5.6.3-5.6.6, and 5.6.9-5.6.12 display some intermediate meshes obtained with the different refinements. As expected, the adaptive algorithms are able to recognize the singular point and the inner layer of Examples 1 and 2, respectively. In addition, we notice that the red refinement (with hanging nodes) behaves so well that it induces, numerically speaking, evidences that our results are still valid on this kind of meshes, being more localized around the singularities than the red-blue-green algorithm.

Likewise to the linear examples, we show in Tables 5.6.5 and 5.6.6 the individual errors, the error estimate and the effectivity index $\mathbf{e}/\boldsymbol{\vartheta}$, and the corresponding experimental rates of convergence for the uniform, red-blue-green, and red refinements as applied to the nonlinear problems given by Examples 3 and 4 with $\mathbf{P}_0 - \mathbf{P}_0 - \mathbf{P}_1 - \mathbf{P}_0$ and $\mathbf{P}_1 - \mathbf{P}_1 - \mathbf{P}_2 - \mathbf{P}_1$ approximations. Moreover, Figures 5.6.13-5.6.14 and 5.6.19-5.6.20 show the global errors $\mathbf{e}$, $\mathbf{e}^{\mathrm{rbg}}$, and $\mathbf{e}^{\mathrm{r}}$ versus the degrees of freedom N, while Figures 5.6.15-5.6.18 and 5.6.21-5.6.24 display some intermediate meshes obtained with the different refinements. The remarks and conclusions here are the same of the linear examples. Again, considering irregular meshes, the corresponding results give us numerical evidences that our analysis is still true for this kind of meshes.

Table 5.6.1: $\mathbf{P}_0 - \mathbf{P}_0 - \mathbf{P}_1 - \mathbf{P}_0$ approximation (Example 1, uniform, red-blue-green, and red refinements).

N	$\mathbf{e}_h(\mathbf{u})$	$\mathbf{e}(\mathbf{t})$	$\mathbf{e}(\boldsymbol{\sigma})$	$\mathbf{e}(p)$	$\boldsymbol{\vartheta}$	$\mathbf{e}/\boldsymbol{\vartheta}$	r
31	3.2988	1.0825	2.6396	1.7023	12.3807	0.3781	——
121	3.0595	0.9845	1.7150	0.9929	8.7290	0.4326	0.3159
481	2.2207	0.7457	1.0757	0.5482	6.0193	0.4378	0.5211
1921	1.4128	0.5100	0.6556	0.2913	3.7565	0.4431	0.6636
7681	0.8147	0.3117	0.3796	0.1531	2.1251	0.4534	0.7890
30721	0.4526	0.1723	0.2050	0.0785	1.1498	0.4624	0.8578
122881	0.2447	0.0899	0.1057	0.0393	0.6066	0.4682	0.9045
31	3.2988	1.0825	2.6396	1.7023	12.3807	0.3781	——
121	3.0595	0.9845	1.7150	0.9929	8.7290	0.4326	0.3159
301	2.3534	0.8207	1.2648	0.6805	6.6965	0.4296	0.5969
601	1.8422	0.6595	0.9351	0.4688	4.7539	0.4667	0.7511
781	1.5107	0.5731	0.8513	0.4452	3.9066	0.4812	1.2654
1711	0.8959	0.3716	0.5399	0.2770	2.5304	0.4521	1.2663
3661	0.6169	0.2591	0.3606	0.1773	1.7282	0.4516	1.0055
8791	0.4278	0.1676	0.2288	0.1102	1.1142	0.4712	0.9053
21391	0.2681	0.1136	0.1584	0.0781	0.7450	0.4572	0.9733
47536	0.1877	0.0753	0.1018	0.0485	0.4969	0.4661	0.9662
81151	0.1426	0.0597	0.0822	0.0399	0.3885	0.4622	0.9509
31	3.2988	1.0825	2.6396	1.7023	12.3807	0.3781	——
121	3.0600	0.9845	1.7150	0.9929	8.7292	0.4326	0.3157
211	2.3905	0.9385	1.5699	0.8899	7.3338	0.4280	0.6650
571	1.6455	0.6553	0.9633	0.4993	4.6509	0.4466	0.8295
661	1.2942	0.5776	0.8887	0.4776	3.8400	0.4530	2.4218
1246	0.9110	0.4758	0.7535	0.4131	3.0259	0.4427	0.8244
2551	0.6593	0.3072	0.4427	0.2254	1.9402	0.4540	1.1703
8851	0.3789	0.1718	0.2387	0.1172	1.0858	0.4548	0.9305
11191	0.3254	0.1519	0.2138	0.1065	0.9492	0.4544	1.1531
36076	0.1942	0.0852	0.1147	0.0543	0.5402	0.4576	0.9512
46696	0.1625	0.0753	0.1050	0.0517	0.4713	0.4540	1.1184

Table 5.6.2: $\mathbf{P}_1 - \mathbf{P}_1 - \mathbf{P}_2 - \mathbf{P}_1$ approximation (Example 1, uniform, red-blue-green, and red refinements).

N	$\mathbf{e}_h(\mathbf{u})$	$\mathbf{e}(\mathbf{t})$	$\mathbf{e}(\boldsymbol{\sigma})$	$\mathbf{e}(p)$	$\boldsymbol{\vartheta}$	$\mathbf{e}/\boldsymbol{\vartheta}$	r
79	2.0925	0.3168	0.4734	0.2488	5.8371	0.3739	——
313	2.3414	0.2944	0.4024	0.1940	5.3663	0.4476	——
1249	1.7969	0.2345	0.3076	0.1408	3.9426	0.4676	0.3824
4993	1.0537	0.1481	0.1841	0.0773	2.1795	0.4967	0.7682
19969	0.4646	0.0706	0.0827	0.0306	0.8904	0.5369	1.1792
79873	0.1737	0.0305	0.0411	0.0217	0.3898	0.4678	1.3907
79	2.0925	0.3168	0.4734	0.2488	5.8371	0.3739	——
313	2.3414	0.2944	0.4024	0.1940	5.3663	0.4476	——
781	1.7977	0.2346	0.3078	0.1409	3.9462	0.4673	0.5777
1249	1.0570	0.1489	0.1852	0.0779	2.1915	0.4956	2.2552
1717	0.4735	0.0733	0.0870	0.0330	0.9336	0.5228	5.0265
2185	0.2009	0.0347	0.0406	0.0149	0.4115	0.5064	7.0626
2965	0.1432	0.0254	0.0305	0.0120	0.2865	0.5204	2.1941
6787	0.0657	0.0121	0.0136	0.0045	0.1346	0.5071	1.8862
8737	0.0446	0.0077	0.0088	0.0030	0.0898	0.5150	3.0877
13495	0.0313	0.0057	0.0063	0.0020	0.0604	0.5384	1.6186
23791	0.0170	0.0029	0.0032	0.0010	0.0315	0.5562	2.1773
35023	0.0114	0.0019	0.0021	0.0006	0.0211	0.5568	2.0605
70279	0.0060	0.0010	0.0011	0.0003	0.0106	0.5787	1.8626
79	2.0925	0.3168	0.4734	0.2488	5.8371	0.3739	——
313	2.3414	0.2944	0.4024	0.1940	5.3663	0.4476	——
547	1.7959	0.2360	0.3089	0.1410	3.9554	0.4659	0.9489
781	1.0598	0.1534	0.1893	0.0784	2.2271	0.4895	2.9482
1015	0.4921	0.0859	0.0993	0.0352	1.0493	0.4865	5.7903
1249	0.2559	0.0596	0.0659	0.0200	0.6609	0.4110	6.0825
3589	0.1011	0.0239	0.0263	0.0078	0.2442	0.4400	1.7576
6397	0.0533	0.0133	0.0146	0.0043	0.1350	0.4220	2.1945
10843	0.0330	0.0081	0.0089	0.0026	0.0832	0.4237	1.8217
18097	0.0203	0.0048	0.0053	0.0015	0.0475	0.4546	1.9154
30967	0.0119	0.0028	0.0030	0.0008	0.0280	0.4497	2.0072
52963	0.0076	0.0017	0.0018	0.0005	0.0166	0.4812	1.7010

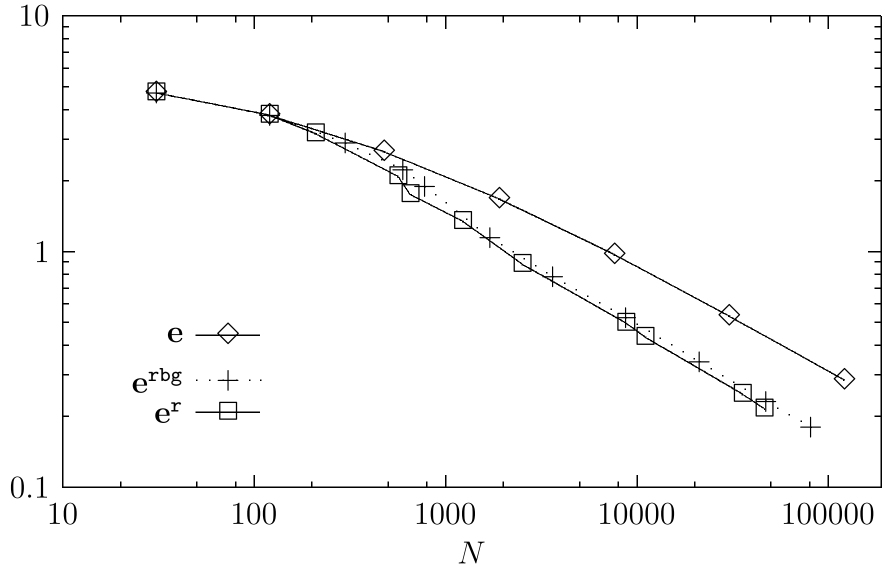

Figure 5.6.1: Example 1 with $\mathbf{P}_0 - \mathbf{P}_0 - \mathbf{P}_1 - \mathbf{P}_0$ approximation: global error $\mathbf{e}$ for the uniform and adaptive refinements.

