UNIVERSIDAD DE CONCEPCIÓN
ESCUELA DE GRADUADOS
CONCEPCIÓN – CHILE

EL PROBLEMA DE EQUILIBRIO MEDIANTE ANÁLISIS DE RECESIÓN

Tesis para optar al título de
Doctor en Ciencias Aplicadas con mención en Ingeniería Matemática

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DEPARTAMENTO DE INGENIERÍA MATEMÁTICA
2005

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Concepción–Enero 2005

A Luisa y Rubén
a Gloria y María Jesús
y a toda mi familia.

Agradecimientos

Quiero agradecer a todas las personas que me han brindado su cariño, amistad y han estado a mi lado durante estos últimos años.

Agradezco a mis padres Luisa y Rubén por todo su amor, comprensión y apoyo. A mis amadas Lolita y María Jesús por su amor, paciencia y confianza en mi.

Agradezco a Fabián F. mi director de tesis por los cursos que nos dictó, por su invitación al ICTP, por el tiempo que me otorgó para discutir los trabajos, por compartir su experiencia y profesionalismo. A los profesores del programa, en especial a mis profesores Mauricio S., Juan M., Freddy P., Fernando C. y Roberto R. A Cecilia, María Estrella, Teresita, René y Claudio por su simpatía y ayuda. A Juanita y demás personas de las bibliotecas de la Universidad por su colaboración. A Rodolfo R. por su gestión y consejo. Al Prof. Osvaldo O. por su simpatía y caballerosidad. A Gabriel G. por los recursos brindados para mi crecimiento profesional. A los profesores Victor A. (UCN), Rafael C. (DIM) y Wilfredo S. (INCA) por su ayuda y hospitalidad en las estadías y congresos. A Sharon (ICTP) por su amabilidad.

Agradezco mucho a mis amigos de la Cabina 6, a: María Eugenia, Chelo, Emilius, Marcela, Antonino, Tomacho, Ramiro y Sandra, Papá y Xime, Lucho, Jessica, por su amistad, apoyo y por los momentos que pasamos juntos. Gracias también a los amigos que pasaron por la cabina, a: Dante, Fernando^{2}, Violeta, Richard, Galina, Gommel, Anibal y Esperanza, Erwin, Mario y Anita, Mauricio, Wilmer, el Pepe, Juan Carlos.

Agradezco mucho a Richi, Juanita, Javiera, Checho y el Mati por su cariño, amistad, compañía y acoger a mi familia. También agradezco mucho a mi amigo el Prepo y su familia por su apoyo.

Agradezco a M. Vidal, Alejandra, Roxana, Vivi, Fernanda, Carolina, Paola, Camilo, Andrés, Roberto, Renato, Sonia y Loreto por su amistad y compartir lindos momentos. Gracias a María Paz y el Tito por su simpatía y colaboración.

Agradezco a los profesores J. Amaya, J.-P. Crouzeix y N. Hadjisavvas por evaluar el trabajo y aceptar ser parte del jurado.

Agradezco a mi programa de doctorado, Escuela de Graduados, Sociedad Matemática de Chile, CONICYT, proyectos MECESUP y FONDAP, International Center of Theoretical Physics (ICTP), Centre International de Matematiques Pures et Appliquees (CIMPA), Universidad Católica del Norte, Universidad Nacional de Ingenieía (Perú), Universidad de Chile, Universidad del Bio-Bio, Universidad de la Frontera, por financiar mi asistencia a talleres, congresos y estadías.

Este trabajo fue financiado por los proyectos MECESUP UCO9907 y FONDAP Matemáticas Aplicadas, a los cuales expreso mi agradecimiento.

Abstract

This thesis is concerned with the study of the equilibrium problem by using asymptotic analysis. As a model of study we consider the complementarity problem (CP).

It is well-known that the (CP) is equivalent to a variational inequality problem (VIP). We employ the latter problem in order to study (CP). Under some continuity assumptions the (VIP) has solutions as soon as is defined on a bounded set. The object of this thesis is to deal with the (VIP) defined on unbounded sets (as in the (CP)). To this end we apply the asymptotic analysis: we approximate the (VIP) with problems defined on bounded sets and that have solutions and then we study the asymptotic properties of the normalized approximate solutions of such problems. With the aid of the obtained information, the reformulated Gowda-Pang existence theorem and by introducing several new classes of mappings, we obtain new existence, stability and sensitivity results. Moreover, we obtain bounds for the solutions sets and the asymptotic cones of the solution sets. The multivalued, piecewise polyhedral, and linear complementarity cases are studied in detail, the results from the literature are recovered and new results are given.

The Lemke's algorithm allows the resolution of the linear complementarity problem in a finite number of steps. For large size problems we have iterative algorithms. Among the latter the splitting method plays an important role. In order for this algorithm to be well-defined some of the matrices involved must be $\mathbf{Q}$-matrices. We study such matrices in detail and characterize them within a new class of matrices we introduced, which enjoys good properties.

Resumen

El objetivo de esta tesis es estudiar el problema de equilibrio mediante análisis de recesión. Tomamos como modelo de estudio al problema de complementariedad (PC).

Es conocido que el (PC) es equivalente a un problema de desigualdad variacional (PDV), el cual empleamos para estudiar nuestro problema. El (PDV) tiene soluciones en dominios acotados bajo ciertas hipótesis de continuidad. El objetivo de esta tesis es estudiar dicho problema en dominios no acotados (como es el caso de (PC)). Para tal efecto usamos análisis de recesión (o análisis asintótico); hacemos una aproximacion de (PDV) mediante problemas en dominios acotados en los cuales se tienen soluciones y luego estudiamos las propiedades asintóticas de sus correspondientes soluciones aproximadas normalizadas. Utilizando la información obtenida, el teorema de existencia reformulado de Gowda-Pang y mediante la introducción de nuevas clases de mapeos obtenemos nuevos resultados de existencia, estabilidad y sensibilidad. Además, obtenemos cota/estimas para los conjuntos solución y los conos asintóticos de los últimos. Los casos multívoco, poliédrico por tramos y lineal son estudiados en detalle, se recuperan los resultados de la literatura y se dan otros nuevos.

El algoritmo de Lemke permite la resolución del problema de complementariedad lineal en un número finito de pasos y para problemas de gran tamaño se tienen a los algoritmos iterativos. Entre los últimos tenemos los algoritmos de descomposición que juegan un rol muy importante. Para que dichos algoritmos esten bien definidos, algunas de las matrices involucradas deben ser **Q**-matrices. Estudiamos dichas matrices con bastante detalle y las caracterizamos dentro de una nueva clase de matrices que introducimos, la cual goza de buenas propiedades.

Glossary of Notations

Spaces	
$\mathbb{R}^n$	real $n-$dimensional space
$\mathbb{R}^n_+$	the nonnegative orthant of $\mathbb{R}^n$
$\mathbb{R}^n_{++}$	the positive orthant of $\mathbb{R}^n$
$\mathbb{R}^{m\times n}$	the space of $m \times n$ matrices
cl-sets$_{\neq\emptyset}(\mathbb{R}^n)$	the space of nonempty closed subsets of $\mathbb{R}^n$
$E^\perp$	the orthogonal subspace of E
Vectors	
$x \in \mathbb{R}^n$	an n$-$dimensional column vector
$x \geq 0$	a nonnegative vector (i.e. in $\mathbb{R}^n_+$)
$x > 0$	a positive vector (i.e. in $\mathbb{R}^n_{++}$)
$\|y\|$	the vector whose $i-$th component is $\|y_i\|$
e^i	the i-th column of the identity matrix in $\mathbb{R}^{n\times n}$
$\mathbb{1}$	the vector of ones
$\langle y, x\rangle$	the standard inner product of vectors in $\mathbb{R}^n$
$\\|\cdot\\|$	the Euclidean norm on $\mathbb{R}^n$
$\|\|y\|\|_d$	the $d-$norm of $y \in \mathbb{R}^n$ for a vector $d > 0$
$x \geq y$	the partial ordering $x_i \geq y_i$, $i = 1, \ldots n$
$x > y$	the strict ordering $x_i > y_i$, $i = 1, \ldots n$
Index sets	
$\mathcal{I}$	an arbitrary index set

α	an index subset of $I \doteq \{1, 2, \ldots, n\}$
$\bar{\alpha}$	the complement of the index subset α
$\mathrm{supp}\{x\}$	the support of a vector x
Matrices	
$M = (a_{ij})$	a matrix with entries a_{ij}
M^{T}	the transpose of a matrix M
$\|M\|$	a norm of a matrix M
$\|M\|_d$	a d-norm of a matrix M
$M_{\alpha\beta}$	$(a_{ij})_{i\in\alpha, j\in\beta}$, a submatrix of a matrix M
$M_{\alpha\cdot}$	$(a_{i\cdot})_{i\in\alpha, \mathrm{all}\, j}$, the rows of M indexed by α
$M_{\cdot\beta}$	$(a_{\cdot j})_{\mathrm{all}\, i, j\in\beta}$, the columns of M indexed by β
Sets	
$\mathrm{co}\, A$	the convex hull of the set A
$\mathrm{int}\, A$	the topological interior of the set A
$\mathrm{cl}\, A = \bar{A}$	the (topological) closure of the set A
$\mathrm{ri}\, A$	the relative interior of the set A
$\mathrm{pos}\, A$	the positive hull of the set A
$\mathrm{pos}^+ A$	the strictly positive hull of the set A
A^*	the (positive) polar cone of the set A
$A^\#$	the strictly (positive) polar cone of the set A
A^∞	the asymptotic cone of the set A
A^∞_d	the $d-$normalized asymptotic cone of the set A
$\mathcal{B}(x, \delta)$	the unit ball with center x and radius δ
$\mathbb{B}$	the unit ball $\mathcal{B}(0, 1)$
$\mathbb{B}_d$	the unit ball $\mathcal{B}_d(0, 1)$ respect to the d-norm
$\mathbb{S}$	the unit sphere with center 0 and radius 1
$\mathbb{S}_d$	the unit sphere respect to the d-norm
Δ_d	$\{x \geq 0 : \langle d, x\rangle = 1\}$ a set in $\mathbb{R}^n$
Δ_J, $J \subseteq I$	$\mathrm{co}\left\{\frac{1}{d_i} e^i : i \in J\right\}$ an extreme face of Δ_d

$d_C(x)$	the distance from x to $C \subseteq \mathbb{R}^n$
$C^k \to C$	the Painlevé-Kuratowski set convergence
$\mathrm{dl}(A, B)$	the integrated set distance from A to B
$\mathrm{dl}_\infty(A, B)$	the Pompeiu-Hausdorff distance from A to B
Mappings	
$\Phi : X \rightrightarrows Y$	a multifunction from X to Y
$\mathrm{dom}\,\Phi$	the domain of the mapping Φ
$\mathrm{gph}\,\Phi$	the graph of the mapping Φ
Φ^{-1}	the inverse of the mapping Φ
$\mathcal{X}$	the set of cuscos on $\mathbb{R}^n_+$
$\mathcal{X}_0$	the set of compact-convex-valued mappings on $\mathbb{R}^n_+$
$\Phi^k \xrightarrow{g} \Phi$	graphical convergence of $\{\Phi^k\}$ to Φ
$\omega(\Phi)$	the $d-$numerical range of Φ
M_Φ, m_Φ	the supremum and infimum of $\omega(\Phi)$
$\\|\Phi\\|$	$\sup\{\\|y\\| : y \in \Phi(x), x \geq 0\}$
$\|\Phi\|_d^+$	the d-outer norm of Φ
δ_C	the indicator function of a set C
σ_C	the support function of a set C
$f : X \to Y$	a function from X to Y
∂f	the subgradient of a function f
usc and lsc	upper and lower semicontinuity property
UL(λ)	upper Lipschitzian property with modulus λ
CP symbols	
$\mathcal{D}(\Phi)$	all vectors q for which MCP(q, Φ) has solutions
$\mathcal{F}(q, \Phi)$	the feasibility set of MCP(q, Φ)
$\mathcal{F}_s(q, \Phi)$	the strict feasibility set of MCP(q, Φ)
$\mathcal{S}(q, \Phi)$	the solution set of MCP(q, Φ).

Introduction

Several problems in optimization and nonlinear analysis can be written in an abstract framework known as the *equilibrium problem*. In the scalar case the above problem reads as follows [12]

$$\text{find } \bar{x} \in K : \; f(\bar{x}, y) \geq 0 \;\; \forall y \in K. \tag{EP}$$

where X is a Hausdorff topological vector space, $K \subset X$ is a nonempty closed convex subset, and $f : K \times K \to \mathbb{R}$ is a function.

Problems like this have a long history, starting with the work of Ky Fan [24, 25] in the early seventies and that of Brezis-Niremberg-Stampacchia [11]. However, it was after the work of Blum-Oettli [12] that such a problem has been in the scope of many researchers (for historical comments see [29]).

We now review some problems that can be written as equilibrium problems by adequately setting the function f (see [12, 29, 44] for instance).

Minimization problem: Let $h : K \to \mathbb{R} \cup \{\infty\}$, it is requested to

$$\text{find } \bar{x} \in K : \; h(x) \geq h(\bar{x}) \;\; \forall x \in K.$$

Set $f(x, y) = h(y) - h(x)$.

Saddle point problem: Let $L : K_1 \times K_2 \to \mathbb{R}$, it is requested to

$$\text{find } (\bar{x}^1, \bar{x}^2) \in K_1 \times K_2 : \; L(\bar{x}^1, y^2) \leq L(y^1, \bar{x}^2) \;\; \forall (y^1, y^2) \in K_1 \times K_2.$$

Set $K = K_1 \times K_2$ and $f((x^1, x^2), (y^1, y^2)) = L(y^1, x^2) - L(x^1, y^2)$.

Fixed-Point problem: Let X be a Hilbert space and $T : K \to K$, it is requested to

$$\text{find } \bar{x} \in K : T(\bar{x}) = \bar{x}.$$

Set $f(x, y) = \langle x - Tx, y - x \rangle$.
If $\Phi : K \rightrightarrows K$ is a multifunction, it is requested to

$$\text{find } \bar{x} \in K : \bar{x} \in \Phi(\bar{x}).$$

Set $f(x, y) = \sup_{z \in \Phi(x)} \langle x - z, y - x \rangle$.

Convex/pseudoconvex differentiable minimization problem: Let $h : X \to \mathbb{R}$ be a convex/pseudoconvex and Gâteaux differentiable function. Consider the minimization problem, it is known (see [29] for instance) that such a problem is equivalent to the problem

$$\text{find } \bar{x} \in K : \langle Dh(\bar{x}), y - \bar{x} \rangle \geq 0 \ \forall y \in K.$$

Set $f(x, y) = \langle Dh(x), y - x \rangle$.

Variational Inequality Problem: Let X be a Banach space and $T : K \to X^*$, it is requested to

$$\text{find } \bar{x} \in K : \langle T(\bar{x}), y - \bar{x} \rangle \geq 0 \ \forall y \in K.$$

Set $f(x, y) = \langle T(x), y - x \rangle$.
If $\Phi : K \rightrightarrows K$ is a multifunction, it is requested to

$$\text{find } \bar{x} \in K : \bar{y} \in \Phi(\bar{x}), \ \langle \bar{y}, y - \bar{x} \rangle \geq 0 \ \forall y \in K.$$

Set $f(x, y) = \sup_{z \in \Phi(x)} \langle z, y - x \rangle$.

Complementarity problem: Let K be a closed convex cone, K^* its positive polar cone, and $T : K \to X^*$, it is requested to

$$\text{find } \bar{x} \in K : T(\bar{x}) \in K^*, \ \langle T(\bar{x}), \bar{x} \rangle = 0.$$

Set $f(x, y) = \langle T(x), y - x \rangle$.

If $\Phi : K \rightrightarrows K$ is a multifunction, it is requested to

$$\text{find } \bar{x} \in K : \bar{y} \in \Phi(\bar{x}),\, \bar{y} \in K^*,\, \langle \bar{y}, \bar{x} \rangle = 0.$$

Set $f(x,y) = \sup_{z\in\Phi(x)}\langle z, y-x\rangle$.

Quasivariational inequality problem: Let $F : K \to X^*$, $Q : K \rightrightarrows X$, $x^* \in X^*$, and K be a closed convex set, it is requested to

$$\text{find } \bar{x} \in Q(\bar{x}) : \langle F(\bar{x}) - x^*, y - \bar{x}\rangle \geq 0 \ \forall y \in Q(\bar{x}).$$

Set $f(x,y) = \langle F(x) - x^*, y - x\rangle + \delta_{Q(x)}(y)$, where δ_C is the indicator function of the set C.

Nash equilibria problem in noncooperative games: Let $\mathcal{I} = \{1, \dots, n\}$ be the set of players. For every player $i \in \mathcal{I}$, let there be given a set K_i (the strategy set of the ith player). Let $K = \prod_{i\in\mathcal{I}} K_i$. For every player $i \in \mathcal{I}$, let there be given a function $f_i : K \to \mathbb{R}$ (the loss function of the ith player, depending on the strategy of all players). For $x = (x_1, \dots, x_n) \in K$, we define $x^i = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$. It is requested to

$$\text{find } \bar{x} \in K : f_i(\bar{x}) \leq f_i(\bar{x}^i, y_i) \ \forall i \in \mathcal{I} \ \forall y_i \in K_i.$$

The vector $\bar{x} \in K$ is said to be a *Nash equilibrium in the noncooperative game*. The above relationship means that no player can reduce his loss by varying his strategy alone.
Set $f(x,y) = \sum_{i\in\mathcal{I}} (f_i(x^i, y_i) - f_i(x))$.

Vector minimization problem: Let $C \subseteq \mathbb{R}^m$ be a closed convex cone, such that both C and C^* have nonempty interior. Consider the partial order in $\mathbb{R}^m$ defined by

$$x \preceq y \Longleftrightarrow y - x \in C \text{ and } x \prec y \Longleftrightarrow y - x \in \operatorname{int} C.$$

It is requested to

$$\text{find } \bar{x} \in K : F(x) \not\prec F(\bar{x}) \ \forall x \in K.$$

Set $f(x, y) = \sup_{||z||=1, z\in C^*}\langle z, F(y) - F(x)\rangle$.

Many authors had contributed to the study of the equilibrium problem when the set K is assumed to be compact or some coerciveness assumptions are imposed on the function f (see [12, 10] and the bibliography therein).

In this thesis, we study the equilibrium problem in a noncoercive framework by approximating the initial problem by problems defined on compact sets and by using techniques of recession analysis similarly as in [8, 70, 34, 13, 27, 14, 29] and the bibliography therein. However, we improve this approach by approximating the mappings as well, using the concept of graphical convergence or convergence by means of the outer norm. This allows us to avoid coercivity conditions and to obtain continuity results for the solution set mappings as well.

We take the complementarity problem in the finite dimensional setting as a model of our study and develop our approach for it.

We begin our study with the multivalued complementarity problem on the nonnegative orthant, which reads as follows

$$\text{find } \bar{x} \geq 0,\ \bar{y} \in \Phi(\bar{x}) : \ \ \bar{y} + q \geq 0,\ \langle \bar{y} + q, \bar{x}\rangle = 0. \tag{MCP}$$

where $\Phi : \mathbb{R}^n_+ \rightrightarrows \mathbb{R}^n$ is a multifunction and $q \in \mathbb{R}^n$ is a column vector. This problem is denoted by $\mathrm{MCP}(q, \Phi)$ and if Φ is a piecewise polyhedral multifunction this problem is called *the polyhedral complementarity problem* whereas if Φ is a linear mapping, it is called *the linear complementarity problem*.

Problem (MCP) is known to be equivalent to the following *multivalued variational inequality problem*:

$$\text{find } \bar{x} \geq 0,\ \bar{y} \in \Phi(\bar{x}) : \ \ \langle \bar{y} + q, x - \bar{x}\rangle \geq 0 \ \ \forall\, x \geq 0 \tag{MVIP}$$

The ASYMPTOTIC ANALYSIS of problem (MCP) consists of approximating the (MVIP) by a sequence of problems $(\mathrm{PMVIP}_\mathrm{k})$ defined on compact sets (in such sets we have solutions) and which have the following form:

$$\text{find } x^k \in D_k,\ y^k \in \Phi^k(x^k):\ \langle y^k + q^k, x - x^k\rangle \geq 0\ \forall x \in D_k. \qquad (\mathrm{PMVIP_k})$$

where $d > 0$, $\{\sigma_k\}$ is an increasing sequence of positive numbers converging to $+\infty$, $D_k = \{x \in \mathbb{R}^n_+ : \langle d, x\rangle \leq \sigma_k\}$ is a compact convex set, $q^k \to q$ and $\mathrm{d\!I}(\Phi^k, \Phi) \to 0$ (graphical convergence) or $|\Phi^k - \Phi|^+_d \to 0$ (convergence in the d-outer norm). Afterwards, we determine the asymptotic behavior of the normalized sequence of the approximate solutions to $(\mathrm{PMVIP_k})$ and we use this information in order to obtain existence and sensibility results for the initial problem (MCP). We have developed this analysis in [30, 32, 53], the last work of which is still in progress.

The outline of the present thesis is as follows:

Chapter 1 is devoted to set up notation and review some facts from matrix analysis, convex analysis, quadratic programming, set-valued analysis and variational inequalities. The notion of asymptotic cone is reviewed as well. It is worth mentioning that in order to realize our approach, we define and employ concepts related to a positive vector d and the simplex Δ_d, as d-norm, d-matrix norm, d-numerical range and d-normalized asymptotic cone among others.

Chapter 2 is devoted to the study of the multivalued complementarity problem. In Section 2.1 we list some classes of multifunctions known in the literature of the complementarity problem and introduce new classes for which we perform the asymptotic analysis. These classes are compared by using the notion of d-numerical range. The asymptotic analysis of the approximate solutions to the variational inequality formulation is performed in Section 2.2 (Basic Lemma) and the abstract Gowda–Pang existence theorem is reformulated therein. In Section 2.3, new classes of multifunctions are introduced and some of their properties are described. In Section 2.4, bounds-estimates for the asymptotic cone of the solution set are obtained. The main specializations of the abstract existence theorem are discussed in Section 2.5, where also some kind of robustness results are established. Section 2.6 is devoted to present some new sensitivity and stability results by using the concept of graphical convergence. In Section 2.7, the asymptotic analysis is carried out by means of the d-outer norm. Finally, precise bounds–estimations for the solution set are obtained in Section 2.8.

Chapter 3 is devoted to the study of the polyhedral complementarity problem. In Section 3.1, we recall the notion of piecewise polyhedral multifunctions, we give examples and establish some of their properties. A new class of multifunctions, suitable for this kind of complementarity problem is introduced in Section 3.2 and the asymptotic analysis for it is performed therein. Section 3.3 is devoted to present existence theorems. Finally, Lipschitzian properties for the solution set multifunction are obtained and the concept of approximable mappings is used in Section 3.4.

Chapter 4 is devoted to the study of the linear complementarity problem. In Section 4.1 we proceed with the asymptotic analysis for arbitrary matrices and establish various equivalent conditions to the non-emptiness of the solution set. Section 4.2 recalls the notion of $\mathbf{G}$-matrices and present some of their characterizations. In Section 4.3, we proceed with the asymptotic analysis for different classes of matrices. The new class of $\mathbf{GT}$-matrices, which contains properly that of $\mathbf{G}^{\#}$-matrices (used in [40]), is introduced as well. Some estimates for the asymptotic cone of the solution set are presented in Section 4.4. In Section 4.5 we prove some existence results for our larger class of matrices, which strengthen part of those in [40], and extend others. New sensitivity results are proved as well. In addition, new characterizations of the nonemptiness and boundedness of the solution set for all vectors q are established when the matrix is either a $\mathbf{G}$-matrix or a positive subdefinite matrix. Moreover, some conditions ensuring the boundedness of the solution set are also provided. Finally, Section 4.6 is devoted to discuss some possible relationship with other existence results, specially those in [40] and [17, 18].

Finally, Chapter 5 is devoted to the study of the class of $\mathbf{Q}$-matrices, which consists of matrices such that the linear complementarity problem has solution for all vectors q. In Section 5.1 we list some classes of matrices arising in the linear complementarity problem and to recall the characterizations of $\mathbf{Q}$-matrices within the class $\mathbf{P}_0$ due to Aganagič and Cottle, and within the class $\mathbf{L}$, which does not contains $\mathbf{P}_0$, due to Pang. In Section 5.2, a new class of matrices, which properly contains $\mathbf{L}$, is introduced. Moreover, some classes of matrices contained in such a class are indicated. In Section 5.3, we further establish the same result of Pang for a this new class of matrices. Furthermore, the equivalence between a $\mathbf{Q}$-matrix and $\mathbf{Q}_b$, which consists of matrices such that the linear complementarity problem has a nonempty and compact solution set for all vectors q, is discussed. Positive subdefinite matrices are analyzed as well.

Introducción

Muchos problemas en optimización y análisis no lineal pueden ser escritos en el mismo marco de referencia, conocido como *problema de equilibrio*. En el caso escalar dicho problema tiene la siguiente formulación [12]:

$$\text{hallar } \bar{x} \in K : \; f(\bar{x}, y) \geq 0 \; \forall y \in K. \tag{EP}$$

donde X es un espacio vectorial topológico de Hausdorff, $K \subset X$ es un subconjunto cerrado convexo no vacio y $f : K \times K \to \mathbb{R}$ es una función.

Los problemas de este tipo tienen una larga historia, comenzando en los trabajos de Ky Fan [24, 25] a comienzos de los años setenta y en los trabajos de Brezis-Niremberg-Stampacchia [11]. Sin embargo, es sólo después del trabjo de Blum-Oettli [12] que tal problema está bajo la lupa de muchos investigadores (para referencias históricas consultar [29]).

A continuación damos un listado de problemas que pueden ser escritos como problemas de equilibrio mediante la eleccción adecuada de la función f (ver [12, 29, 44] por ejemplo)

Problema de minimización: Sea $h : K \to \mathbb{R} \cup \{\infty\}$, se pide

$$\text{hallar } \bar{x} \in K : \; h(x) \geq h(\bar{x}) \; \forall x \in K.$$

Escoger $f(x, y) = h(y) - h(x)$.

Problema de punto silla: Sea $L : K_1 \times K_2 \to \mathbb{R}$, se pide

$$\text{hallar } (\bar{x}^1, \bar{x}^2) \in K_1 \times K_2 : \; L(\bar{x}^1, y^2) \leq L(y^1, \bar{x}^2) \; \forall (y^1, y^2) \in K_1 \times K_2.$$

Escoger $K = K_1 \times K_2$ y $f((x^1, x^2), (y^1, y^2)) = L(y^1, x^2) - L(x^1, y^2)$.

Problema de punto fijo: Sea X un espacio de Hilbert y $T : K \to K$, se pide

$$\text{hallar } \bar{x} \in K : \; T(\bar{x}) = \bar{x}.$$

Escoger $f(x,y) = \langle x - Tx, y - x\rangle$.
Si $\Phi : K \rightrightarrows K$ es una multifunción, se pide

$$\text{hallar } \bar{x} \in K : \ \bar{x} \in \Phi(\bar{x}).$$

Escoger $f(x,y) = \sup_{z\in\Phi(x)}\langle x - z, y - x\rangle$.

Problema de minimización convexa/pseudoconvexa diferenciable: Sea $h : X \to \mathbb{R}$ una función convexa/pseudoconvexa y diferenciable según Gâteaux. Consideremos el problema de minimización, se sabe (ver [29] por ejemplo) que dicho problema es equivalenta al problema

$$\text{hallar } \bar{x} \in K : \ \langle Dh(\bar{x}), y - \bar{x}\rangle \geq 0 \ \forall y \in K.$$

Escoger $f(x,y) = \langle Dh(x), y - x\rangle$.

Problema de desigualdad variacional: Sea X un espacio de Banach y $T : K \to X^*$, se pide

$$\text{hallar } \bar{x} \in K : \ \langle T(\bar{x}), y - \bar{x}\rangle \geq 0 \ \forall y \in K.$$

Escoger $f(x,y) = \langle T(x), y - x\rangle$.
Si $\Phi : K \rightrightarrows K$ es una multifunción, se pide

$$\text{hallar } \bar{x} \in K : \ \bar{y} \in \Phi(\bar{x}), \ \langle \bar{y}, y - \bar{x}\rangle \geq 0 \ \forall y \in K.$$

Escoger $f(x,y) = \sup_{z\in\Phi(x)}\langle z, y - x\rangle$.

Problema de complementariedad: Sea K un cono cerrado convexo, K^* su cono polar positivo y $T : K \to X^*$, se pide

$$\text{hallar } \bar{x} \in K : \ T(\bar{x}) \in K^*, \ \langle T(\bar{x}), \bar{x}\rangle = 0.$$

Escoger $f(x,y) = \langle T(x), y - x\rangle$.
Si $\Phi : K \rightrightarrows K$ es una multifunción, se pide

$$\text{hallar } \bar{x} \in K : \ \bar{y} \in \Phi(\bar{x}), \ \bar{y} \in K^*, \ \langle \bar{y}, \bar{x}\rangle = 0.$$

Escoger $f(x,y) = \sup_{z\in\Phi(x)}\langle z, y-x\rangle$.

Problema de desigualdad casivariacional: Sea $F : K \to X^*$, $Q : K \rightrightarrows X$, $x^* \in X^*$ y K un conjunto cerrado convexo, se pide

$$\text{hallar } \bar{x} \in Q(\bar{x}) : \langle F(\bar{x}) - x^*, y - \bar{x}\rangle \geq 0 \ \forall y \in Q(\bar{x}).$$

Escoger $f(x,y) = \langle F(x) - x^*, y - x\rangle + \delta_{Q(x)}(y)$, donde δ_C es la función indicatriz del conjunto C.