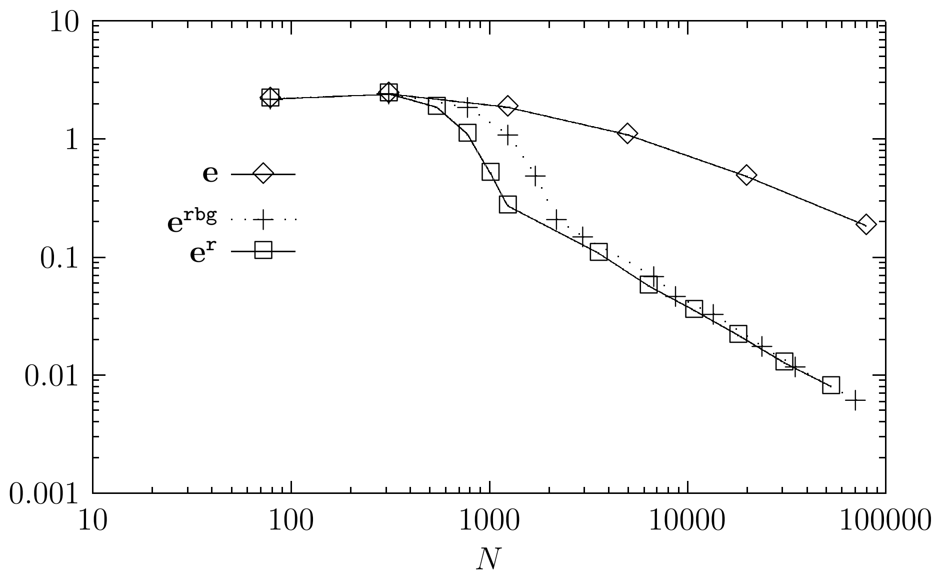

Figure 5.6.2: Example 1 with $\mathbf{P}_1 - \mathbf{P}_1 - \mathbf{P}_2 - \mathbf{P}_1$ approximation: global error $\mathbf{e}$ for the uniform and adaptive refinements.

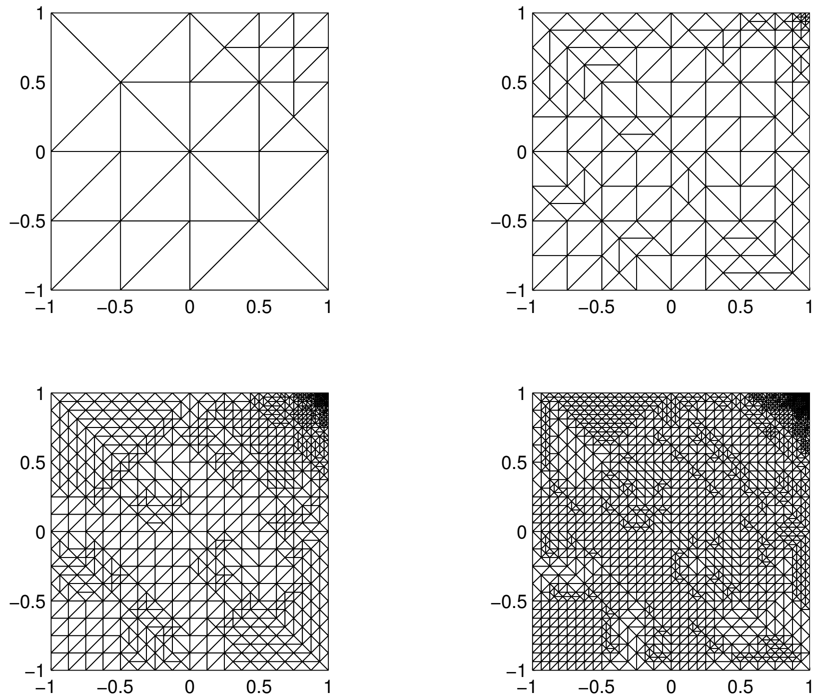

Figure 5.6.3: Example 1 with $\mathbf{P}_0 - \mathbf{P}_0 - \mathbf{P}_1 - \mathbf{P}_0$ approximation, without hanging nodes: adapted intermediate meshes with 601, 3661, 21391 and 47536 degrees of freedom.

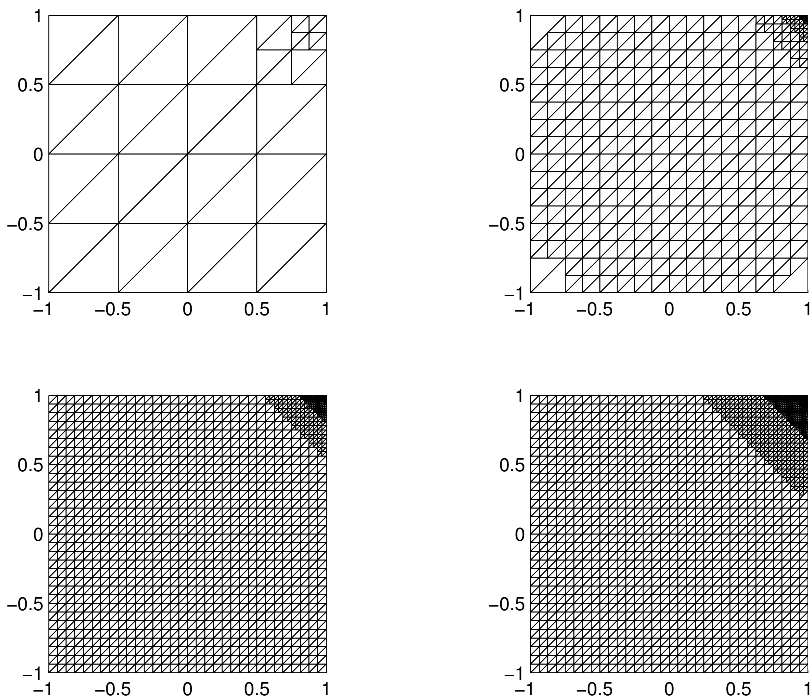

Figure 5.6.4: Example 1 with $\mathbf{P}_0 - \mathbf{P}_0 - \mathbf{P}_1 - \mathbf{P}_0$ approximation, with hanging nodes: adapted intermediate meshes with 661, 8851, 36076 and 46696 degrees of freedom.

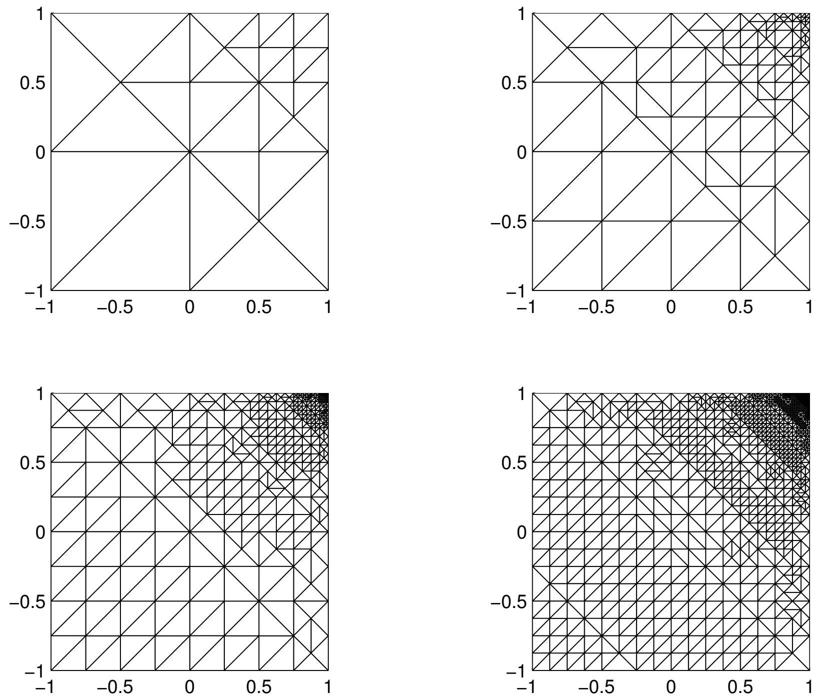

Figure 5.6.5: Example 1 with $\mathbf{P}_1 - \mathbf{P}_1 - \mathbf{P}_2 - \mathbf{P}_1$ approximation, without hanging nodes: adapted intermediate meshes with 1249, 6787, 23791 and 70279 degrees of freedom.