Problema de equilibrio de Nash en juegos no coperativos: Sea $\mathcal{I} = \{1, \ldots, n\}$ el conjunto de jugadores. Para cada jugador $i \in \mathcal{I}$ es dado el conjunto K_i (la estrategia de cada jugador i-ésimo). Sea $K = \prod\limits_{i\in\mathcal{I}} K_i$. Para cada jugador $i \in \mathcal{I}$ es también dada la función $f_i : K \to \mathbb{R}$ (la función de pérdida del jugador i-ésimo, dependiendo de la estrategia de todos los jugadores). Para $x = (x_1, \ldots, x_n) \in K$, definimos $x^i = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$. Se pide

$$\text{hallar } \bar{x} \in K : \ f_i(\bar{x}) \leq f_i(\bar{x}^i, y_i) \ \forall i \in \mathcal{I} \ \forall y_i \in K_i.$$

El vector $\bar{x} \in K$ se llama *punto de equilibrio de Nash en juegos no coperativos*. Las anteriores relaciones significan que ningún jugador puede reducir sus pérdidas cambiando el sólo su estrategia.
Escoger $f(x,y) = \sum\limits_{i\in\mathcal{I}} (f_i(x^i, y_i) - f_i(x))$.

Problema de minimización vectorial: Sea $C \subseteq \mathbb{R}^m$ un cono cerrado convexo, tal que C y C^* tienen interior no vacio. Consideramos el orden parcial en $\mathbb{R}^m$ definido por

$$x \preceq y \Longleftrightarrow y - x \in C \quad \text{y} \quad x \prec y \Longleftrightarrow y - x \in \operatorname{int} C.$$

Se pide

$$\text{hallar } \bar{x} \in K : \ F(x) \not\prec F(\bar{x}) \ \forall x \in K.$$

Escoger $f(x,y) = \sup_{||z||=1, z\in C^*}\langle z, F(y) - F(x)\rangle$.

Muchos investigadores han contribuido al estudio del problema de equilibrio en el caso cuando el conjunto K es compacto o cuando ciertas condiciones de coercividad son impuestas sobre la función f (ver [12, 10] y la bibliografía contenida ahí).

En esta tesis se estudia el problema de equilibrio en un marco de referencia no coercivo mediante la aproximación del problema original por problemas definidos en conjuntos compactos y usando técnicas de análisis de recesión (o análisis asintótico) de manera similar como en los trabajos [8, 70, 34, 13, 27, 14, 29] y la bibliografía contenida ahí). Sin embargo, mejoramos dicho enfoque aproximando también a los mapeos, usando el concepto de convergencia gráfica o de convergencia mediante la norma exterior. Esto nos permite evitar las condiciones de coercividad y obtener además resultados de continuidad para los mapeos de conjunto solución.

Tomamos al problema de complementariedad en dimensión infinita como modelo de estudio y desarrollamos nuestro enfoque para dicho problema.

Comenzamos nuestro estudio con el problema de complementariedad multívoco en el octante no negativo (primer octante), dicho problema se enuncia de la siguiente manera:

$$\text{hallar } \bar{x} \geq 0,\ \bar{y} \in \Phi(\bar{x}) : \ \ \bar{y} + q \geq 0,\ \langle \bar{y} + q, \bar{x} \rangle = 0. \tag{MCP}$$

donde $\Phi : \mathbb{R}^n_+ \rightrightarrows \mathbb{R}^n$ es una multifunción y $q \in \mathbb{R}^n$ es un vector columna. Denotamos a este problema por $\mathrm{MCP}(q, \Phi)$ y si Φ es una multifunción poliédrica por tramos este problema se llama *problema de complementarity poliédrico* mientras que, si Φ es un mapeo lineal unívoco se llama *problema de complementariedad lineal*.

Se sabe que el problema (MCP) es equivalente al *problema de desigualdad variacional multívoco*:

$$\text{hallar } \bar{x} \geq 0,\ \bar{y} \in \Phi(\bar{x}) : \ \ \langle \bar{y} + q, x - \bar{x} \rangle \geq 0 \ \ \forall\, x \geq 0 \tag{MVIP}$$

El ANÁLISIS DE RECESIÓN O ASINTÓTICO para el problema (MCP) consiste en aproximar el problema (MVIP) por una sucesión de problemas $(\mathrm{PMVIP_k})$ definidos en conjuntos compactos (en los cuales hay soluciones) que tienen la siguiente forma:

$$\text{hallar } x^k \in D_k,\ y^k \in \Phi^k(x^k) :\ \langle y^k + q^k, x - x^k \rangle \geq 0\ \forall x \in D_k. \qquad (\mathrm{PMVIP_k})$$

donde $d > 0$, $\{\sigma_k\}$ es una sucesión creciente de números positivos que convergen a $+\infty$, $D_k = \{x \in \mathbb{R}^n_+ : \langle d, x \rangle \leq \sigma_k\}$ es un conjunto compacto convexo, $q^k \to q$ y $\mathrm{d\!I}(\Phi^k, \Phi) \to 0$ (convergencia gráfica) o $|\Phi^k - \Phi|^+_d \to 0$ (convergencia en la d-norma exterior). A continuación, determinamos el comportamiento asintótico de la sucesión de soluciones aproximadas normalizadas del problema $(\mathrm{PMVIP_k})$ y usamos esta información para obtener resultados de existencia y sensibilidad para el problema original (MCP). Hemos desarrollado este análisis en los siguientes trabajos:

1. FLORES-BAZÁN F., LÓPEZ R., The linear complementarity problem under asymptotic analysis, Pre-print 2003-23, Depto. Ingeniería Matemática, Universidad de Concepción, será publicado en la revista ISI *Mathematics of Operations Research* (2005).

2. FLORES-BAZÁN F., LÓPEZ R., Characterizing Q-matrices beyond L-matrices, Pre-print 2004-11, Depto. Ingeniería Matemática, Universidad de Concepción, será publicado en la revista ISI *Journal of Optimization Theory and Applications* (2005).

3. FLORES-BAZÁN F., LÓPEZ, R., Asymptotic analysis, existence and sensitivity results for a class of multivalued complementarity problems. Pre-print 2004-19, Depto. Ingeniería Matemática, Universidad de Concepción. Sometido a publicación (2004).

4. LÓPEZ R. The polyhedral complementarity problem, en preparación (2004).

A continuación damos una breve descripción de la tesis:

El Capítulo 1 está dedicado a fijar la notación y revisar algunos resultados del análisis matricial, análisis convexo, programación cuadrática, análisis multívoco y desigualdades variacionales. La noción de cono asintótico es revisada también. Es importante mencionar que para llevar a cabo nuestro enfoque se han definido y utilizado conceptos relacionados a un vector positivo d y al simplejo Δ_d, como ser d-norma, d-norma matricial, d-rango numérico y d-cono asintótico normalizado entre otros.

El Capítulo 2 está dedicado al estudio del problema de complementariedad multívoco. En la Sección 2.1 listamos algunas clases de multifunciones conocidas en la literatura del problema de complementariedad e introducimos nuevas clases de multifunciones para las cuales se efectúa el análsis asintótico. Estas clases son comparadas usando la noción de d-rango numérico. El análisis asintótico de las soluciones aproximadas de la desigualdad variacional (MVIP) es realizado en la Sección 2.2 (Lema Básico) y el teorema de existencia abstracto de Gowda–Pang es reformulado ahí mismo. En la Sección 2.3, se introducen nuevas clases de multifunciones y se estudian algunas de sus propiedades. En la Sección 2.4, se obtienen cotas/estimas para el cono asintótico del conjunto solución. En la Sección 2.5 se discuten las principales especializaciones del teorema abstracto de existencia, también se obtienen resultados en los cuales se observa cierta clase de robusticidad. La Sección 2.6 está dedicada a la obtención de algunos nuevos resultados de sensibilidad y estabilidad mediante el uso de la convergencia gráfica. En la Sección 2.7, se realiza el análisis asintótico mediante la d-norma exterior. Finalmente en la Sección 2.8 se obtienen cotas/estimas para los conjuntos solución.

El Capítulo 3 está dedicado al estudio del problema de complementariedad poliédrico. En la Sección 3.1, repasamos la noción de multifunción poliédrica por tramos, damos algunos ejemplos y establecemos algunas de sus propiedades. En la Sección 3.2 introducimos una nueva clase de multifunciones que es adecuada para este tipo de problema de complementariedad, además realizamos el análisis asintótico para dicha clase. En la Sección 3.3 se presentan teoremas de existencia. Finalmente, en la Sección 3.4 se establecen propiedades Lipschitzianas para la multifunción conjunto-solución y el concepto de mapeo aproximable es usado.

El Capítulo 4 está dedicado al estudio del problema de complementariedad lineal. En la Sección 4.1 realizamos el análisis asintótico para matrices arbitrarias y establecemos condiciones equivalentes para la existencia de soluciones. En la Sección 4.2 se repasa la noción de $\mathbf{G}$-matriz y se presentan algunas de sus caracterizaciones. En la Sección 4.3, se realiza el análisis asintótico para diferentes clases de matrices. Además se introduce la nueva clase de $\mathbf{GT}$-matrices, la cual contiene propiamente a la clase de $\mathbf{G}^{\#}$-matrices (usada en [40]). En la Sección 4.4 se hallan algunas estimas para el cono asintótico del conjunto solución. En la Sección 4.5 se demuestran algunos resultados de existencia para clases más amplias de matrices y que refuerzan parte de aquellos obtenidos en [40] y se extienden otros. Se prueban nuevos resultados de sensibilidad. Además, se dan nuevas caracterizaciones de la no vacuidad y la acotación del conjunto solución para todos los vectores q cuando la matriz es o bien $\mathbf{G}$-matriz o bien positivamente subdefinida. También se dan condiciones que aseguran la acotación del conjunto solución. Finalmente, en la Sección 4.6 se discuten posibles relaciones con otros resultados de existencia, especialmente con aquellos de los trabajos [40] y [17, 18].

Finalmente, el Capítulo 5 está dedicado al estudio de la clase de $\mathbf{Q}$-matrices, la cual consiste en las matrices tales que el problema de complementariedad lineal tiene solución para todos los vectores q. En la Sección 5.1 listamos algunas clases de matrices que aparecen en el problema de complementariedad lineal y repasamos algunas caracterizaciones de las $\mathbf{Q}$-matrices dentro de la clase $\mathbf{P}_0$ debido a Aganagič y Cottle, y dentro de la clase $\mathbf{L}$, que no contiene a $\mathbf{P}_0$, debido a Pang. En la Sección 5.2, se introduce una nueva clase de matrices que contiene propiamente a $\mathbf{L}$. Además, algunas clases de matrices contenidas en tal clase son indicadas. En la Sección 5.3, generalizamos el resultado de Pang para la nueva clase de matrices. Se discute también la equivalencia entre **Q**-matrices y $\mathbf{Q}_b$-matrices, la última consiste en las matrices tal que el problema de complementariedad lineal tiene conjunto de solución no vacio y compacto para todo vector q. Las matrices positivamente subdefinidas son analizadas en detalle.

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Chapter 1

Notation and Preliminary Facts

In this thesis, we use the following notation: $x \geq 0$ (resp. $x > 0$) whenever $x \in \mathbb{R}^n_+$ (resp. $x \in \mathbb{R}^n_{++} = \operatorname{int} \mathbb{R}^n_+$); $|y| = (|y_1|, \ldots, |y_n|)$ and $||y||_d := \langle d, |y| \rangle$ whenever $y \in \mathbb{R}^n$ and $d > 0$ (in particular $||y||_d = \langle d, y \rangle$ for $y \geq 0$), $\| \cdot \|_d$ is a vector norm on $\mathbb{R}^n$; given $x \in \mathbb{R}^n$, the index set $\operatorname{supp}\{x\} := \{i \in I : \ x_i \neq 0\}$ is the support of x where $I \doteq \{1, \ldots, n\}$.

We denote by $\mathbb{R}^{m \times n}$ the space of matrices with real entries of order $m \times n$; by M^{T} we mean the transpose of $M \in \mathbb{R}^{m \times n}$. For $M \in \mathbb{R}^{m \times n}$, $\alpha \subseteq I$ and $\beta \subseteq I$, $M_{\alpha\beta}$ denotes the submatrix of M consisting of the rows and columns of M whose indices are in α and β respectively.

Given a matrix $M \in \mathbb{R}^{n \times n}$ and the vector norm $\| \cdot \|_d$ for a positive vector $d > 0$, the nonnegative number defined by

$$\|M\|_d \doteq \max_{w \neq 0} \frac{\|Mw\|_d}{\|w\|_d} = \max_{\|w\|_d = 1} \|Mw\|_d$$

is called the *matrix norm subordinated to the vector norm* $\|\cdot\|_d$ (see [21] for the definition of subordinated norm). By an standard argument we can find the following formula for this norm.

$$\|M\|_d = \max_{1 \leq j \leq n} \left(\sum_{i=1}^{n} \frac{d_i}{d_j} |a_{ij}| \right) \tag{1.1}$$

for instance, if $d = \mathbb{1}$ is the vector of ones, we have that $\|x\|_{\mathbb{1}} = |x_1| + \cdots + |x_n|$ is the sum norm and $\|M\|_{\mathbb{1}} = \max_{1\le j\le n} \sum_{i=1}^{n} |a_{ij}|$ is the maximum column-sum norm. Unless otherwise stated, by $\|x\|$ and $\|M\|$ we mean the usual Euclidean norm of x and the matrix norm of M subordinated to it.

Let $d > 0$ and $\mu_d = \sqrt{n}\,\frac{\max_{1\le i\le n} d_i}{\min_{1\le i\le n} d_i}$, we have the following relationship between these matrix norms

$$\frac{1}{\mu_d}\,\|M\|_d \le \|M\| \le \mu_d\,\|M\|_d\,.$$

1.1 Convex analysis

Let $A \subset \mathbb{R}^n$ be a subset, we denote by $\operatorname{co} A$ the convex hull of the set A; $\operatorname{ri} A$ the relative interior of A, that is, the interior with respect to its affine hull. The sets $\operatorname{pos} A = \{tx : t \ge 0, x \in A\}$ and $\operatorname{pos}^+ A = \{tx : t > 0, x \in A\}$ are the positive hull and strictly positive hull of A respectively; $A^* \doteq \{x : \langle u, x\rangle \ge 0\ \forall\, u \in A\}$, $A^\# \doteq \{x : \langle u, x\rangle > 0\ \forall\, u \in A \setminus \{0\}\}$ are the (positive) polar cone and the strictly (positive) polar cone of A respectively. Let $d > 0$, the sets $\mathbb{B}_d = \{x : \|x\|_d \le 1\}$ and $\mathbb{S}_d = \{x : \|x\|_d = 1\}$ are the unit ball and sphere with center 0 respect to the vector norm $\|\cdot\|_d$, the notations $\mathbb{B}$ and $\mathbb{S}$ are used when the Euclidean norm is employed; Δ_d is the simplex $\{x \ge 0 : \langle d, x\rangle = 1\}$, clearly $\Delta_d = \mathbb{S}_d \cap \mathbb{R}^n_+$; if $J \subseteq I$, $\Delta_J = \Delta_J(d) \doteq \operatorname{co}\{\frac{1}{d_i} e^i : i \in J\}$ is an extreme face of Δ_d, where e^i is the i-th column of the identity matrix in $\mathbb{R}^{n\times n}$. In particular, $\Delta_I = \Delta_d$.

The convex hull of a finite set of points is called a *polytope*. A *polyhedral set* or *polyhedron* is the intersection of a finite number of closed half-spaces. For a union of such sets we have the following result (see [68, Lem 2.50])

Lemma 1.1.1 *If a convex set C is the union of a finite collection of polyhedral sets C_k, it must itself be polyhedral. Moreover if $\operatorname{int} C \ne \emptyset$, the sets C_k with $\operatorname{int} C_k = \emptyset$ are superfluous in the representation. In fact C can be given a refined expression as the union of a finite collection of polyhedral sets $\{D_i\}_{i=1}^m$ such that*

(a) *each set D_i is included in one of the sets C_k;*

(b) $\mathrm{int}\, D_i \neq \emptyset$, *so* $D_i = \mathrm{cl}(\mathrm{int}\, D_j)$*;*

(c) $\mathrm{int}\, D_{i_1} \cap \mathrm{int}\, D_{i_2} = \emptyset$ *when* $i_1 \neq i_2$*.*

In Chapter 3 we employ the following result [51, Lem 23.3]

Lemma 1.1.2 *If* $C \subseteq \mathbb{R}^n$ *is a compact convex set with nonempty interior, then for each* $\varepsilon > 0$ *there exists a polytope* P *such that* $P \subseteq C \subseteq P_\varepsilon$*, where* $P_\varepsilon = P + \varepsilon\mathbb{B}$*.*

Let $K \subset \mathbb{R}^n$ be a convex cone, then

$$w \in \mathrm{int}\, K \iff \langle v, w\rangle > 0 \text{ for all nonzero } v \in K^*,$$

moreover if $K \subset \mathbb{R}^n$ is a closed cone, not necessarily convex, then

$$w \in \mathrm{int}\, K^* \iff \langle v, w\rangle > 0 \text{ for all nonzero } v \in K,$$

thus in this case we can write $\mathrm{int}\, K^* = K^{\#}$ (see [68] for instance).

The *indicator function* of a set $C \subset \mathbb{R}^n$, denoted by δ_C, is defined by

$$\delta_C(x) \doteq \begin{cases} 0 & \text{if } x \in C, \\ +\infty & \text{if } x \notin C. \end{cases}$$

If $C \subset \mathbb{R}^n$ is a nonempty closed convex subset, then δ_C is a proper, lsc, and convex function.

A *subgradient* of a function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ at a point x with $f(x)$ finite is any vector $\xi \in \mathbb{R}^n$ satisfying $f(y) \geq f(x) + \langle \xi, y - x\rangle$ for all y. The set of all subgradients of f at $x \in \mathrm{dom}\, f$ is called the *subdifferential* of f at x and is denoted by $\partial f(x)$, such a set is closed and convex.

The *support function* of a set $C \subset \mathbb{R}^n$, denoted by σ_C, is defined by

$$\sigma_C(x) \doteq \sup\{\langle y, x\rangle : \; y \in C\}.$$

Let $D \subset \mathbb{R}^n$ be a nonempty closed convex set then (see [68, Cor. 8.25])

$$\partial \sigma_D(x) = \arg\max_{y \in D} \langle y, x\rangle.$$

We shall use the following theorem of the alternative.

Lemma 1.1.3 (Farkas) *Let $A \in \mathbb{R}^{n\times m}$ and $c \in \mathbb{R}^n$ be given. Then exactly one of the following two systems has a solution:*
1. $Ax \leq 0$ *and* $\langle c, x\rangle > 0$ *for some* $x \in \mathbb{R}^n$*;*
2. $A^{\mathrm{T}}y = c$ *and* $y \geq 0$ *for some* $y \in \mathbb{R}^m$*.*

1.2 Optimality conditions

Consider the following quadratic programming problem

$$\begin{array}{ll} \text{Minimize} & \langle c, x\rangle + \frac{1}{2}\langle x, Qx\rangle \\ \text{subject to} & Ax \geq b \\ & x \geq 0. \end{array} \tag{QP}$$

where $Q \in \mathbb{R}^{n\times n}$ is symmetric, $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m\times n}$, and $b \in \mathbb{R}^m$. If x is a locally optimal solution of the program (QP), then there exists a vector $y \in \mathbb{R}^m$ such that the pair (x, y) satisfies the *Karush-Kuhn-Tucker conditions* (see [15] for instance)

$$\begin{array}{ll} u = c + Qx - A^{\mathrm{T}}y \geq 0, & x \geq 0,\ \langle x, u\rangle = 0, \\ v = -b + Ax \geq 0, & y \geq 0,\ \langle y, v\rangle = 0. \end{array} \tag{KKT}$$

If, in addition, Q is positive semi-definite, i.e., if the objective function is convex, then the conditions in (KKT) are in fact, sufficient for the vector x to be a globally optimal solution of (QP).

In particular, for $Q = 0$ we get the linear programming problem.

$$\begin{array}{ll} \text{Minimize} & \langle c, x\rangle \\ \text{subject to} & Ax \geq b \\ & x \geq 0 \end{array} \tag{LP}$$

where $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, and $A \in \mathbb{R}^{m\times n}$. A vector $x \in \mathbb{R}^n$ is an optimal solution to (LP) if and only if there exists a vector $y \in \mathbb{R}^m$ such that the pair (x, y) satisfies the Karush-Kuhn-Tucker conditions

$$\begin{array}{ll} u = c - A^{\mathrm{T}}y \geq 0, & x \geq 0,\ \langle x, u\rangle = 0, \\ v = -b + Ax \geq 0, & y \geq 0,\ \langle y, v\rangle = 0. \end{array}$$

1.3 Set-valued analysis

Set convergence

We introduce the following collection of subsets of $\mathbb{N}$:

$$\begin{aligned}
\mathcal{N}_\infty &\doteq \{N \subseteq \mathbb{N} : \mathbb{N} \setminus N \text{ finite}\} \\
&= \{\text{subsequences of } \mathbb{N} \text{ containing all } v \text{ beyond some } \bar{v}\} \\
\mathcal{N}_\infty^{\#} &\doteq \{N \subseteq \mathbb{N} : N \text{ infinite}\} = \{\text{all subsequences of } \mathbb{N}\}.
\end{aligned}$$

For a sequence $\{C^k\}$ of subsets of $\mathbb{R}^n$, the *outer limit* is the set

$$\limsup_{k\to+\infty} C^k \doteq \left\{x : \exists N \in \mathcal{N}_\infty^{\#},\ \exists x^v \in C^v\,(v \in N) \text{ with } x^v \underset{N}{\longrightarrow} x\right\}$$

the *inner limit* is the set

$$\liminf_{k\to+\infty} C^k \doteq \left\{x : \exists N \in \mathcal{N}_\infty,\ \exists x^v \in C^v\,(v \in N) \text{ with } x^v \underset{N}{\longrightarrow} x\right\}$$

The *limit* of the sequence exists if the outer and inner limits sets are equal:

$$\lim_{k\to+\infty} C^k = \limsup_{k\to+\infty} C^k = \liminf_{k\to+\infty} C^k.$$

The above limits can be expressed in terms of distance functions:

$$\liminf_{k\to+\infty} C^k = \left\{x : \limsup_{k\to+\infty} d_{C^k}(x) = 0\right\},\ \limsup_{k\to+\infty} C^k = \left\{x : \liminf_{k\to+\infty} d_{C^k}(x) = 0\right\}$$

where $d_C(x) \doteq d(C, x)$ stands for the distance from x to $C \subseteq \mathbb{R}^n$ (for $C = \emptyset$, we have $d_C(x) = \infty$).

When $\lim_k C^k$ exists and equals C, the sequence $\{C^k\}$ is said to *converge to C in Painvelé-Kuratowski sense*, written $C^k \to C$.

Metric characterization of set convergence

Let $A, B \subseteq \mathbb{R}^n$ be two sets, the *integrated set distance* between them is defined by

$$\mathrm{d\!l}(A, B) := \int_0^\infty \mathrm{d\!l}_\rho(A, B) e^{-\rho} d\rho.$$

where for $\rho \geq 0$,

$$d\!\!I_\rho(A,B) := \max_{\|x\|\leq\rho} |d_A(x) - d_B(x)|.$$

The expression $d\!\!I$ gives a metric on $\mathrm{cl}-\mathrm{sets}_{\neq\emptyset}(\mathbb{R}^n)$-the space of all nonempty closed subsets of $\mathbb{R}^n$, which characterizes ordinary set convergence

$$C^k \to C \iff d\!\!I(C^k, C) \to 0.$$

Another notion of convergence for sets regards the *Pompeiu-Hausdorff* distance which is defined by

$$\begin{aligned} d\!\!I_\infty(A,B) &:= \max_{x\in\mathbb{R}^n} |d_A(x) - d_B(x)| \\ &= \inf\{\eta \geq 0 : C \subseteq D + \eta\mathbb{B}, D \subseteq C + \eta\mathbb{B}\}. \end{aligned}$$

where $A, B \subset \mathbb{R}^n$ are two nonempty and closed sets. The convergence with respect to Pompieu-Hausdorff distance entails the ordinary set convergence $C^k \to C$ and is equivalent to it when there is a bounded set $X \subseteq \mathbb{R}^n$ such that $C^k, C \subseteq X$ for all k. But convergence with respect to Pompieu-Hausdorff distance is not equivalent to ordinary set convergence without this boundedness restriction. Indeed, it is possible to have $C^k \to C$ with $d\!\!I_\infty(C^k, C) \to +\infty$ (even for C, C^k being compact sets) as is shown in [68, p. 118].

The Pompieu-Hausdorff distance is unsuitable for analyzing sequences of unbounded sets or even unbounded sequences of bounded sets. That is why in this thesis we employ the integrated set distance, since we deal with distances between graphics of multifunctions, which are unbounded sets.

Let $C \subseteq \mathbb{R}^n$ be a compact convex set with nonempty interior, using Lemma1.1.2 we prove that there is a sequence $\{C^k\}$ of polytopes such that $d\!\!I_\infty(C^k, C) \to 0$ and by the above remark we obtain $d\!\!I(C^k, C) \to 0$.

Multifunctions

Given a set $C \subset \mathbb{R}^n$, a *multifunction or set-valued map* Φ from C to $\mathbb{R}^m$, denoted by $\Phi : C \rightrightarrows \mathbb{R}^m$, is a mapping that associates to any $x \in C$ a subset $\Phi(x)$

of $\mathbb{R}^m$, called the image of x under Φ. A single-valued mapping $f : C \to \mathbb{R}^m$ can be treated in terms of the multifunction $F : C \rightrightarrows \mathbb{R}^m$ defined by $F(x) = \{f(x)\}$, we simply write $F(x) = f(x)$.
The following sets:

$$\operatorname{dom}\Phi \doteq \{x \in C : \; \Phi(x) \neq \emptyset\}\,,\; \operatorname{gph}\Phi \doteq \{(x, y) \in C \times \mathbb{R}^m : y \in \Phi(x)\}\,,$$

$$\operatorname{rge}\Phi \doteq \cup_{x\in\mathbb{R}^n}\Phi(x),\; \Phi^{-1}(y) \doteq \{x \in \mathbb{R}^n : \; y \in \Phi(x)\}\,.$$

denote the *domain*, the *graph*, the *range* and the *inverse* of Φ respectively. We may consider that $\Phi(x) = \emptyset$ for $x \notin \operatorname{dom}\Phi$.
Minkowski addition and *scalar multiplication* are defined as follows:

$$(\Phi + \Psi)(x) \doteq \Phi(x) + \Psi(x) \text{ for } x \in \operatorname{dom}\Phi \cap \operatorname{dom}\Psi,$$

$$(\lambda\Phi)(x) \doteq \lambda\,\Phi(x) \text{ for } x \in \operatorname{dom}\Phi,\; \lambda \in \mathbb{R}.$$

Classes of multifunctions

A multifunction $\Phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is said to be:

- *compact* (*resp. convex, closed, bounded*) *valued* if for each $x \in \operatorname{dom}\Phi$, $\Phi(x)$ is compact (resp. convex, closed, bounded);
- *lower semicontinuous* (lsc) if for any $x \in \operatorname{dom}\Phi$, $y \in \Phi(x)$ and any sequence $\{x^k\} \subseteq \operatorname{dom}\Phi$ converging to x, there exists a sequence $\{y^k\}$ such that $y^k \in \Phi(x^k)$ and $y^k \to y$;
- *upper semicontinuous* (usc) if for any $x \in \operatorname{dom}\Phi$ and any open set $V \subset \mathbb{R}^n$ containing $\Phi(x)$, there is a neighborhood U of x such that $\Phi(U) \subseteq V$;
- a *cusco* if it is usc and compact convex valued;
- *locally upper Lipschitzian* at $\bar{x} \in \operatorname{dom}\Phi$ *with modulus* λ (UL(λ)) if there is a neighborhood U of $\bar{x}$ such that $\Phi(x) \subseteq \Phi(\bar{x}) + \lambda\,\|x - \bar{x}\|\,\mathbb{B}$ for all $x \in U$.

- *sequentially bounded* if for any bounded sequence $\{x^k\} \subseteq \operatorname{dom}\Phi$, it follows that any sequence $\{y^k\}$ with $y^k \in \Phi(x^k)$ for all k, is bounded;
- *superadditive* if $\Phi(x) + \Phi(y) \subseteq \Phi(x+y)$ for all $x, y \in \operatorname{dom}\Phi$;
- *uniformly bounded* if there exists a bounded set C such that $\Phi(x) \subseteq C$ for all $x \in \operatorname{dom}\Phi$;
- *graph-convex* (resp. *graph-closed*) if its graph is convex (resp. closed).
- $\mathbb{R}^n_+$-*convex* if $t\Phi(x)+(1-t)\Phi(y) \in \Phi(tx+(1-t)y)+\mathbb{R}^n_+$ for all $x, y \in \operatorname{dom}\Phi$, $t \in]0,1[$.

In this thesis, we use repeatedly the following property which is related to the upper semicontinuity property of multifunctions.