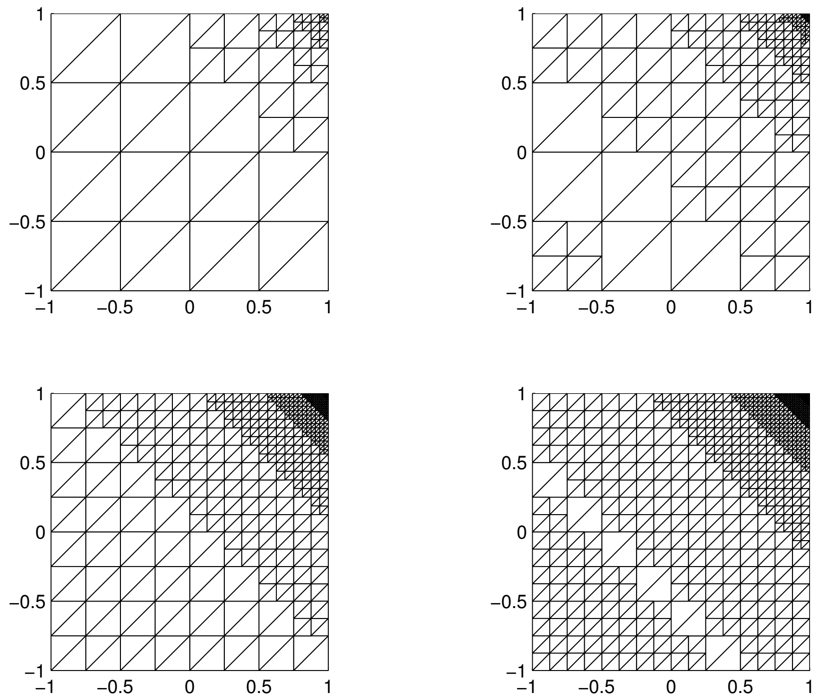

Figure 5.6.6: Example 1 with $\mathbf{P}_1 - \mathbf{P}_1 - \mathbf{P}_2 - \mathbf{P}_1$ approximation, with hanging nodes: adapted intermediate meshes with 3589, 10843, 30967 and 52963 degrees of freedom.

Table 5.6.3: $\mathbf{P}_0 - \mathbf{P}_0 - \mathbf{P}_1 - \mathbf{P}_0$ approximation (Example 2, uniform, red-blue-green, and red refinements).

N	$\mathbf{e}_h(\mathbf{u})$	$\mathbf{e}(\mathbf{t})$	$\mathbf{e}(\boldsymbol{\sigma})$	$\mathbf{e}(p)$	$\boldsymbol{\vartheta}$	$\mathbf{e}/\boldsymbol{\vartheta}$	r
91	27.7530	4.5119	8.6094	5.1848	104.5580	0.2856	——
361	81.5164	18.2308	30.1984	17.0232	300.2485	0.3012	——
1441	90.3410	25.9842	33.2659	14.6875	265.7586	0.3793	——
5761	61.3113	20.9879	24.0423	8.2928	151.1932	0.4604	0.5341
23041	34.9934	12.5703	13.9487	4.2752	82.1369	0.4863	0.8016
92161	19.2402	6.7162	7.4175	2.2261	44.6317	0.4885	0.8736
91	27.7530	4.5119	8.6094	5.1848	104.5580	0.2856	——
271	81.5063	18.2332	30.0414	16.8825	300.2306	0.3009	——
451	90.6101	25.9775	33.2114	14.6316	265.6911	0.3802	——
631	61.4417	20.9911	24.0254	8.2642	151.2558	0.4610	2.2073
811	35.5645	12.7541	14.1600	4.3496	83.4416	0.4864	4.3129
991	23.5086	8.0024	9.0209	2.9443	53.1132	0.5005	4.2206
2281	12.4999	4.3924	4.9620	1.6322	29.1683	0.4883	1.4974
5011	7.7918	2.5289	2.8891	0.9878	17.5121	0.4992	1.2401
13636	4.7580	1.5850	1.8217	0.6350	10.9793	0.4894	0.9724
22321	3.6281	1.2182	1.3953	0.4810	8.4037	0.4881	1.0957
52306	2.3582	0.8008	0.9246	0.3269	5.5424	0.4829	1.0026
85216	1.8248	0.6250	0.7164	0.2476	4.3067	0.4812	1.0482
91	27.7530	4.5119	8.6094	5.1848	104.5580	0.2856	——
181	81.2676	18.2260	30.0517	16.8955	300.3732	0.3001	——
271	90.2975	25.9945	33.1657	14.5645	265.7890	0.3789	——
361	61.2733	20.9934	23.9869	8.2052	151.3981	0.4594	2.5811
451	36.1299	13.1283	14.5617	4.4548	86.7567	0.4766	4.6732
541	26.2382	9.5676	10.6339	3.2819	64.8448	0.4636	3.5030
1441	14.3907	5.1411	5.6798	1.7073	34.7893	0.4712	1.2382
2971	9.7602	3.3815	3.7685	1.1762	23.3350	0.4739	1.0880
6841	6.3317	2.1388	2.3950	0.7621	14.8890	0.4796	1.0490
14941	4.2436	1.4197	1.5912	0.5081	9.9563	0.4797	1.0294
30826	2.9866	0.9939	1.1213	0.3672	7.0034	0.4800	0.9701
64801	2.0519	0.6803	0.7640	0.2458	4.8001	0.4804	1.0146

Table 5.6.4: $\mathbf{P}_1 - \mathbf{P}_1 - \mathbf{P}_2 - \mathbf{P}_1$ approximation (Example 2, uniform, red-blue-green, and red refinements).

N	$\mathbf{e}_h(\mathbf{u})$	$\mathbf{e}(\mathbf{t})$	$\mathbf{e}(\boldsymbol{\sigma})$	$\mathbf{e}(p)$	$\boldsymbol{\vartheta}$	$\mathbf{e}/\boldsymbol{\vartheta}$	r
235	25.8203	8.3819	13.2488	7.2551	106.6109	0.2914	——
937	76.7277	14.0911	23.8389	13.5966	301.5960	0.2742	——
3745	79.0125	14.8175	20.6097	10.1293	237.0900	0.3526	——
14977	36.4816	7.7792	9.2608	3.5529	94.4255	0.4088	1.1152
59905	10.7937	2.5907	2.8672	0.8686	27.1349	0.4237	1.7473
235	25.8203	8.3819	13.2488	7.2551	106.6109	0.2914	——
703	76.6581	14.0920	23.8200	13.5795	301.6019	0.2740	——
1171	79.1601	14.8104	20.5901	10.1145	237.0435	0.3533	——
1639	36.5359	7.7802	9.2614	3.5526	94.4537	0.4092	4.5989
2107	11.0779	2.6417	2.9362	0.9063	27.9778	0.4216	9.4498
2575	5.2572	1.1055	1.2487	0.4106	12.0434	0.4592	7.5523
5305	2.0662	0.3651	0.4107	0.1330	3.9782	0.5385	2.6245
8035	1.4901	0.2517	0.2829	0.0913	2.6754	0.5757	1.5892
16732	0.6428	0.1099	0.1241	0.0407	1.1777	0.5648	2.2897
21529	0.4937	0.0798	0.0892	0.0281	0.8543	0.5956	2.1250
41692	0.2781	0.0435	0.0498	0.0173	0.4703	0.6090	1.7394
62323	0.1755	0.0273	0.0306	0.0098	0.2922	0.6178	2.2958
235	25.8203	8.3819	13.2488	7.2551	106.6109	0.2914	——
469	76.6090	14.1048	23.8151	13.5686	301.6354	0.2738	——
703	79.0296	14.8224	20.5882	10.1037	237.0999	0.3527	——
937	36.5155	7.7958	9.2642	3.5390	94.5687	0.4085	5.3744
1171	12.8417	3.0479	3.3829	1.0378	32.4836	0.4207	9.3240
2107	6.2777	1.5374	1.6880	0.4929	15.8892	0.4216	2.4275
2809	3.7224	0.9200	1.0275	0.3235	9.7074	0.4103	3.6154
4681	2.3065	0.5519	0.6094	0.1827	5.6086	0.4378	1.8943
7840	1.2740	0.2883	0.3206	0.0991	2.8517	0.4729	2.3237
12988	0.7608	0.1751	0.1922	0.0560	1.7232	0.4677	2.0399
20125	0.5138	0.1131	0.1240	0.0359	1.1099	0.4881	1.8143
32878	0.3149	0.0678	0.0732	0.0195	0.6576	0.5032	2.0084
51247	0.2050	0.0434	0.0468	0.0123	0.4205	0.5116	1.9407

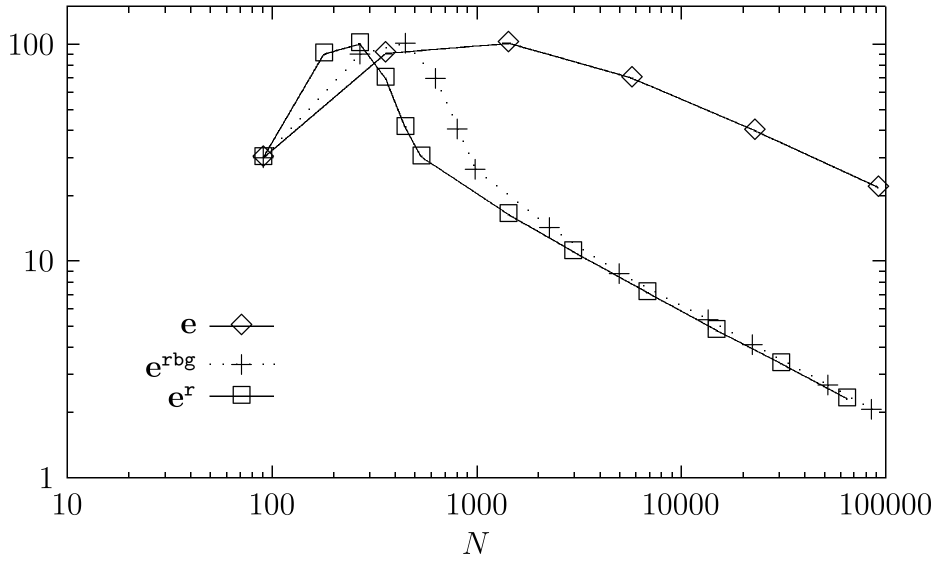

Figure 5.6.7: Example 2 with $\mathbf{P}_0 - \mathbf{P}_0 - \mathbf{P}_1 - \mathbf{P}_0$ approximation: global error $\mathbf{e}$ for the uniform and adaptive refinements.