Proposition 1.3.1 [3, Prop. 2,3] *Let X and Y be two Hausdorff topological spaces, and $\Phi : X \rightrightarrows Y$ be a multifunction.*

(a) *If Φ is usc and closed-valued, then it is graph-closed;*

(b) *If Φ is usc and compact-valued from the compact space X to Y, then $\Phi(X)$ is compact.*

Generalized equations and graphical convergence

For a sequence of mappings $\Phi^k : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, the *graphical outer limit* (resp. *graphical inner limit)*, denoted by $g\text{-}\limsup_k \Phi^k$ (resp. $g\text{-}\liminf_k \Phi^k$), is the mapping with graph $\limsup_k(\operatorname{gph}\Phi^k)$ (resp. $\liminf_k(\operatorname{gph}\Phi^k)$). If these outer and inner limits agree, the *graphical limit* $g\text{-}\lim_k \Phi^k$ exists. In this case the notation $\Phi^k \xrightarrow{g} \Phi$ is used.

Graphical convergence of mappings can be characterized with the metric of set convergence. Indeed, on the set of all cuscos we consider the metric (that we shall denote also by $\mathrm{d\!I}$)

$$\mathrm{d\!I}(\Phi^1, \Phi^2) \doteq \mathrm{d\!I}(\mathrm{gph}\,\Phi^1, \mathrm{gph}\,\Phi^2),$$

such a metric characterizes the graphical convergence "$\xrightarrow{g}$", i.e.

$$\Phi^k \xrightarrow{g} \Phi \iff \mathrm{d\!I}(\Phi^k, \Phi) \to 0,$$

In our analysis we employ repeatedly the following two properties of the graphical convergence [68, Ex. 5.34(b), Th. 5.37].

Theorem 1.3.2 (Uniformity in graphical convergence) *Let the mappings $\Phi, \Phi^k : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$. Suppose the mappings Φ^k are connected-valued (e.g. convex-valued). If $g\text{-}\limsup_k \Phi^k = \Phi$, and if $\Phi(\bar{x})$ is nonempty and bounded, then there exist $k_0 \in \mathbb{N}$, a neighborhood V of $\bar{x}$ and a bounded set B such that*

$$\Phi(x) \subseteq B \ \ \textit{and} \ \ \Phi^k(x) \subset B \ \ \forall x \in V, \ \ k \geq k_0.$$

Theorem 1.3.3 (Generalized equations) *For two closed-valued mappings Φ, $\Phi^k : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ and two vectors $\bar{u}, \bar{u}^k \in \mathbb{R}^m$, consider the generalized equation $\bar{u}^k \in \Phi^k(x)$ as an approximation to the generalized equation $\bar{u} \in \Phi(x)$, the respective solutions being $(\Phi^k)^{-1}(\bar{u}^k)$ and $\Phi^{-1}(\bar{u})$, with the elements of the former referred to as approximate solutions and the elements of the latter as true solutions.*

(a) *As long as $g\text{-}\limsup_k \Phi^k \subseteq \Phi$, one has that for every choice of $\bar{u}^k \to \bar{u}$ that $\limsup_k (\Phi^k)^{-1}(\bar{u}^k) \subseteq \Phi^{-1}(\bar{u})$. Thus, any cluster point of a sequence of approximate solutions is a true solution.*

(b) *If $g\text{-}\liminf_k \Phi^k \supseteq \Phi$, one has $\Phi^{-1}(\bar{u}) \subseteq \bigcap_\varepsilon \liminf_k (\Phi^k)^{-1}(\mathcal{B}(\bar{u}, \varepsilon))$. In this case, therefore, every true solution is the limit of approximate solutions corresponding to some choice of $\bar{u}^k \to \bar{u}$.*

(c) *When $\Phi^k \xrightarrow{g} \Phi$, both conclusions hold.*

A sequence of matrices $\{M^k\} \subseteq \mathbb{R}^{n\times n}$ is said to converge to a matrix M if and only if its components converge $a_{ij}^k \to a_{ij}$ for all i and j, or equivalently if $\|M^k - M\| \to 0$. Let $\Phi^k(x) = M^k x$ and $\Phi(x) = Mx$ with $M^k, M \in \mathbb{R}^{n\times n}$. For such mappings $\Phi^k \xrightarrow{g} \Phi$ if and only if $\|M^k - M\| \to 0$.

1.4 Variational inequalities

In the existence theory for the complementarity problem, we employ the following well-known existence theorems for variational inequalities (see [50, 69] for instance).

Theorem 1.4.1 (Hartman-Stampacchia) *Let $K \subseteq \mathbb{R}^n$ be a compact convex set and let $f : K \to \mathbb{R}^n$ be continuous. Then there exists an $\bar{x} \in K$ such that*

$$\langle f(\bar{x}), x - \bar{x}\rangle \geq 0 \quad \forall x \in K.$$

It is worth mentioning that under the above assumptions for f, if K is unbounded, the variational inequality problem does not always admit a solution, take for instance $K = \mathbb{R}$ and $f(x) = e^x$.

Theorem 1.4.2 (Saigal) *Let $K \subseteq \mathbb{R}^n$ be a compact convex set and let $\Phi : K \rightrightarrows \mathbb{R}^n$ be an usc compact contractible (in particular convex) valued multifunction. Then there exists an $\bar{x} \in K$ and $\bar{y} \in \Phi(\bar{x})$ such that*

$$\langle \bar{y}, x - \bar{x}\rangle \geq 0 \quad \forall x \in K.$$

1.5 Asymptotic cones

Given a nonempty set $C \subseteq \mathbb{R}^n$ and a vector $d > 0$. We define the *d-normalized asymptotic cone* of C as the set

$$C_d^\infty \doteq \left\{ v \in \mathbb{R}^n :\ \exists\, x^k \in C,\ ||x^k||_d \to +\infty,\ \frac{x^k}{||x^k||_d} \to v \right\},$$

and the *asymptotic cone* of C as the set

$$C^\infty \doteq \left\{ v \in \mathbb{R}^n : \exists\, x^k \in C,\ t_k \downarrow 0,\ t_k x^k \to v \right\}.$$

The ``*recession*´´ term instead of asymptotic is employed when convex sets are considered.

Now, we list some properties of the asymptotic cones (see [7, 29, 68] for instance):

(a) C is a closed cone if and only if $C^\infty = C$;

(b) $C^\infty = \operatorname{pos} C_d^\infty$ (where by convention $\operatorname{pos} \emptyset = \{0\}$);

(c) $C^\infty = \{0\}$ if and only if C is bounded, or equivalently if $C_d^\infty = \emptyset$;

(d) $C^\infty = (\overline{C})^\infty$;

(e) $C_1 \subseteq C_2$ implies $C_1^\infty \subseteq C_2^\infty$;

(f) Let C be a non-empty closed convex set, $x_0 \in C$. Then,

$$C^\infty = \Big\{ u \in \mathbb{R}^n : \ x_0 + tu \in C \quad \forall\, t > 0 \Big\}.$$

Such a cone is independent on x_0;

(g) Let $\{C_i\}$, $i \in \mathcal{I}$, be any family of non-empty sets in $\mathbb{R}^n$, then

$$\Big(\bigcap_{i \in \mathcal{I}} C_i \Big)^\infty \subseteq \bigcap_{i \in \mathcal{I}} C_i^\infty.$$

If, in addition, each C_i is closed and convex and $\bigcap_{i \in \mathcal{I}} C_i \neq \emptyset$, then

$$\Big(\bigcap_{i \in \mathcal{I}} C_i \Big)^\infty = \bigcap_{i \in \mathcal{I}} C_i^\infty.$$

Chapter 2

The multivalued complementarity problem

A great variety of problems arising in most applications in Sciences and Engineering have the same mathematical formulation known as a *multivalued complementarity problem* which may be stated as follows: given a multifunction $\Phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ such that $\mathbb{R}^n_+ \subseteq \operatorname{dom} \Phi$, and a vector $q \in \mathbb{R}^n$, it is requested to

$$\text{find } \bar{x} \geq 0,\ \bar{y} \in \Phi(\bar{x}) \ \text{ such that } \ \bar{y} + q \geq 0,\ \langle \bar{y} + q, \bar{x} \rangle = 0. \tag{MCP}$$

This problem denoted by $\mathrm{MCP}(q, \Phi)$ generalizes substantially the so-called linear complementarity problem largely studied since 1958 (see [15, p. 218]), where Φ is assumed to be a linear mapping.

Problem (MCP) is known to be equivalent to the following *multivalued variational inequality problem* $\mathrm{MVI}(\mathbb{R}^n_+, \Phi + q)$:

$$\text{find } \bar{x} \geq 0,\ \bar{y} \in \Phi(\bar{x}) \ \text{ such that } \ \langle \bar{y} + q, x - \bar{x} \rangle \geq 0 \ \ \forall\, x \geq 0. \tag{MVIP}$$

In this chapter we present a method which allows us to develop a general theory yielding new existence and sensitivity results and unifying the ones found in the literature. Our method is based on the asymptotic description of a sequence of approximate solutions to (MVIP). Thus, problems possibly allowing an unbounded solution set are also treated. Another advantage of our approach is that all requirements arise in a natural way. Several examples are discussed illustrating the wide applicability of our results. We follow the line of reasoning carried out in [27, 29].

We denote by $\mathcal{S}(q,\Phi)$ the solution set of $\mathrm{MCP}(q,\Phi)$ and by

$$\mathcal{D}(\Phi) \doteq \left\{q \in \mathbb{R}^n:\ q \in w - \Phi(x),\ (x,w) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+,\ \langle w, x\rangle = 0\right\}$$

the set of vectors for which $\mathrm{MCP}(q,\Phi)$ has solutions. More precisely,

$$q \in \mathcal{D}(\Phi) \Longleftrightarrow \mathcal{S}(q,\Phi) \neq \emptyset \Longleftrightarrow \Phi \in \mathcal{D}^{-1}(q).$$

Here $\mathcal{D}^{-1}$ denotes the inverse multifunction of $\mathcal{D}$. Moreover, the sets

$$\mathcal{F}\ (q,\Phi) \doteq \{x \geq 0 : y \in \Phi(x),\ y + q \geq 0\},$$

$$\mathcal{F}_s(q,\Phi) \doteq \{x \geq 0 : y \in \Phi(x),\ y + q > 0\}$$

denote the *feasible* and *strict feasible sets* of $\mathrm{MCP}(q,\Phi)$ respectively.

2.1 Definitions and preliminaries

Here and in the subsequent sections we shall deal with *multifunctions* $\Phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ such that $\operatorname{dom}\Phi = \mathbb{R}^n_+$, we shall assume that $\Phi(x) = \emptyset$ for all $x \notin \operatorname{dom}\Phi$. We denote such multifunctions by $\Phi : \mathbb{R}^n_+ \rightrightarrows \mathbb{R}^n$.

Before introducing our main classes of multifunctions, we need the following notation:

$$\mathcal{X} \doteq \{\Phi : \mathbb{R}^n_+ \rightrightarrows \mathbb{R}^n :\ \Phi \text{ is a cusco}\}.$$

$$\mathcal{C} \doteq \{c : \mathbb{R}_{++} \to \mathbb{R}_{++},\ c(0) \geq 0,\ \lim_{t\to+\infty} c(t) = +\infty\}.$$

Definition 2.1.1 *For $c \in \mathcal{C}$, $d > 0$, the mapping $\Phi : \mathbb{R}^n_+ \rightrightarrows \mathbb{R}^n$ such that $0 \in \Phi(0)$, is said to be*

- c-homogeneous (*on* Δ_d) *if* $\Phi(\lambda x) = c(\lambda)\Phi(x) \;\; \forall x \in \Delta_d,\; \lambda > 0$*;*

- c-subhomogeneous (*on* Δ_d) *if* $\Phi(\lambda x) \subseteq c(\lambda)\Phi(x) \;\; \forall x \in \Delta_d,\; \lambda > 0$*;*

- zero-subhomogeneous (*on* Δ_d) *if* $\Phi(\lambda x) \subseteq \Phi(x) \;\; \forall x \in \Delta_d,\; \lambda > 0$*;*

- c-Moré (*on* Δ_d) *if* $\forall \lambda \geq 1,\; x \in \Delta_d,\; y \in \Phi(\lambda x) \;\; \exists z \in \Phi(x)$ *such that* $\langle y, x\rangle \geq c(\lambda)\langle z, x\rangle$.

We have to point out that our notion of (sub)homogeneity lies on the compact set Δ_d in place of the standard requirement lying on $\mathbb{R}^n_+$. We can weaken this definitions, by considering that the equalities and inclusions hold for all $\lambda \geq r$ for some $r \geq 1$, without changing most of the results. However, in order to present a unified theory we restrict us to the above definitions, unless otherwise is stated.

Example 2.1.2 **1.**[41] *A (positively) homogeneous multifunction* Φ *of degree* $\gamma > 0$*, i.e. such that*

$$\Phi(\lambda x) = \lambda^\gamma \Phi(x) \text{ for all } x \geq 0 \text{ and } \lambda > 0,$$

is λ^γ*-homogeneous on* Δ_d *for any* $d > 0$ *provided* $0 \in \Phi(0)$*. The multifunctions* $\Phi_1(x) = Mx$*, where* $M \in \mathbb{R}^{n\times n}$*;* $\Phi_2(x) = (f_1(x), \ldots, f_n(x))^T$*, where* $f_i(x) = \max\{\langle w_{ij}, x\rangle : j \in \Lambda_i\}$ *with* $w_{ij} \in \mathbb{R}^n$ *and* Λ_i *being a finite index set;* $\Phi_3(x) = \{My : Ax + Qy \leq 0\}$*, where* $M \in \mathbb{R}^{n\times n}$ *and* $A, Q \in \mathbb{R}^{m\times n}$*, are all homogeneous of degree 1. The mapping* $\Phi_4(x) = \|x\|\, Mx$ *is homogeneous of degree 2.*
2.[72] *A (positively) generalized homogeneous multifunction* Φ*, i.e. such that for some* $c \in \mathcal{C}$*,*

$$\Phi(\lambda x) = c(\lambda)\Phi(x) \text{ for all } x \geq 0 \text{ and } \lambda > 0,$$

is c*-homogeneous on* Δ_d *for any* $d > 0$ *provided* $0 \in \Phi(0)$*.*

3. *The mappings* $\Phi_5(x) = [x + x^2, 2x + 2x^2]$ *and* $\Phi_6(x) = [e^x - 1, 2e^x - 2]$ *are* c*-homogeneous on* Δ_1 *with* $c(\lambda) = \frac{\lambda+\lambda^2}{2}$ *and* $c(\lambda) = \frac{e^\lambda - 1}{e-1}$ *respectively, and if*

$$\Phi_7(x) = \begin{cases} [0,1], & \text{if } 0 \leq x \leq 1; \\ [x-1, x], & \text{if } x > 1. \end{cases} \quad \text{and} \quad \bar{c}(\lambda) = \begin{cases} 1, & \text{if } 0 \leq \lambda \leq 1; \\ \lambda, & \text{if } \lambda > 1. \end{cases},$$

then Φ_7 *is* $\bar{c}$*-subhomogeneous on* Δ_1 *but not* c*-homogeneous for any* $c \in \mathcal{C}$.