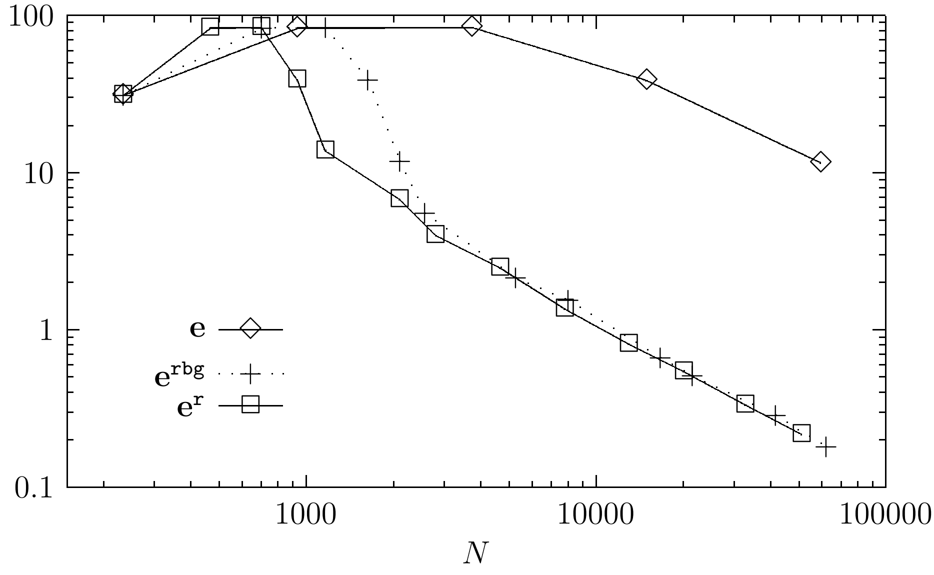

Figure 5.6.8: Example 2 with $\mathbf{P}_1 - \mathbf{P}_1 - \mathbf{P}_2 - \mathbf{P}_1$ approximation: global error $\mathbf{e}$ for the uniform and adaptive refinements.

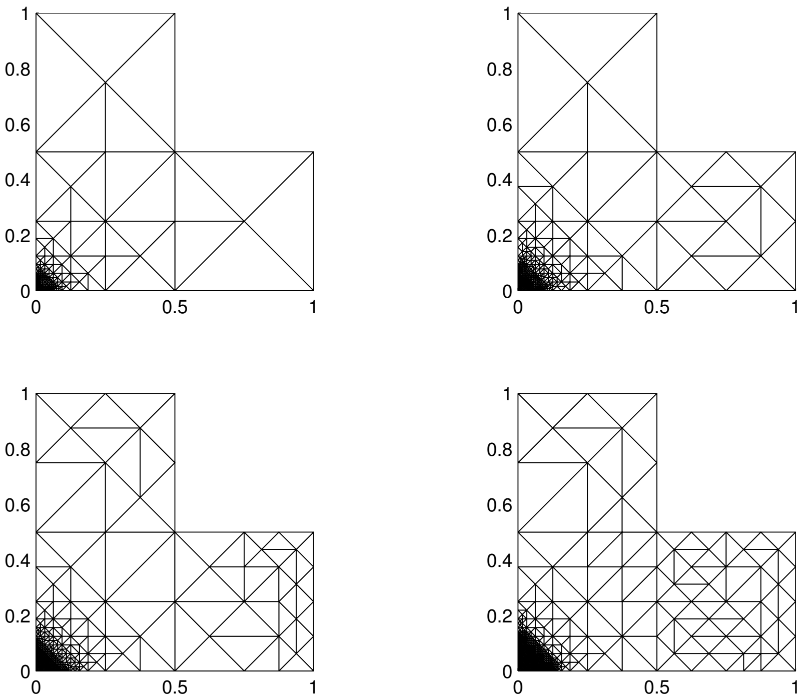

Figure 5.6.9: Example 2 with $\mathbf{P}_0 - \mathbf{P}_0 - \mathbf{P}_1 - \mathbf{P}_0$ approximation, without hanging nodes: adapted intermediate meshes with 5011, 13636, 22321 and 52306 degrees of freedom.

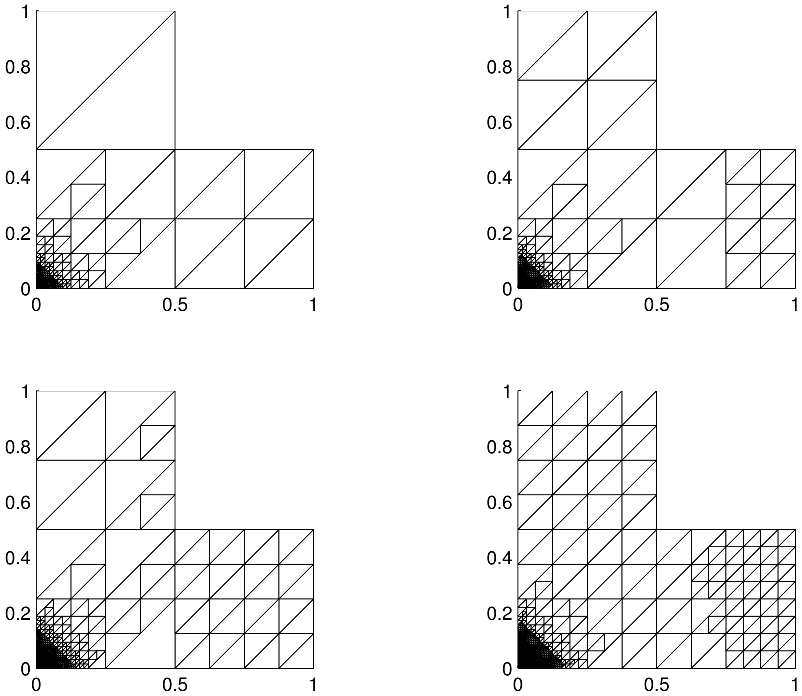

Figure 5.6.10: Example 2 with $\mathbf{P}_0 - \mathbf{P}_0 - \mathbf{P}_1 - \mathbf{P}_0$ approximation, with hanging nodes: adapted intermediate meshes with 6841, 14941, 30826 and 64801 degrees of freedom.

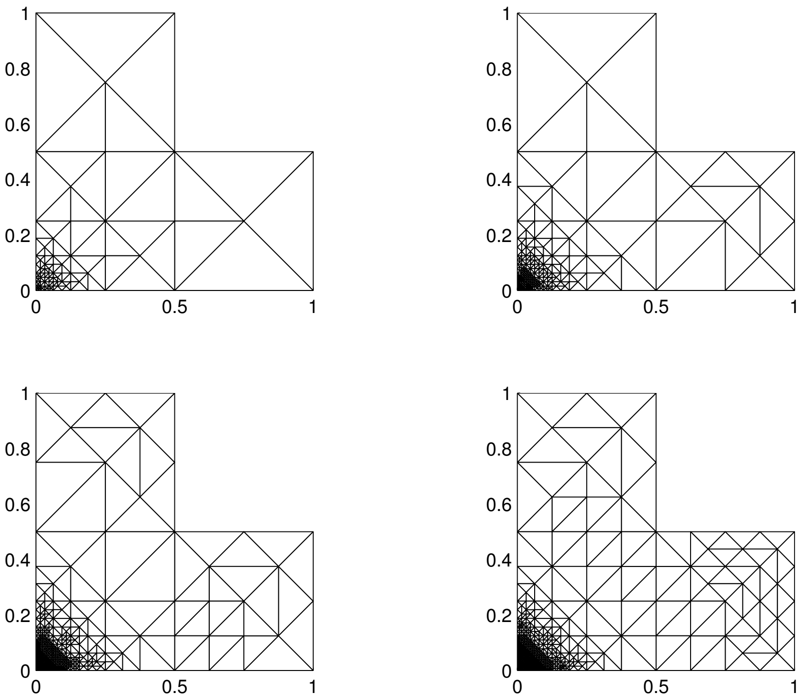

Figure 5.6.11: Example 2 with $\mathbf{P}_1 - \mathbf{P}_1 - \mathbf{P}_2 - \mathbf{P}_1$ approximation, without hanging nodes: adapted intermediate meshes with 5305, 16732, 41692 and 62323 degrees of freedom.

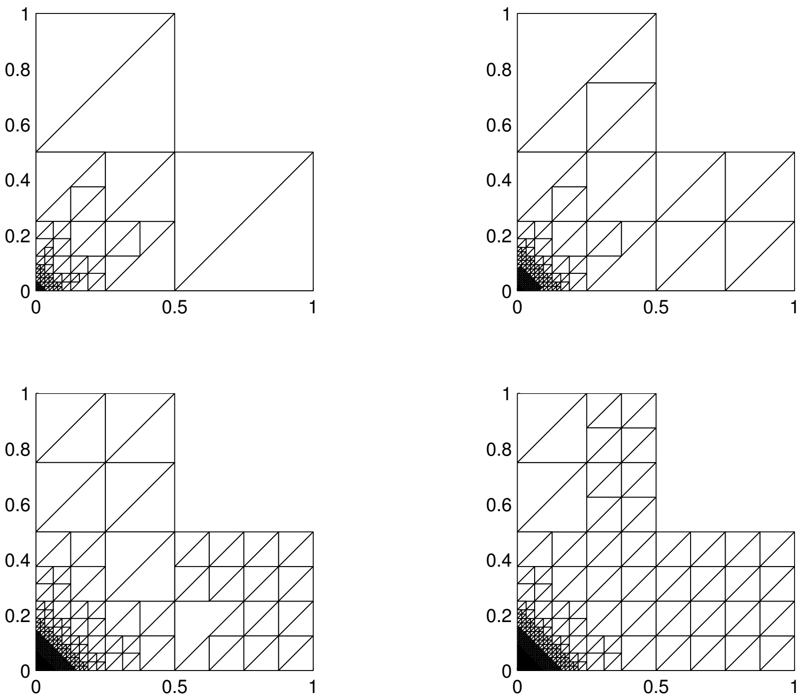

Figure 5.6.12: Example 2 with $\mathbf{P}_1 - \mathbf{P}_1 - \mathbf{P}_2 - \mathbf{P}_1$ approximation, with hanging nodes: adapted intermediate meshes with 4681, 12988, 32878 and 51247 degrees of freedom.