4.[57, 58, 72] *Let* $f : \mathbb{R}^n_+ \to \mathbb{R}^n$ *and* $c \in \mathcal{C}$ *such that*

$$\langle x, f(\lambda x) - f(0)\rangle \geq c(\lambda)\langle x, f(x) - f(0)\rangle \text{ for all } x \geq 0 \text{ and } \lambda \geq 1,$$

The mapping $\Phi(x) = f(x) - f(0)$ *is* c*-Moré on* Δ_d *for any* $d > 0$. *In particular, if* $f : \mathbb{R}_+ \to \mathbb{R}$ *is convex then* Φ *is* $\hat{c}$*-Moré for* $\hat{c}(\lambda) = \lambda$. *In connection with this result see (d) of Proposition 2.1.5.*

5. *The mapping* $\Phi_8(x) = [x, 2x]$ *is* $\frac{\lambda}{2}$*-Moré on* Δ_1.

6. *The mapping* $\Phi_9(x) = [0, 1/\|x\|_d]$ *if* $\|x\|_d \geq 1$ *and* $\Phi_9(x) = [0, \|x\|_d]$ *if* $\|x\|_d \leq 1$ *is zero-subhomogeneous on* Δ_d *for* $d > 0$.

7.[41] *A (positively) homogeneous multifunction* Φ *of degree 0, i.e. such that*

$$\Phi(\lambda x) = \Phi(x) \text{ for all } x \geq 0 \text{ and } \lambda > 0,$$

is zero-subhomogeneous on Δ_d *for any* $d > 0$ *provided* $0 \in \Phi(0)$. *For instance,* $\Phi_{10}(x) = \partial h(x)$ *where* $h(x) = \sup_{y \in C}\langle x, y\rangle$ *for* $C \subseteq \mathbb{R}^n$ *a nonempty compact convex set such that* $0 \in C$.

Proposition 2.1.3 *Let* $c \in \mathcal{C}$ *and* $\Phi : \mathbb{R}^n_+ \rightrightarrows \mathbb{R}^n$ *be a multifunction.*

(a) *If* Φ *is usc with compact values, then it is sequentially bounded and graph-closed;*

(b) *If* Φ *is a zero-subhomogeneous cusco, then it is uniformly bounded;*

(c) *If* Φ *is either* c*-homogeneous with* $c(\mathbb{R}_+) = \mathbb{R}_+$, $\Phi(0) = \{0\}$ *and superadditive or simply graph-convex, then the set* $\Phi(\mathbb{R}^n_+)$ *is convex;*

(d) *if* Φ *is* c*-homogeneous such that* $\Phi(0)$ *is bounded, then* $\Phi(0) = \{0\}$.

Proof. (a): It follows from Proposition 1.3.1
(b): For $0 \neq x \geq 0$, we have $\Phi(x) = \Phi(\|x\|_d \frac{x}{\|x\|_d}) \subseteq \Phi(\frac{x}{\|x\|_d}) \subseteq \Phi(\Delta_d)$, which implies the desired result since $\Phi(\Delta_d)$ is compact by Proposition 1.3.1.
(c) and (d) are straightforward. ■

Notice that within cuscos, uniformly bounded does not imply zero-subhomogeneity as the function $\Phi(x) = 1/(1+x)$ shows.

The (nonlinear) multivalued version of the classes of mappings introduced in the study of linear complementarity problems (see [15, 30] for example) arise in a natural way in the present setting. We now recall some of them. Let $d > 0$ and $\Phi : \mathbb{R}^n_+ \rightrightarrows \mathbb{R}^n$ be a multifunction. We say that Φ is:

- *copositive* if $\langle y, x\rangle \geq 0 \quad \forall\, (x, y) \in \operatorname{gph}\Phi$;
- *strictly copositive* if $\langle y, x\rangle > 0 \quad \forall\, (x, y) \in \operatorname{gph}\Phi$ with $x \neq 0$;
- *strongly copositive* if $\exists\, \alpha > 0$ such that $\langle y, x\rangle \geq \alpha\, \|x\|^2 \quad \forall\, (x, y) \in \operatorname{gph}\Phi$;
- *semimonotone* if $\mathcal{S}(p, \Phi) = \{0\} \quad \forall\, p > 0$;
- a $\mathbf{R}(d)$-*mapping*, or $\Phi \in \mathbf{R}(d)$, if $\mathcal{S}(\tau d, \Phi) = \{0\} \quad \forall\, \tau \geq 0$;
- a $\mathbf{G}(d)$-*mapping*, or $\Phi \in \mathbf{G}(d)$, if $\mathcal{S}(\tau d, \Phi) = \{0\} \quad \forall\, \tau > 0$;
- *monotone* if $\langle y^1 - y^2, x^1 - x^2\rangle \geq 0 \ \ \forall (x^1, y^1), (x^2, y^2) \in \operatorname{gph}\Phi$;
- *q-pseudomonotone* if $\langle y^1 + q, x^2 - x^1\rangle \geq 0 \Rightarrow \langle y^2 + q, x^2 - x^1\rangle \geq 0\; \forall (x^1, y^1)$, $(x^2, y^2) \in \operatorname{gph}\Phi$,

The following definition generalizes that for linear mappings used in [43].

Definition 2.1.4 *For $d > 0$ and $\Phi \in \mathcal{X}$. The d-*numerical range *of Φ is by definition, the set* $\omega(\Phi) \doteq \{\langle y, x\rangle :\ x \in \Delta_d,\ y \in \Phi(x)\}$.

We denote $M_\Phi \doteq \sup \omega(\Phi) < +\infty$ and $m_\Phi \doteq \inf \omega(\Phi) > -\infty$.

Proposition 2.1.5 *Let $d > 0$, $c \in \mathcal{C}$, and $\Phi \in \mathcal{X}$.*

(a) *If Φ is c-subhomogeneous and $m_\Phi > 0$ (in particular if Φ is strictly copositive), then Φ is $\bar{c}$-Moré for $\bar{c}(\lambda) = \frac{m_\Phi}{M_\Phi} c(\lambda)$;*

(b) *If Φ is copositive c-subhomogeneous and the following implication holds: $(v \geq 0,\ w \in \Phi(v),\ \langle w, v\rangle = 0 \Longrightarrow v = 0)$, then Φ is $\bar{c}$-Moré;*

(c) *If Φ is strongly copositive and $0 \in \Phi(0)$, then it is $\tilde{c}$-Moré for $\tilde{c}(\lambda) = \frac{\alpha\lambda}{M_\Phi \|d\|^2}$.*

(d) *If Φ is $\mathbb{R}^n_+$-convex and $0 \in \Phi(0)$, then it is $\hat{c}$-Moré for $\hat{c}(\lambda) = \lambda$.*

Proof. (a): Let $\lambda \geq 1$, $x \in \Delta_d$, and $y \in \Phi(\lambda x)$ be given, by hypothesis $\frac{1}{c(\lambda)} y \in \Phi(x)$ and for any $z \in \Phi(x)$ we get $\langle \frac{1}{c(\lambda)} y, x\rangle \geq m_\Phi \geq \frac{m_\Phi}{M_\Phi}\langle z, x\rangle$. Thus, setting $\bar{c}(\lambda) = \frac{m_\Phi}{M_\Phi} c(\lambda)$ we obtain the desired result.
(b): Since Φ is copositive, $m_\Phi \geq 0$. Suppose that $m_\Phi = 0$, then there exist $x \in \Delta_d$ and $y \in \Phi(x)$ such that $m_\Phi = \langle y, x\rangle = 0$, contradicting the hypothesis. The result follows from (a).
(c): Let $\lambda \geq 1$ and $x \in \Delta_d$, then $\|x\| \geq \frac{1}{\|d\|}$ and if $y \in \Phi(\lambda x)$, then $\langle y, \lambda x\rangle \geq \alpha \|\lambda x\|^2$ for some $\alpha > 0$, and thus $\langle y, x\rangle \geq \frac{\alpha\lambda}{\|d\|^2}$. Clearly if $z \in \Phi(x)$ then $\frac{1}{M_\Phi}\langle z, x\rangle \leq 1$. Therefore $\langle y, x\rangle \geq \frac{\alpha\lambda}{\|d\|^2} \geq \tilde{c}(\lambda)\langle z, x\rangle$.
(d): Let $\lambda \geq 1$ and $x \in \Delta_d$, by definition $\frac{1}{\lambda}\Phi(\lambda x) + (1 - \frac{1}{\lambda})\Phi(0) \subseteq \Phi(x) + \mathbb{R}^n_+$, since $0 \in \Phi(0)$ we conclude that $\frac{1}{\lambda}\Phi(\lambda x) \subseteq \Phi(x) + \mathbb{R}^n_+$, thus, if $y \in \Phi(\lambda x)$ then there exists $z \in \Phi(x)$ such that $\frac{1}{\lambda} y \geq z$. Thus, $\langle y, x\rangle \geq \hat{c}(\lambda)\langle z, x\rangle$. ■

2.2 Asymptotic analysis and existence theorem

We approximate problem (MVIP), which is the variational inequality formulation to (MCP), by the following sequence of problems

$$\text{find } x^k \in D_k,\ y^k \in \Phi(x^k):\ \langle y^k + q, x - x^k\rangle \geq 0\ \forall x \in D_k. \qquad (\mathrm{MVIP_k})$$

where $d > 0$, $\{\sigma_k\}$ is an increasing sequence of positive numbers converging to $+\infty$, and

$$D_k = \left\{x \in \mathbb{R}^n_+ : \langle d, x\rangle \le \sigma_k\right\}.$$

If Φ is a cusco, the existence of $(x^k, y^k) \in \mathrm{gph}\,\Phi$ satisfying $(\mathrm{MVIP_k})$ is guaranteed by Theorem 1.4.2.

It is clear that (x^k, y^k) solves $(\mathrm{MVIP_k})$ if and only if x^k is an optimal solution of the linear program

$$\inf_x [\langle y^k + q, x\rangle : \ x \ge 0, \ \langle d, x\rangle \le \sigma_k]. \tag{P}$$

Applying usual optimality conditions we conclude that (x^k, y^k) solves $(\mathrm{MVIP_k})$ if and only if there exists $\theta_k \in \mathbb{R}$ such that (x^k, y^k, θ_k) solves the so-called *augmented multivalued complementarity problem*

$$\begin{gathered}\text{find } \ x^k \ge 0, \ \theta_k \ge 0, \ y^k \in \Phi(x^k) \ \text{ such that} \\ y^k + q + \theta_k d \ge 0, \ \ \langle d, x^k\rangle \le \sigma_k, \\ \langle y^k + q + \theta_k d, x^k\rangle = 0, \ \ \theta_k(\sigma_k - \langle d, x^k\rangle) = 0.\end{gathered} \tag{$\mathrm{MCP_k}$}$$

In particular,

$$x^k \in \mathcal{S}(q + \theta_k d, \Phi) \text{ and } x^k \in \mathcal{S}(q, \Phi + \theta_k d). \tag{2.1}$$

Clearly, we observe that

$$\langle d, x^k\rangle < \sigma_k \implies \theta_k = 0 \implies x^k \in \mathcal{S}(q, \Phi).$$

This line of reasoning was also applied in [66].

We introduce the following definition: a subset M of a metric space X is said to be *closed at* x, if whenever a sequence $\{x^k\} \subseteq M$ converges to x, one has $x \in M$. Obviously, if M is closed then M is closed at every point $x \in M$.

The next theorem was established in [41]. We reformulate it in terms of the above definition.

Theorem 2.2.1 *Let $d > 0$, $\{\sigma_k\}$ be an increasing sequence of positive numbers converging to $+\infty$, $\Phi \in \mathcal{X}$, and $\{(x^k, y^k, \theta_k)\}$ be a sequence of solutions to problem* $(\mathrm{MCP_k})$. *Assume $\liminf_{k\to+\infty} \theta_k = 0$. Then, the following assertions are equivalent:*

(a) *$\mathcal{S}(q, \Phi)$ is nonempty;*

(b) *$\mathcal{D}(\Phi)$ is closed at q.*

Proof. (a)⇒(b): It is obvious.
(b)⇒(a): Without loss of generality we may assume that $\theta_k \to 0$. By (2.1), we conclude that $\mathcal{S}(q + \theta_k d, \Phi)$ is nonempty, thus $q + \theta_k d \in \mathcal{D}(\Phi)$. By hypothesis $q \in \mathcal{D}(\Phi)$, thus $\mathcal{S}(q, \Phi)$ is nonempty. ■

An important class of multifunctions Φ for which $\mathcal{D}(\Phi)$ is closed at every q is that of polyhedral ones (see Chapter 3). However, we look for new classes of multifunctions such that $\mathcal{D}(\Phi)$ is closed at some particular q. These classes are introduced in Section 2.3.

Remark 2.2.2 *As pointed above, if $\theta_k = 0$ for some k, then $\mathcal{S}(q, \Phi)$ is nonempty. Theorem 2.2.1 yields existence of solutions when $\theta_k > 0$ for all k. Indeed, let $\Phi(x_1, x_2) = [-x_1, x_1]\times[-x_2, x_2]$, $d = (1, 1)^{\mathrm{T}}$, $\sigma_k = k$, and $q = (0, -1)^{\mathrm{T}}$. We get that $\{(x^k, y^k, \theta_k)\}$ solves $(\mathrm{MCP_k})$ for $x^k = (0, k)^{\mathrm{T}}$, $y^k = (0, 1 - \frac{1}{k})^{\mathrm{T}}$, and $\theta_k = \frac{1}{k} > 0$. Since $\liminf_{k\to+\infty} \theta_k = 0$ and $\mathcal{D}(\Phi)$ is closed being Φ polyhedral (see* [41, Prop. 3]*), the above theorem asserts that $\mathcal{S}(q, \Phi)$ is nonempty.*

In our opinion, Theorem 2.2.1 was established only by taking $x^k \in \mathcal{S}(q + \theta_k d, \Phi)$ into account in (2.1); in this respect the closedness of $\mathcal{D}(\Phi)$ at q plays a certain role since $q + \theta^k d \in \mathcal{D}(\Phi)$ and, $\mathcal{S}(q, \Phi) \neq \emptyset \iff q \in \mathcal{D}(\Phi)$. However, if instead we look at $x^k \in \mathcal{S}(q, \Phi + \theta_k d)$ in (2.1), we have to analize the closedness of $\mathcal{D}^{-1}(q)$ relative to some particular class of approximating mappings. Thus, we first need a good notion of convergence for multifunctions, and secondly, to find the particular approximating mappings. Just to give an idea to be developed presently, we observe that the c-subhomogeneity of Φ does not imply the c-subhomegeneity of $\Phi + \theta_k d$; however, the multifunction Ψ^k given by $\Psi^k(x) = \theta_k d$ is copositive and uniformly bounded whenever $\theta_k \to 0$. On the other hand, in order to discuss sensitivity and stability results we also substitute Φ by an approximating mapping Φ^k. These considerations give rise to the notion of *approximable mappings* to be discussed in Section 2.6.