Table 5.6.5: $\mathbf{P}_0 - \mathbf{P}_0 - \mathbf{P}_1 - \mathbf{P}_0$ approximation (Example 3, uniform, red-blue-green, and red refinements).

N	$\mathbf{e}_h(\mathbf{u})$	$\mathbf{e}(\mathbf{t})$	$\mathbf{e}(\boldsymbol{\sigma})$	$\mathbf{e}(p)$	$\boldsymbol{\vartheta}$	$\mathbf{e}/\boldsymbol{\vartheta}$	r
61	4.5347	1.9138	1.6448	0.5247	14.4397	0.3612	——
241	3.5361	1.6316	1.4170	0.5113	9.3919	0.4446	0.3239
961	2.2715	1.1714	0.9768	0.3391	5.3454	0.5158	0.6002
3841	1.2286	0.6968	0.5742	0.2032	2.8604	0.5378	0.8423
15361	0.6225	0.3863	0.3128	0.1097	1.5025	0.5352	0.9359
61441	0.3153	0.2008	0.1618	0.0568	0.7722	0.5326	0.9673
61	4.5347	1.9138	1.6448	0.5247	14.4397	0.3612	——
241	3.5361	1.6316	1.4170	0.5113	9.3919	0.4446	0.3239
541	2.3534	1.2073	1.0500	0.3949	5.5821	0.5147	0.9247
841	1.6625	0.8501	0.8255	0.3635	3.7751	0.5493	1.4781
2566	0.9421	0.5139	0.4801	0.1979	2.1595	0.5521	0.9924
6796	0.5978	0.3204	0.2887	0.1104	1.3454	0.5540	0.9645
12256	0.4458	0.2408	0.2170	0.0820	1.0153	0.5488	0.9865
26416	0.3078	0.1669	0.1482	0.0540	0.6976	0.5505	0.9695
48766	0.2297	0.1243	0.1110	0.0401	0.5181	0.5532	0.9546
61	4.5347	1.9138	1.6448	0.5247	14.4397	0.3612	——
241	3.5351	1.6316	1.4169	0.5112	9.3914	0.4445	0.3242
421	2.4278	1.2770	1.2800	0.5989	5.8571	0.5268	1.0836
646	1.6615	0.9839	1.1135	0.5763	4.2845	0.5374	1.3682
1861	1.0131	0.6419	0.6674	0.3157	2.6764	0.5262	0.9289
3661	0.7328	0.4668	0.4320	0.1753	1.9381	0.5088	1.0539
7036	0.5211	0.3340	0.3051	0.1184	1.3870	0.5048	1.0482
13111	0.3820	0.2480	0.2235	0.0838	1.0290	0.4997	0.9920
25531	0.2712	0.1773	0.1572	0.0558	0.7316	0.4982	1.0330
48616	0.1998	0.1296	0.1137	0.0396	0.5372	0.4968	0.9675

Table 5.6.6: $\mathbf{P}_1 - \mathbf{P}_1 - \mathbf{P}_2 - \mathbf{P}_1$ approximation (Example 3, uniform, red-blue-green, and red refinements).

N	$\mathbf{e}_h(\mathbf{u})$	$\mathbf{e}(\mathbf{t})$	$\mathbf{e}(\boldsymbol{\sigma})$	$\mathbf{e}(p)$	$\boldsymbol{\vartheta}$	$\mathbf{e}/\boldsymbol{\vartheta}$	r
157	4.8596	1.1398	1.0115	0.3834	11.3194	0.4512	——
625	2.6623	0.8856	0.7320	0.2560	6.1436	0.4738	0.8139
2497	1.3167	0.5058	0.3941	0.1302	2.2758	0.6460	0.9862
9985	0.4296	0.1942	0.1462	0.0466	0.7035	0.7047	1.5688
39937	0.1275	0.0589	0.0450	0.0159	0.1959	0.7571	1.7407
157	4.8596	1.1398	1.0115	0.3834	11.3184	0.4512	——
625	2.6623	0.8856	0.7320	0.2560	6.1436	0.4738	0.8139
1405	1.3320	0.5136	0.4039	0.1336	2.3107	0.6447	1.6541
2185	0.5238	0.2335	0.1907	0.0624	0.8976	0.6768	4.0620
4954	0.2529	0.1073	0.0958	0.0344	0.4148	0.7063	1.7820
7216	0.1749	0.0771	0.0637	0.0226	0.2755	0.7362	1.9564
12676	0.0968	0.0402	0.0359	0.0127	0.1578	0.7069	2.1212
20047	0.0610	0.0260	0.0215	0.0073	0.0976	0.7182	2.0290
35101	0.0358	0.0145	0.0127	0.0046	0.0552	0.7417	1.9161
49453	0.0256	0.0104	0.0090	0.0032	0.0393	0.7450	1.9671
157	4.8596	1.1398	1.0115	0.3834	11.3184	0.4512	——
625	2.6623	0.8856	0.7320	0.2560	6.1436	0.4738	0.8139
1093	1.3733	0.5413	0.4352	0.1440	2.4275	0.6367	2.2651
1561	0.6726	0.3189	0.2790	0.0963	1.3023	0.6148	3.6906
3784	0.3690	0.1732	0.1506	0.0509	0.6861	0.6377	1.3653
4954	0.2374	0.1118	0.0943	0.0321	0.4277	0.6563	3.2943
8113	0.1595	0.0730	0.0612	0.0212	0.2671	0.6999	1.6482
12793	0.0938	0.0432	0.0377	0.0130	0.1651	0.6706	2.3009
20398	0.0609	0.0273	0.0229	0.0075	0.0997	0.7123	1.9036
30109	0.0427	0.0190	0.0161	0.0052	0.0682	0.7285	1.8310
43447	0.0290	0.0130	0.0111	0.0037	0.0471	0.7183	2.0948
58891	0.0214	0.0095	0.0080	0.0026	0.0344	0.7229	2.0235

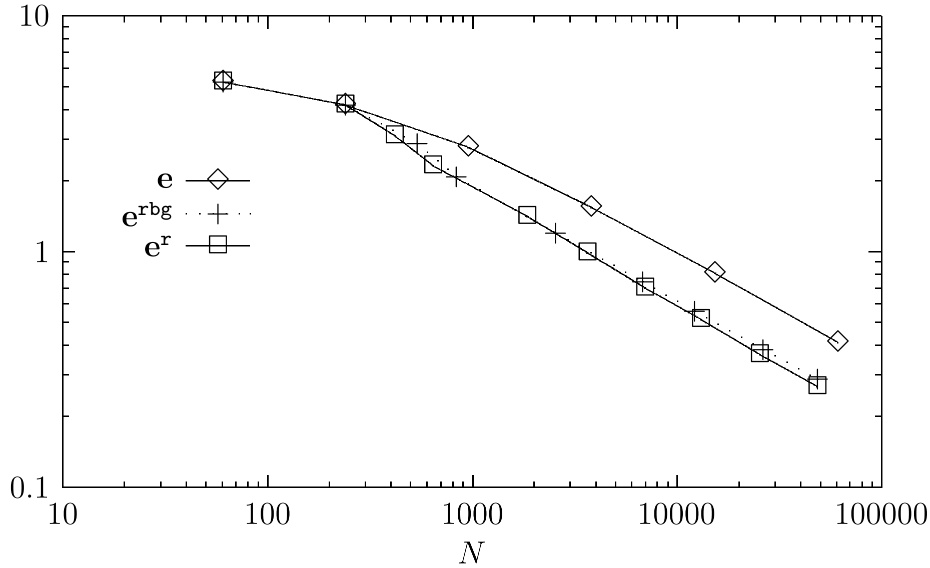

Figure 5.6.13: Example 3 with $\mathbf{P}_0 - \mathbf{P}_0 - \mathbf{P}_1 - \mathbf{P}_0$ approximation: global error $\mathbf{e}$ for the uniform and adaptive refinements.

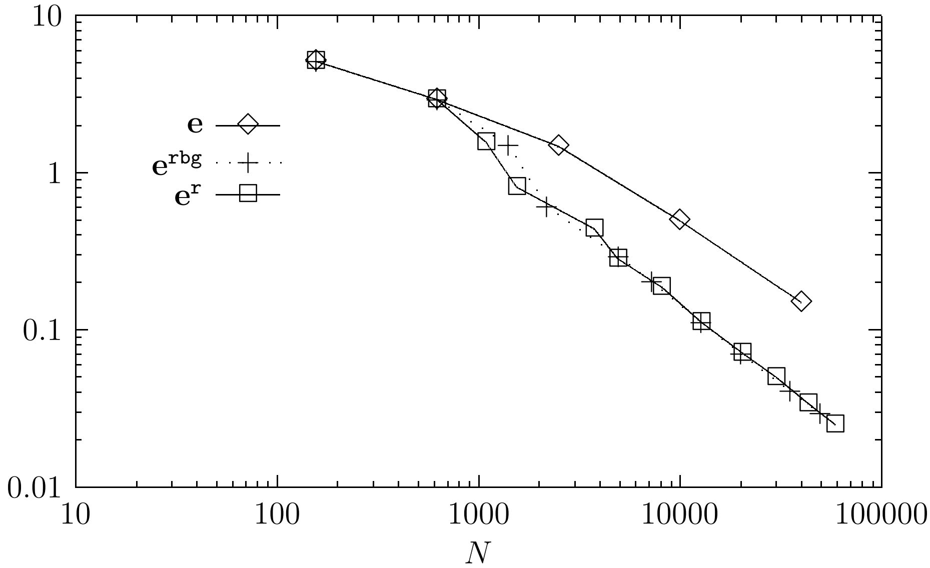

Figure 5.6.14: Example 3 with $\mathbf{P}_1 - \mathbf{P}_1 - \mathbf{P}_2 - \mathbf{P}_1$ approximation: global error $\mathbf{e}$ for the uniform and adaptive refinements.

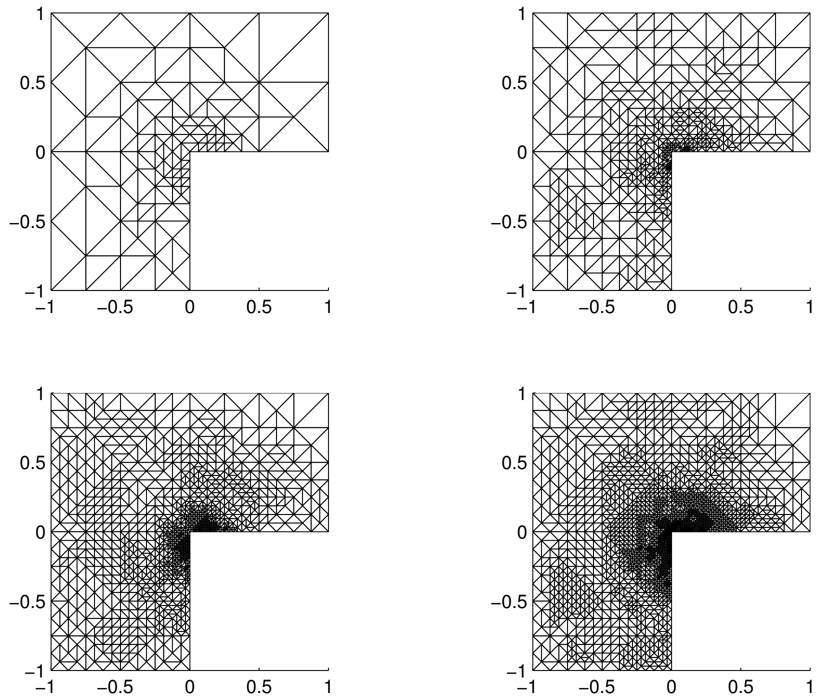

Figure 5.6.15: Example 3 with $\mathbf{P}_0 - \mathbf{P}_0 - \mathbf{P}_1 - \mathbf{P}_0$ approximation, without hanging nodes: adapted intermediate meshes with 2566, 12256, 26416 and 48766 degrees of freedom.

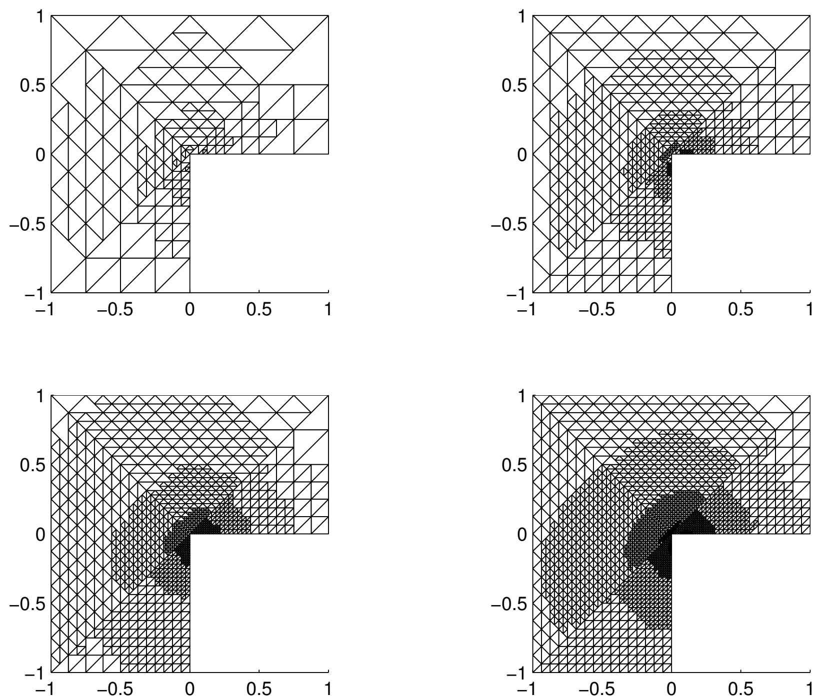

Figure 5.6.16: Example 3 with $\mathbf{P}_0 - \mathbf{P}_0 - \mathbf{P}_1 - \mathbf{P}_0$ approximation, with hanging nodes: adapted intermediate meshes with 3661, 13111, 25531 and 48616 degrees of freedom.

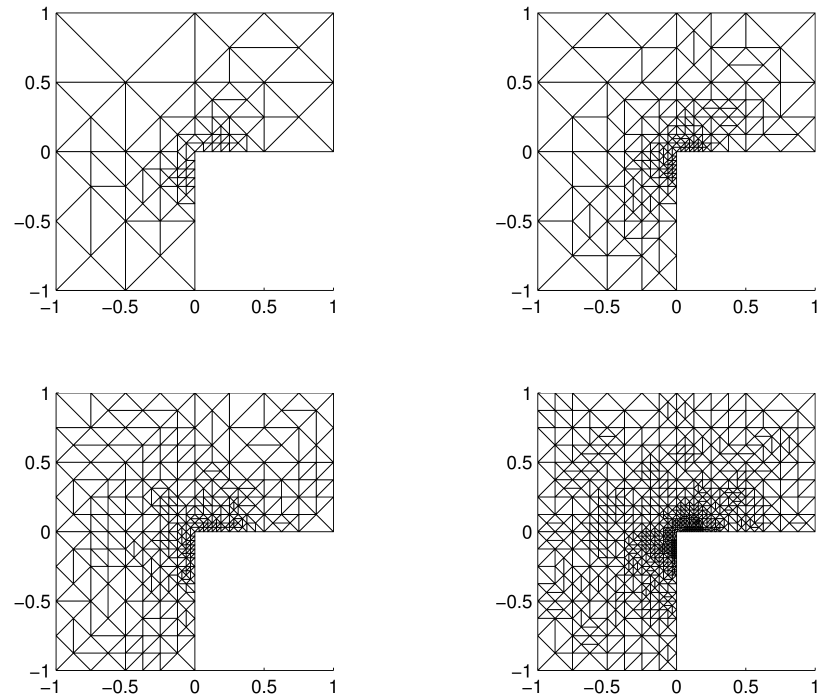

Figure 5.6.17: Example 3 with $\mathbf{P}_1 - \mathbf{P}_1 - \mathbf{P}_2 - \mathbf{P}_1$ approximation, without hanging nodes: adapted intermediate meshes with 4954, 12676, 20097 and 49453 degrees of freedom.

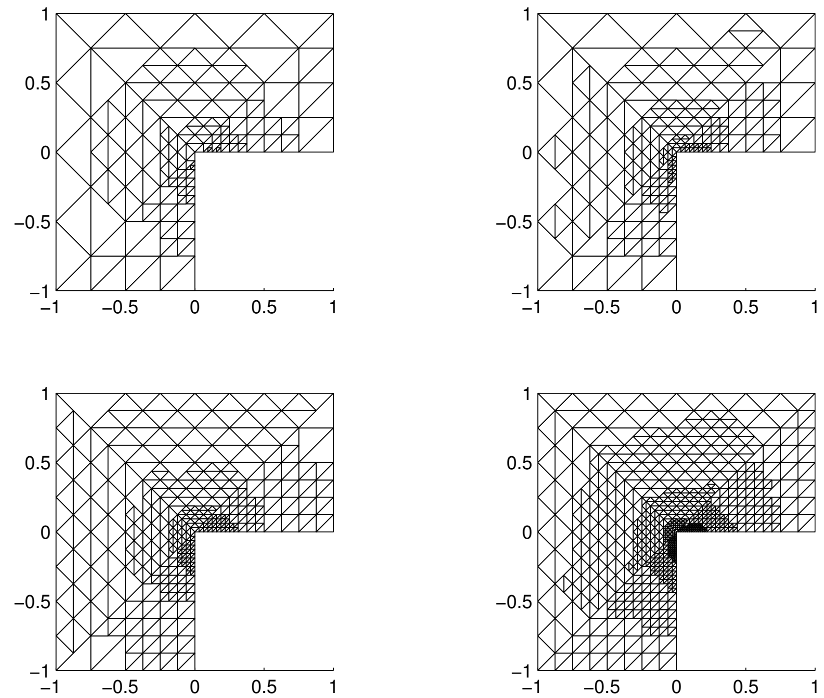

Figure 5.6.18: Example 3 with $\mathbf{P}_1 - \mathbf{P}_1 - \mathbf{P}_2 - \mathbf{P}_1$ approximation, with hanging nodes: adapted intermediate meshes with 8113, 12793, 20398 and 43447 degrees of freedom.