The next (Basic) lemma describes the asymptotic behavior of the corresponding solutions to the problems associated to the approximating mappings (see $(\mathrm{PMVIP_k})$ below). Existence of solutions will require further assumptions on the original and/or the approximating mappings. The latter is also analyzed in Section 2.6.

Lemma 2.2.3 (Basic Lemma) *Let $d > 0$, $\{\sigma_k\}$ be an increasing sequence of positive numbers converging to $+\infty$; $q, q^k \in \mathbb{R}^n$; $\Phi, \Psi, \Phi^k, \Psi^k \in \mathcal{X}$ be such that $\Phi^k \xrightarrow{g} \Phi$, $\Psi^k \xrightarrow{g} \Psi$, $q^k \to q$ and $\{(x^k, y^k, r^k)\}$ be a sequence of solutions to*

$$\textit{find } x^k \in D_k: \; y^k \in \Phi^k(x^k), \; r^k \in \Psi^k(x^k), \; \langle y^k + r^k + q^k, x - x^k\rangle \geq 0 \; \forall x \in D_k. \tag{$\mathrm{PMVIP_k}$}$$

such that $\langle d, x^k\rangle = \sigma_k$ and $\frac{x^k}{\sigma_k} \to v$. Then, there exist subsequences $\{(x^{k_m}, y^{k_m}, r^{k_m})\}$, $\{\sigma_{k_m}\}$, numbers k_0, $m_0 \in \mathbb{N}$, and an index set $\emptyset \neq J_v \subseteq I$ such that

(a) *for all $k \geq k_0$, $x^k - \frac{\sigma_k}{2}v \geq 0$ and $0 < ||x^k - \frac{\sigma_k}{2}v||_d < \sigma_k$;*

(b) *for all $m \geq m_0$, $\frac{x^{k_m}}{\sigma_{k_m}} \in \mathrm{ri}(\Delta_{J_v})$, thus $\mathrm{supp}\{x^{k_m}\} = J_v$, hence $\mathrm{supp}\{v\} \subseteq J_v$;*

(c) *for all $m \geq m_0$, $z \in \Delta_{J_v}$: $\langle y^{k_m} + r^{k_m} + q^{k_m}, \sigma_{k_m} z - x^{k_m}\rangle = 0$.*

Moreover,

(d) *if each Φ^k is c-subhomogeneous and each Ψ^k is uniformly bounded with respect to the same set, then the subsequences $\{y^{k_m}\}$, $\{r^{k_m}\}$, $\{\sigma_{k_m}\}$ may be chosen in such a way that there are vectors w and r such that $\frac{1}{c(\sigma_{k_m})}y^{k_m} \to w \in \Phi(v)$, $r^{k_m} \to r$, $\langle w, v\rangle \leq 0$, $\langle w, y\rangle \geq \langle d, y\rangle\langle w, v\rangle$ for all $y \geq 0$, and $\langle w, z\rangle = \langle w, v\rangle$, for all $z \in \Delta_{J_v}$;*

(e) *if each Φ^k is c-Moré and each Ψ^k is uniformly bounded with respect to the same set, then there exist vectors w, r and sequences $\{w^k\}$ and $\{r^{k_m}\}$ such that $w^k \in \Phi^k(\frac{x^k}{\sigma_k})$, $w^{k_m} \to w \in \Phi(v)$, $r^{k_m} \to r$, and $\langle w, v\rangle \leq 0$;*

(f) *if each Φ^k is q^k-pseudomonotone and each $\Psi^k = 0$, then $v \in \mathbb{R}^n_+ \cap [-\Phi(\mathbb{R}^n_+) - q]^*$. Hence, $0 \leq v \in -[\Phi(\mathbb{R}^n_+)]^*$ and $\langle q, v\rangle \leq 0$ provided Φ is c-homogeneous and $\Phi(0) = \{0\}$ as well.*

(g) *if each Φ^k is monotone and each Ψ^k is uniformly bounded with respect to the same set and copositive, then $v \in \mathbb{R}^n_+ \cap [-\Phi(\mathbb{R}^n_+) - q]^*$.*

Proof. By Theorems 5.19 and 5.51(b) from [68], $\Phi^k + \Psi^k \in \mathcal{X}$, and problem $(\mathrm{PMVIP_k})$ has solutions by Theorem 1.4.2.
(a): As $\frac{1}{\sigma_k}x^k \to v$, for $\varepsilon = \min\{\frac{v_i}{2} : v_i > 0\} > 0$ there exists k_0 such that for all $k \geq k_0$, $\sum_{i=1}^n |\frac{x_i^k}{\sigma_k} - v_i| < \varepsilon$. This implies $\frac{v_i}{2} < \frac{x_i^k}{\sigma_k}$ for $i \in \operatorname{supp}\{v\}$. Thus $0 \neq x^k - \frac{\sigma_k}{2}v \geq 0$, and then (a) holds.
(b): Clearly $\Delta_d = \Delta_I = \operatorname{co}\{\frac{1}{d_i}e_i : i \in I\}$ may be written as the disjoint union of the relative interior of its extreme faces. More precisely, if we denote its extreme faces by $\Delta_{J_1}, \Delta_{J_2}, \ldots, \Delta_{J_{2^n-1}}$, then

$$\Delta_d = \bigcup_{i=1}^{2^n-1} \operatorname{ri}(\Delta_{J_i}).$$

As $\frac{1}{\sigma_k}x^k \in \Delta_d$, $k \in \mathbb{N}$, there exist an $i_0 \in \{1, 2, \ldots, 2^n-1\}$, m_0, and a subsequence $\{x^{k_m}\}$ such that $\frac{1}{\sigma_{k_m}}x^{k_m} \in \operatorname{ri}(\Delta_{J_{i_0}})$ for all $m \geq m_0$. By setting $J_v \doteq J_{i_0}$, one obtains $\operatorname{supp}\{x^{k_m}\} = J_v$ and $\operatorname{supp}\{v\} \subseteq J_v$.
(c): We analyze two cases, whether J_v is a singleton or not. In the first case, we have $\frac{1}{\sigma_{k_m}}x^{k_m} = v$ for all $m \geq m_0$ because of $\operatorname{ri}(\Delta_{J_v}) = \Delta_{J_v}$, and therefore (c) obviously holds. In the second case, for all $z \in \Delta_{J_v}$ and all $m \geq m_0$, by virtue of (b) there exists $\varepsilon_z > 0$ such that for all t, $|t| < \varepsilon_z$, one obtains

$$\frac{1}{\sigma_{k_m}}x^{k_m} + t\left(z - \frac{1}{\sigma_{k_m}}x^{k_m}\right) \in \Delta_{J_v}.$$

Because of the choice of x^{k_m}, we have

$$\langle y^{k_m} + r^{k_m} + q^{k_m}, \sigma_{k_m}(\frac{x^{k_m}}{\sigma_{k_m}} + t(z - \frac{x^{k_m}}{\sigma_{k_m}})) - x^{k_m}\rangle \geq 0, \ \ \forall\, |t| < \varepsilon_z.$$

Then

$$\langle y^{k_m} + r^{k_m} + q^{k_m}, t(\sigma_{k_m} z - x^{k_m})\rangle \geq 0, \ \ \forall\, |t| < \varepsilon_z.$$

Hence

$$\langle y^{k_m} + r^{k_m} + q^{k_m}, \sigma_{k_m} z - x^{k_m}\rangle = 0, \ \ \forall\, z \in \Delta_{J_v}.$$

(d): By assumption $\frac{y^k}{c(\sigma_k)} \in \Phi^k(\frac{x^k}{\sigma_k})$. Since $\left\{\frac{x^k}{\sigma_k}\right\}$ is bounded, by Theorem 1.3.2 as $\Phi^k \xrightarrow{g} \Phi$, we may also assume that $\frac{y^k}{c(\sigma_k)} \to w$ up to subsequences. From Theorem 1.3.3(a), it follows in particular that $w \in \Phi(v)$.
Moreover, from $r^k \in \Psi^k(x^k)$, since Ψ^k is uniformly bounded with respect to the same set, the sequence $\{r^k\}$ is bounded and $r^{k_m} \to r$.
On dividing the inequality in $(\mathrm{PMVIP_k})$ by $c(\sigma_k)\sigma_k$ and letting $k \to +\infty$ for $x = 0$ and $x = \sigma_k \frac{y}{\|y\|_d}$ with $0 \neq y \geq 0$ respectively, we obtain $\langle w, v\rangle \leq 0$ and $\langle w, y\rangle \geq \langle d, y\rangle\langle w, v\rangle$ for all $y \geq 0$. Dividing (c) by $c(\sigma_{k_m})\sigma_{k_m}$ and letting $m \to +\infty$ we obtain the last part of (d).
(e): By assumption, $\langle y^k, \frac{x^k}{\sigma_k}\rangle \geq c(\sigma_k)\langle w^k, \frac{x^k}{\sigma_k}\rangle$ for some $w^k \in \Phi^k(\frac{x^k}{\sigma_k})$. As in (d), we obtain $w^{k_m} \to w$ and $w \in \Phi(v)$. On dividing $(\mathrm{PMVIP_k})$ (for $x = 0$) by $c(\sigma_k)$, we get

$$-\langle \frac{r^k + q^k}{c(\sigma_k)}, \frac{x^k}{\sigma_k}\rangle \geq \langle \frac{y^k}{c(\sigma_k)}, \frac{x^k}{\sigma_k}\rangle \geq \langle w^k, \frac{x^k}{\sigma_k}\rangle.$$

Taking the limit we obtain $\langle w, v\rangle \leq 0$.
(f): Let us fix $x \geq 0$ and $y \in \Phi(x)$. Since Φ^k, Φ are closed-valued and $\Phi^k \xrightarrow{g} \Phi$, by Theorem 1.3.3 we consider x as the limit of a sequence $\{a^j\}$, corresponding to some choice of $\{b^j\}$ satisfying $b^j \in \Phi^j(a^j)$ and $b^j \to y$ as $j \to +\infty$. Obviously there is j_0 such that $a^j \in D_{j_0}$ for all j. In particular, for $j \geq j_0$ we have that $a^j \in D_{j_0} \subseteq D_j$. By q^j-pseudomonotonicity of Φ^j, $(\mathrm{PMVIP_j})$ implies $\langle b^j + q^j, a^j - x^j\rangle \geq 0$ for all j sufficiently large, dividing by σ_j and taking the limit we conclude that $\langle y + q, v\rangle \leq 0$. Thus $v \in [-\Phi(\mathbb{R}^n_+) - q]^*$. The remaining part is obvious.

(g): Since Ψ^k is uniformly bounded and $r^k \in \Psi^k(x^k)$, proceeding as in (d), we obtain $r^{k_m} \to r$ and by copositivity $\langle r^k, x^k\rangle \geq 0$, thus $\langle r, v\rangle \geq 0$.
Let us fix $x \geq 0$ and $y \in \Phi(x)$, as in (f) we obtain x as the limit of a sequence $\{a^j\}$, corresponding to some choice of $\{b^j\}$ satisfying $b^j \in \Phi^j(a^j)$ and $b^j \to y$ as $j \to +\infty$, and $a^j \in D_{j_0} \subseteq D_j$ for $j \geq j_0$. By $(r^j + q^j)$-pseudomonotonicity of Φ^j, $(\mathrm{PMVIP_j})$ implies $\langle b^j + r^j + q^j, a^j - x^j\rangle \geq 0$ for all j sufficiently large; dividing by σ_j and taking the limit we conclude that $0 \geq \langle y + r + q, v\rangle \geq \langle y + q, v\rangle$. Thus $v \in [-\Phi(\mathbb{R}^n_+) - q]^*$. ■

Remark 2.2.4 *Clearly $\langle d, v\rangle = 1$. Moreover, when Φ^k are c-subhomogeneous, by choosing $y = e^i$, $i \in I$ in* (d) *and setting $\tau = -\langle w, v\rangle \geq 0$, we obtain $w + \tau d \geq 0$ and $\langle w + \tau d, v\rangle = 0$. Thus $0 \neq v \in \mathcal{S}(\tau d, \Phi)$.*

We now exhibit an instance where our Basic Lemma is applicable. Let us consider $\Phi^k(x) = M^k x$, $\Psi^k(x) = \partial\sigma_{C^k}(x)$, where $M^k \in \mathbb{R}^{n\times n}$ converges to $M \in \mathbb{R}^{n\times n}$, and $\partial\sigma_{C^k}$ is the subdifferential of the support function of the nonempty compact convex set C^k, which converges (in the sense of Painlevé-Kuratowski) to the nonempty compact convex C. It is known that $M^k \xrightarrow{g} M$, and by Corollaries 11.5 and 8.24, and Theorem 12.35 of [68], one obtains, $C^k \to C \iff \partial\sigma_{C^k} \xrightarrow{g} \partial\sigma_C$. Moreover, as mentioned in Example 2.1.2, each $\partial\sigma_{C^k}$ and $\partial\sigma_C$ is zero-subhomogeneous, and we may consider that $\partial\sigma_{C^k}$ are uniformly bounded respect to the same set. In addition, if $0 \in C^k \cap C$ (this is not a stringent assumption), then $\partial\sigma_{C^k}$ and $\partial\sigma_C$ are copositive.

Let $\{(x^k, y^k)\}$ be a sequence of solutions to $(\mathrm{MVIP_k})$, if $\langle d, x^k\rangle < \sigma_k$ for some k, then $x^k \in \mathcal{S}(q, \Phi)$. Thus, we are interested in the case when $\langle d, x^k\rangle = \sigma_k$ for all k. Therefore, the following set of sequences will play an essential role in our analysis.

Definition 2.2.5 *Let $d > 0$ and $\{\sigma_k\}$ be an increasing sequence of positive numbers converging to $+\infty$. Let $\mathcal{W}$ be the set of sequences $\{(x^k, y^k)\}$ in $\mathbb{R}^n_+ \times \mathbb{R}^n$, satisfying* *for each* k,

$$(x^k, y^k) \text{ solves problem } (\mathrm{MVIP_k}) \tag{2.2}$$

$$\langle d, x^k \rangle = \sigma_k. \tag{2.3}$$

We point out that the requirement (2.3) is verified if $\mathcal{S}(q, \Phi)$ is either empty or unbounded. Indeed, if $\mathcal{S}(q, \Phi) = \emptyset$ then $\langle d, x^k \rangle = \sigma_k$ for all k by the above reasoning. If $\mathcal{S}(q, \Phi)$ is nonempty and unbounded, then for all k, there exists $x^k \in \mathcal{S}(q, \Phi)$ such that $\langle d, x^k \rangle \geq k$, and then we put $\sigma_k = \langle d, x^k \rangle$.
Under requirement (2.3), there exists a vector v such that, up to subsequences, $\frac{x^k}{\sigma_k} \to v$, and if $\{\theta_k\}$ is such that (x^k, y^k, θ_k) solves $(\mathrm{MCP_k})$ for each k, by (c) of the Basic Lemma (for $\Phi^k = \Phi$, $\Psi^k = 0$ and $q^k = q$ for all k) we get.

$$\theta_{k_m} = -\langle y^{k_m} + q, \frac{x^{k_m}}{\sigma_{k_m}} \rangle = -\langle y^{k_m} + q, v \rangle. \tag{2.4}$$

2.3 New classes of multifunctions

We introduce the following new classes of multifunctions, which generalize those introduced in [30] for the linear complementarity problem.