Table 5.6.7: $\mathbf{P}_0 - \mathbf{P}_0 - \mathbf{P}_1 - \mathbf{P}_0$ approximation (Example 4, uniform, red-blue-green, and red refinements).

N	$\mathbf{e}_h(\mathbf{u})$	$\mathbf{e}(\mathbf{t})$	$\mathbf{e}(\boldsymbol{\sigma})$	$\mathbf{e}(p)$	$\boldsymbol{\vartheta}$	$\mathbf{e}/\boldsymbol{\vartheta}$	r
91	9.2999	2.0266	2.6848	1.5977	24.0305	0.4169	——
361	7.7723	2.2596	2.3950	1.2706	18.4319	0.4631	0.2323
1441	6.6210	2.1460	1.8748	0.8721	13.8864	0.5229	0.2338
5761	5.5502	1.9470	1.4139	0.5316	10.6885	0.5681	0.2579
23041	4.0926	1.6475	1.0679	0.3190	7.2802	0.6250	0.4164
91	9.2999	2.0266	2.6848	1.5977	24.0305	0.4169	0.4030
151	7.6738	2.2881	2.5494	1.3567	19.1942	0.4435	0.6431
601	7.0024	2.2164	2.0080	0.9351	15.0205	0.5107	0.1506
1441	6.0212	2.0920	1.6023	0.6301	12.1643	0.5428	0.3432
1981	5.1854	1.9548	1.5229	0.6442	10.1537	0.5696	0.8329
2521	4.0010	1.6850	1.4065	0.6498	7.8272	0.5889	1.8819
2941	3.0695	1.3621	1.2974	0.6587	6.3106	0.5800	2.9944
3946	2.3603	1.1156	1.0284	0.5028	4.7364	0.6019	1.7001
5476	1.9541	0.8959	0.8945	0.4484	4.0282	0.5886	1.1239
14311	1.2336	0.5882	0.5500	0.2618	2.5071	0.5968	0.9586
30286	0.8324	0.4209	0.3819	0.1720	1.7441	0.5863	1.0157
58306	0.6252	0.3041	0.2717	0.1218	1.2784	0.5916	0.9207
91	9.2999	2.0266	2.6848	1.5977	24.0305	0.4169	——
136	7.8666	2.3018	2.5023	1.3009	19.5785	0.4427	0.7203
496	7.0898	2.3309	2.1768	1.0330	15.5965	0.5028	0.1547
1036	6.2889	2.2420	1.8157	0.7571	12.7759	0.5448	0.3240
1306	5.5945	2.1510	1.7763	0.7809	10.9284	0.5765	0.8607
1576	4.5476	1.9310	1.6943	0.7919	8.7614	0.6030	1.8744
1801	3.7327	1.6846	1.6247	0.8088	7.6311	0.5870	2.4720
2566	3.0437	1.4010	1.3076	0.6200	5.8273	0.6263	1.1571
3781	2.6053	1.1756	1.0615	0.4887	4.8795	0.6328	0.8625
9271	1.7748	0.7662	0.6885	0.3130	3.1381	0.6615	0.8856
16741	1.3377	0.5519	0.4940	0.2204	2.3413	0.6598	0.9998
31411	0.9979	0.4150	0.3614	0.1546	1.7145	0.6708	0.9379
57376	0.7468	0.3000	0.2623	0.1120	1.2706	0.6720	0.9885

Table 5.6.8: $\mathbf{P}_1 - \mathbf{P}_1 - \mathbf{P}_2 - \mathbf{P}_1$ approximation (Example 4, uniform, red-blue-green, and red refinements).

N	$\mathbf{e}_h(\mathbf{u})$	$\mathbf{e}(\mathbf{t})$	$\mathbf{e}(\boldsymbol{\sigma})$	$\mathbf{e}(p)$	$\boldsymbol{\vartheta}$	$\mathbf{e}/\boldsymbol{\vartheta}$	r
235	11.2569	1.3855	2.5978	1.6797	19.4592	0.6041	——
937	8.0976	1.2593	1.4422	0.8134	13.2850	0.6293	0.4929
3745	5.8418	1.1927	1.0402	0.5114	9.8167	0.6187	0.4612
14977	4.5820	1.1203	0.7805	0.3071	7.6189	0.6288	0.3423
59905	3.1715	0.9091	0.5735	0.1874	4.7975	0.6991	0.5145
235	11.2569	1.3855	2.5978	1.6797	19.4592	0.6041	——
391	7.7162	1.3267	1.9996	1.2351	14.2108	0.5752	1.4273
1561	6.0490	1.2800	1.2729	0.6773	10.9116	0.5818	0.3652
2965	5.4469	1.2465	1.0177	0.4753	9.5063	0.5996	0.3364
4369	4.5954	1.0963	0.8193	0.3499	7.5857	0.6338	0.8780
5461	3.5023	0.8427	0.6567	0.2927	5.3446	0.6873	2.4125
6826	1.9690	0.6159	0.4188	0.1580	2.6850	0.7863	4.9650
8464	1.1428	0.2966	0.2538	0.1205	1.6572	0.7323	5.1478
12910	0.4847	0.1622	0.1342	0.0591	0.7527	0.7064	3.9094
25468	0.2518	0.0940	0.0760	0.0325	0.4107	0.6847	1.8757
35608	0.1745	0.0572	0.0473	0.0207	0.2692	0.7086	2.3166
55420	0.1286	0.0394	0.0333	0.0142	0.1871	0.7446	1.4201
235	11.2569	1.3855	2.5978	1.6797	19.4592	0.6041	——
352	8.0017	1.4132	2.1072	1.2923	14.8019	0.5738	1.6093
1288	6.5150	1.3518	1.3646	0.7214	11.3569	0.6014	0.3359
1990	5.8613	1.3316	1.1134	0.5180	9.9964	0.6137	0.4938
2692	4.9078	1.1846	0.9422	0.4209	8.1356	0.6334	1.1543
3160	3.5570	1.0168	0.8412	0.3877	6.5021	0.5865	3.7560
4447	2.0263	0.6982	0.5258	0.2203	3.3609	0.6599	3.1733
5617	1.3036	0.3821	0.3174	0.1407	1.9539	0.7176	3.9263
7957	0.8613	0.3030	0.2504	0.1080	1.3152	0.7245	2.2179
14041	0.5171	0.1569	0.1295	0.0566	0.7749	0.7208	1.8812
23635	0.2992	0.1059	0.0834	0.0344	0.4531	0.7282	2.0220
33346	0.2023	0.0655	0.0531	0.0220	0.3055	0.7211	2.3469
49843	0.1463	0.0462	0.0370	0.0152	0.2082	0.7615	1.6377

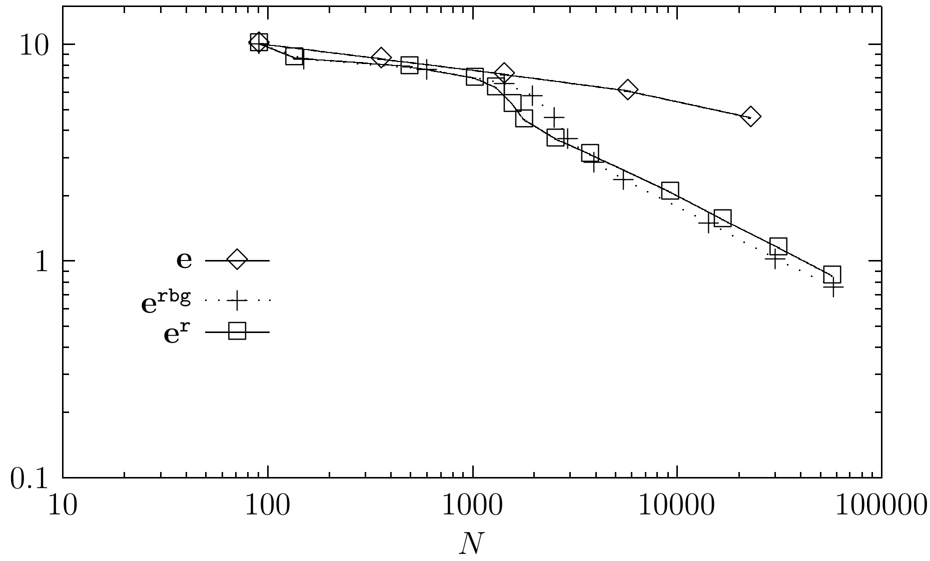

Figure 5.6.19: Example 4 with $\mathbf{P}_0 - \mathbf{P}_0 - \mathbf{P}_1 - \mathbf{P}_0$ approximation: global error $\mathbf{e}$ for the uniform and adaptive refinements.

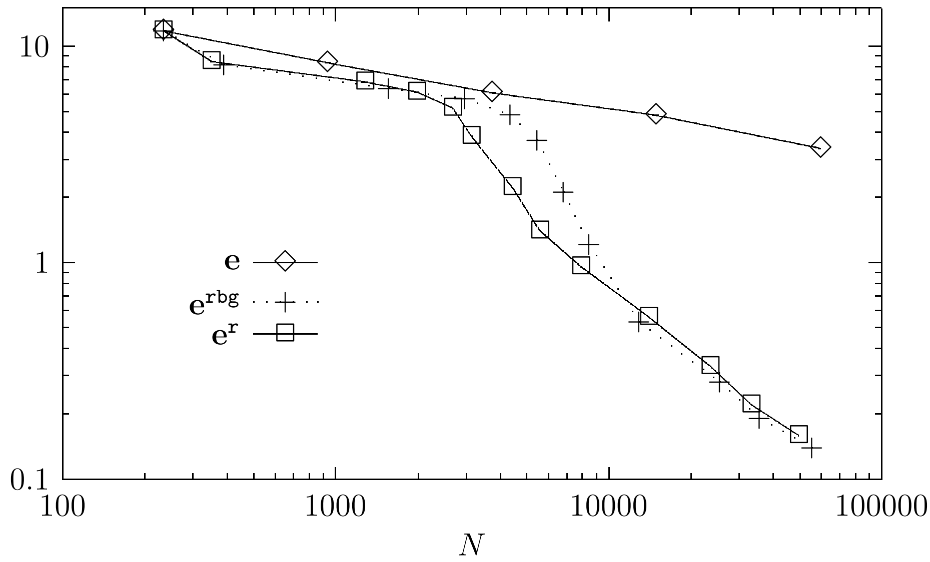

Figure 5.6.20: Example 4 with $\mathbf{P}_1 - \mathbf{P}_1 - \mathbf{P}_2 - \mathbf{P}_1$ approximation: global error $\mathbf{e}$ for the uniform and adaptive refinements.

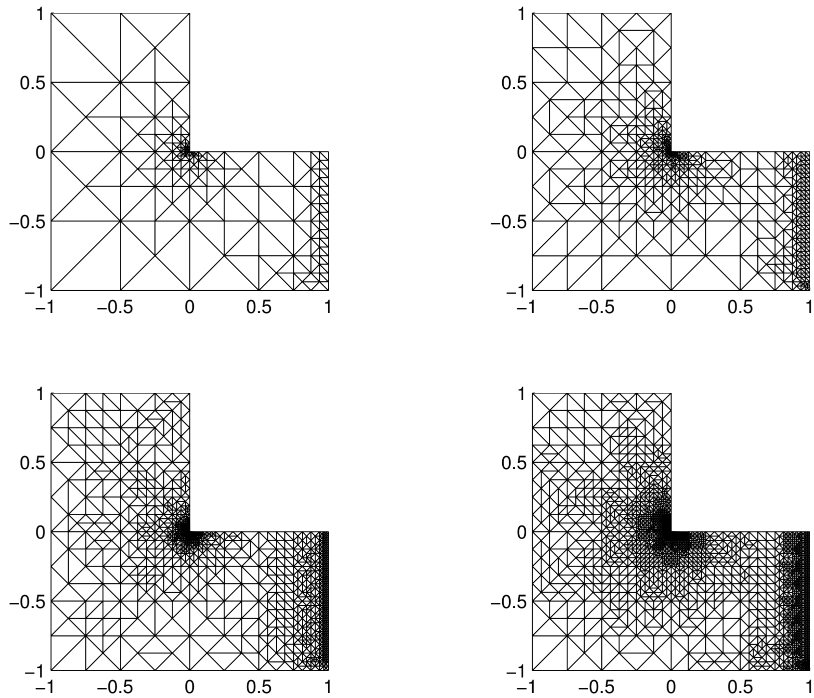

Figure 5.6.21: Example 4 with $\mathbf{P}_0 - \mathbf{P}_0 - \mathbf{P}_1 - \mathbf{P}_0$ approximation, without hanging nodes: adapted intermediate meshes with 3946, 14311, 30286 and 58306 degrees of freedom.

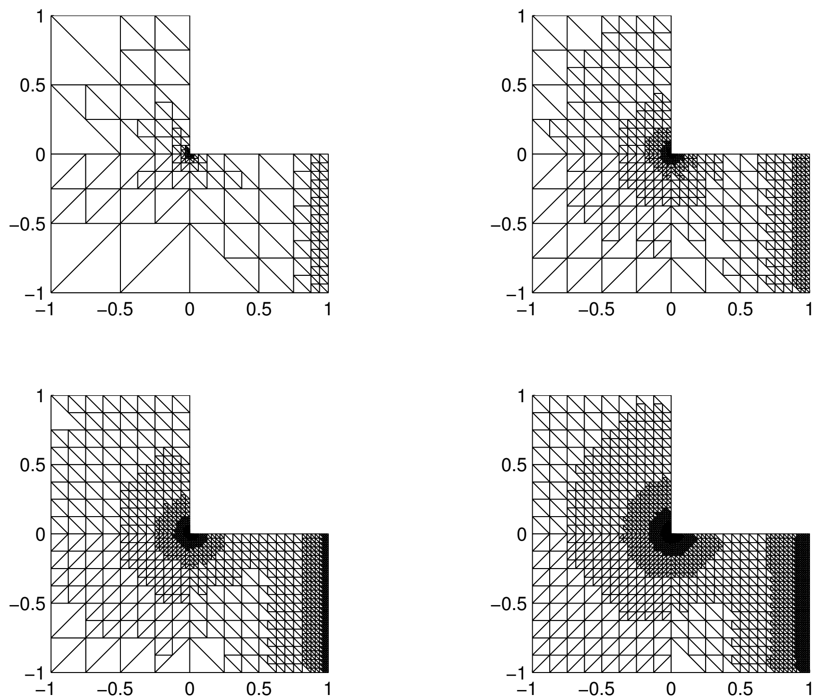

Figure 5.6.22: Example 4 with $\mathbf{P}_0 - \mathbf{P}_0 - \mathbf{P}_1 - \mathbf{P}_0$ approximation, with hanging nodes: adapted intermediate meshes with 3781, 16741, 31411 and 57376 degrees of freedom.

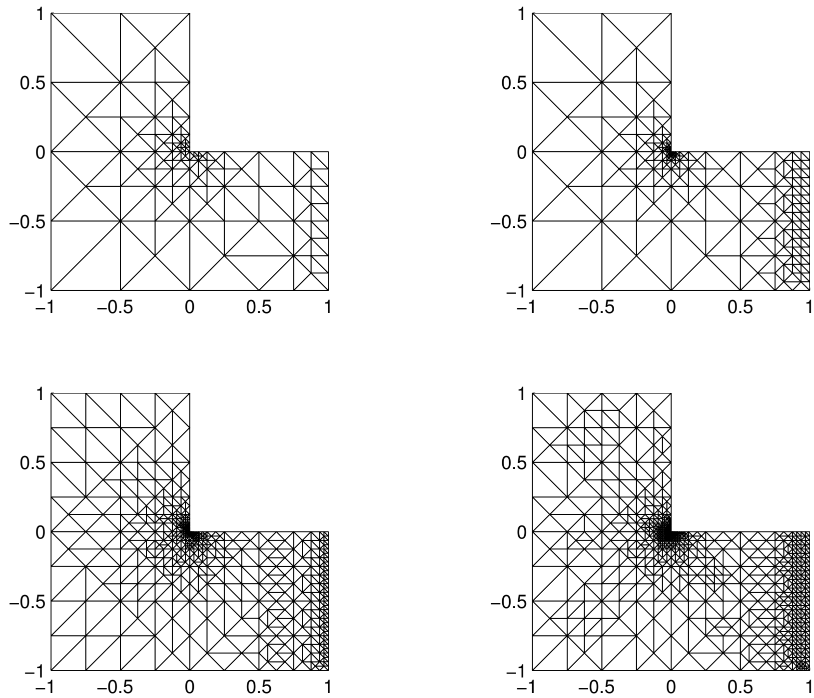

Figure 5.6.23: Example 4 with $\mathbf{P}_1 - \mathbf{P}_1 - \mathbf{P}_2 - \mathbf{P}_1$ approximation, without hanging nodes: adapted intermediate meshes with 6826, 12910, 35608 and 55420 degrees of freedom.

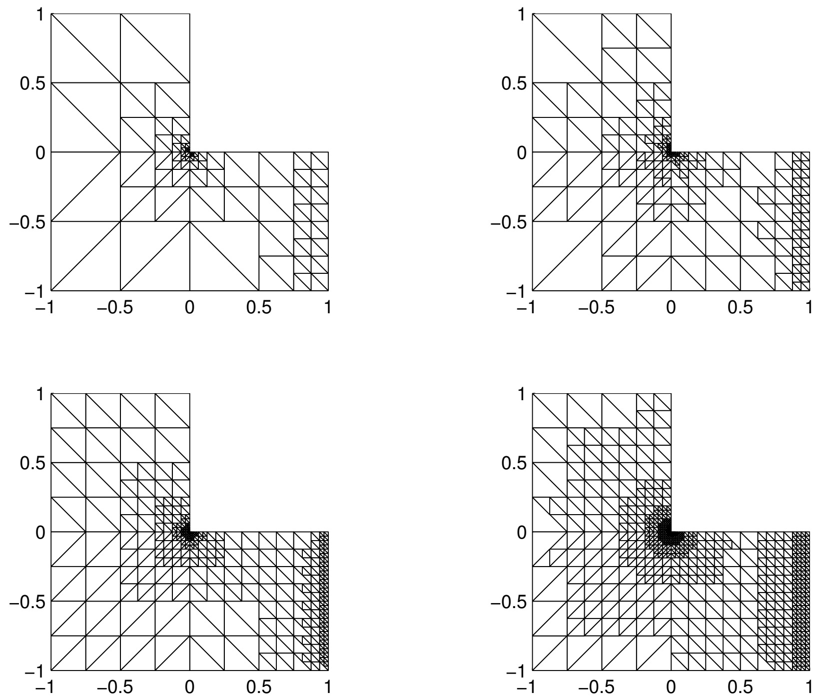

Figure 5.6.24: Example 4 with $\mathbf{P}_1 - \mathbf{P}_1 - \mathbf{P}_2 - \mathbf{P}_1$ approximation, with hanging nodes: adapted intermediate meshes with 5617, 14041, 23635 and 49843 degrees of freedom.

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