Definition 2.3.1 *Let* $d > 0$, $\Phi : \mathbb{R}^n_+ \rightrightarrows \mathbb{R}^n$, *s.t.* $0 \in \Phi(0)$. *We say that* Φ *is a*

- $\mathbf{T}(d)$-mapping, *if for any index subset* $J \subseteq I$, *one has*

$$\left.\begin{array}{l} v \geq 0,\ w \geq 0,\ w \in \Phi(v) \\ w_J = 0,\ \emptyset \neq \mathrm{supp}\{v\} \subseteq J \end{array}\right\} \Longrightarrow v \in \left[\Phi(\mathrm{pos}^+ \Delta_J)\right]^*. \tag{2.5}$$

- $\tilde{\mathbf{T}}(d)$-mapping, *if for any index subset* $J \subseteq I$, *one has*

$$\left.\begin{array}{l} v \geq 0,\ w \geq 0,\ w \in \Phi(v) \\ w_J = 0,\ \emptyset \neq \mathrm{supp}\{v\} \subseteq J \end{array}\right\} \Longrightarrow \begin{array}{l} \langle y, x \rangle \geq 0, \\ \forall x \in \mathrm{pos}^+ \Delta_J,\ y \in \Phi(x). \end{array} \tag{2.6}$$

- $\mathbf{GT}(d)$ (resp. $\mathbf{G}\tilde{\mathbf{T}}(d)$)-mapping *if it is* $\mathbf{G}(d)$ *and* $\mathbf{T}(d)$ (*resp.* $\tilde{\mathbf{T}}(d)$).

Remark 2.3.2 *We observe that if Φ is c-subhomogeneous for some $c \in \mathcal{C}$, then*

$$\Phi \in \mathbf{T}(d) \iff \Phi \in \mathbf{T}(d') \;\forall\, d' > 0;$$

$$\Phi \in \tilde{\mathbf{T}}(d) \iff \Phi \in \tilde{\mathbf{T}}(d') \;\forall\, d' > 0.$$

Moreover, $\Phi \in \mathbf{T}(d)$ (resp. $\Phi \in \tilde{\mathbf{T}}(d)$) if and only if (2.5) (resp. (2.6)) holds with Δ_J instead of $\mathrm{pos}^+\Delta_J$.

Proposition 2.3.3 *Let $d > 0$, $c \in \mathcal{C}$, and $\Phi : \mathbb{R}^n_+ \rightrightarrows \mathbb{R}^n$ be a multifunction. The following assertions hold:*

(a) *$0 \in \mathcal{S}(p, \Phi)$ for all $p \geq 0$ provided $0 \in \Phi(0)$;*

(b) *if Φ is copositive then it is semimonotone (hence $\mathbf{G}(p)$ for all $p > 0$) and $\tilde{\mathbf{T}}(p)$ for all $p > 0$;*

(c) *if Φ is superadditive c-homogeneous (in particular if Φ is single-valued and linear) and $\tilde{\mathbf{T}}(d)$, then it is $\mathbf{T}(d)$.*

Proof. (a): It is obvious.
(b): For $p > 0$ fixed, we take any $x \in \mathcal{S}(p, \Phi)$. Then, $y + p \geq 0$ and $\langle y + p, x\rangle = 0$ for some $y \in \Phi(x)$. By copositivity $\langle p, x\rangle \leq 0$, which implies $x = 0$, proving that Φ is semimonotone. The remaining assertion is obvious.
(c): Let v, w be such that the left-hand side of (2.5) holds, then $\langle y, x\rangle \geq 0$ for all $x \in \mathrm{pos}^+\Delta_J$ and $y \in \Phi(x)$. By hypothesis, $y + c(t)w \in \Phi(x + tv)$ for all $t > 0$ since $w \in \Phi(v)$. Thus $\langle y + c(t)w, x + tv\rangle \geq 0$ for all $t > 0$. It follows that $\langle y, v\rangle \geq 0$ since $\langle w, x\rangle = \langle w, v\rangle = 0$, proving (2.5). ■

Example 2.3.4 **1.** *We must point to out that there is no relationship between $\mathbf{G}$ and $\mathbf{T}$-mappings, even in the linear case (see Chapter 4). Analogously, there is no relationship between $\mathbf{G}$ and $\tilde{\mathbf{T}}$-mappings. Indeed, take*

$$M_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, M_2 = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}.$$

Then the mapping $\Phi_1(x) = M_1 x \in \tilde{\mathbf{T}}(d) \setminus \mathbf{G}(d)$ for any $d > 0$, whereas the mapping $\Phi_2(x) = M_2 x \in \mathbf{G}(d) \setminus \tilde{\mathbf{T}}(d)$ for any $d > 0$.

2. *Let $M_3 = \begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix}$, $M_4 = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$, and consider the mappings $\Phi_i(x) = M_i x$, $i = 3, 4$. Then Φ_3 is not copositive but it is semimonotone (hence $\Phi_3 \in \mathbf{G}(p)$ for any $p > 0$) and $\tilde{\mathbf{T}}(p)$ for any $p > 0$; whereas $\Phi_4 \in \mathbf{T}(p) \setminus \tilde{\mathbf{T}}(p)$ for any $p > 0$. This shows that $\mathbf{T}$ and $\tilde{\mathbf{T}}$ does not coincide even in the linear case. The same properties hold for $\Phi_i(x) = ||x||M_i x$, $i = 3, 4$.*

One can check directly that

$$\mathcal{F}(q,\Phi) \neq \emptyset \Longrightarrow [v \geq 0,\ v \in -[\Phi(\mathbb{R}^n_+)]^*,\ \langle q, v\rangle \leq 0 \Longrightarrow \langle q, v\rangle = 0]. \qquad (2.7)$$

And the reverse implication holds whenever Φ is c-homogeneous for some $c \in \mathcal{C}$, $\Phi(0) = \{0\}$, and the set $\mathbb{R}^n_+ - \Phi(\mathbb{R}^n_+)$ is convex and closed. It should be notice that the right-hand side of (2.7) amounts to writing

$$v \geq 0,\ v \in -[\Phi(\mathbb{R}^n_+)]^* \Longrightarrow \langle q, v\rangle \geq 0.$$

Proposition 2.3.5 *Let $d > 0$, $c \in \mathcal{C}$, $q \in \mathbb{R}^n$, and $\Phi : \mathbb{R}^n_+ \rightrightarrows \mathbb{R}^n$ be q-pseudomonotone c-homogeneous. Assume that $\mathcal{F}(q,\Phi) \neq \emptyset$. Then,*

(a) *Φ is copositive on $\mathbb{R}^n_{++}$;*

(b) *Φ is copositive, if in addition it is either lsc or superadditive;*

(c) *if Φ is superadditive, then for all $J \subseteq I$*

$$\left.\begin{array}{l} v \geq 0,\ w \in \Phi(v),\ w_J \leq 0 \\ \emptyset \neq \operatorname{supp}\{v\} \subseteq J \end{array}\right\} \Longrightarrow v \in [\Phi(\Delta_J)]^*. \qquad (2.8)$$

Proof. (a): Let $x^0 \geq 0$, $y^0 \in \Phi(x^0)$ such that $y^0 + q \geq 0$. For any $x > 0$ there exists $t_x > 0$ such that for all $t > t_x$, $\frac{tx}{\|x\|_d} - x^0 \geq 0$. Thus $\langle y^0 + q, \frac{t}{\|x\|_d} x - x^0\rangle \geq 0$

for all $t > t_x$. If $y \in \Phi(x)$, by c-homogeneity $\frac{c(t)}{c(\|x\|_d)}y \in \Phi(\frac{t}{\|x\|_d}x)$, and by q-pseudomonotonicity,

$$\langle \frac{c(t)}{c(\|x\|_d)}y + q, t\frac{x}{\|x\|_d} - x^0 \rangle \geq 0 \quad \forall\ t > t_x.$$

On dividing by $c(t)t$ and taking the limit as $t \to +\infty$, we get $\langle y, x\rangle \geq 0$, proving (a).

(b): Let x be on the boundary of $\mathbb{R}^n_+$ and $y \in \Phi(x)$. Then, there exists $\{x^k\} \subseteq \mathbb{R}^n_{++}$ such that $x^k \to x$. Suppose first that Φ is lsc, then there exists $\{y^k\}$ such that $y^k \in \Phi(x^k)$ and $y^k \to y$. By (a), $\langle y^k, x^k\rangle \geq 0$ and then $\langle y, x\rangle \geq 0$. We now suppose that Φ is superadditive. If $e > 0$, $tx + e > 0$ for all $t > 0$. Let $y \in \Phi(x)$ and $u \in \Phi(e)$. Clearly $c(t)y + u \in \Phi(tx + e)$, and by (a) $\langle c(t)y + u, tx + e\rangle \geq 0$ for all $t > 0$. After dividing by $c(t)t$ and taking the limit as $t \to +\infty$, we get $\langle y, x\rangle \geq 0$. This completes the proof that Φ is copositive in either case.

(c): Let v, w satisfy the left-hand side of (2.8). For $z \in \Delta_J$ and $y \in \Phi(z)$ we get $y + c(t)w \in \Phi(z + tv)\ \forall t > 0$, and by (b) we obtain $\langle y + c(t)w, z + tv\rangle \geq 0$. Since $\langle w, v\rangle = \langle w, z\rangle \leq 0$, we deduce that $\langle y, v\rangle \geq 0$, proving (2.8). ■

The next result describes the asymptotic behavior of the normalized approximate solutions to problem (MCP), for the mappings introduced recently.

Lemma 2.3.6 *Let $d > 0$, $c \in \mathcal{C}$; $\Phi, \Psi \in \mathcal{X}$, and $\{\sigma_k\}$ be an increasing sequence of positive numbers converging to $+\infty$. Assume there exist a sequence $\{(x^k, y^k + r^k)\} \in \mathcal{W}$ for $\Phi + \Psi$ such that $\frac{x^k}{\sigma_k} \to v$. Then, in addition to the existence of w, r, $\{w^k\}$ and subindex set $\emptyset \neq J_v \subseteq I$ satisfying the properties established in the Basic Lemma (for $\Phi^k = \Phi$, $\Psi^k = \Psi$, and $q^k = q$ for all k), we also obtain the following:*

- **(a)** *for Φ to be c-subhomogeneous and Ψ copositive uniformly bounded:*
  - **(a.1)** $\Phi \in \mathbf{G}(d)$ *implies* $w \geq 0,\ w_{J_v} = 0$ *(hence $\langle w, v\rangle = 0$);*
  - **(a.2)** $\Phi \in \mathbf{G}\tilde{\mathbf{T}}(d)$ *implies* $w \geq 0,\ w_{J_v} = 0$, $\langle q, v\rangle \leq 0$, $\langle r, v\rangle \geq 0$, *and* $\langle y, x\rangle \geq 0\ \forall\, x \in \mathrm{pos}^+\Delta_{J_v}$, $y \in \Phi(x)$;

(a.3) $\Phi \in \mathbf{GT}(d)$ *implies* $w \geq 0,\ w_{J_v} = 0$, $\langle q, v\rangle \leq 0$, $\langle r, v\rangle \geq 0$, *and* $v \in [\Phi(\mathrm{pos}^+\Delta_{J_v})]^*$;

(b) *for* Φ *to be a copositive* c*-Moré and* Ψ *copositive uniformly bounded:* $\langle w, v\rangle = 0$, $\langle r, v\rangle \geq 0$, *and* $\langle q, v\rangle \leq 0$.

Proof. We set $\Phi^k = \Phi$, $\Psi^k = \Psi$ and $q^k = q$ for all k in the Basic Lemma.
(a.1): By (d) of the Basic Lemma, $\langle w, v\rangle \leq 0$. If, on the contrary $\langle w, v\rangle < 0$, then from Remark 2.2.4, $0 \neq v \in \mathcal{S}(\tau d, \Phi)$ with $\tau = -\langle w, v\rangle > 0$. This contradicts the fact that $\Phi \in \mathbf{G}(d)$. Hence $\langle w, v\rangle = 0$. We then apply again (d) to obtain the desired result. Moreover, since Ψ is copositive $\langle r^k, x^k\rangle \geq 0$, thus $\langle r, v\rangle \geq 0$.
(a.2): If $\Phi \in \mathbf{G\tilde{T}}(d)$, then (a.1) holds, and by (2.6) we get $\langle y, x\rangle \geq 0$ for all $x \in \mathrm{pos}^+\Delta_{J_v}$ and $y \in \Phi(x)$, which in turn implies $\langle y^{k_m}, x^{k_m}\rangle \geq 0$. From $\langle y^k + r^k + q, x^k\rangle \leq 0$ for all k (set $x = 0$ in $(\mathrm{MVIP_k})$ for $\Phi + \Psi$), we deduce that $\langle q, x^{k_m}\rangle \leq 0$, thus $\langle q, v\rangle \leq 0$.
(a.3): If $\Phi \in \mathbf{GT}(d)$, then (a.1) holds, and by (2.5) we get $v \in [\Phi(\mathrm{pos}^+\Delta_{J_v})]^*$, which in turn implies $\langle y^{k_m}, v\rangle \geq 0$. From (c) of the Basic Lemma (for $z = v$), and setting $x = 0$ in $(\mathrm{MVIP_k})$ for $\Phi+\Psi$ we get $\langle q + r^{k_m}, v\rangle \leq \langle y^{k_m} + r^{k_m} + q, v\rangle = \langle y^{k_m} + r^{k_m} + q, \frac{x^{k_m}}{\sigma_{k_m}}\rangle \leq 0$, thus $\langle r + q, v\rangle \leq 0$. Since $\langle r, v\rangle \geq 0$, we get $\langle q, v\rangle \leq 0$.
(b): By copositivity of Φ and Ψ, and (e) of the Basic Lemma, $\langle w, v\rangle = 0$ and setting $x = 0$ in $(\mathrm{MVIP_k})$ for $\Phi + \Psi$, $\langle q, x^k\rangle \leq \langle y^k + r^k + q, x^k\rangle \leq 0$. Thus $\langle q, v\rangle \leq 0$. As above we prove $\langle r, v\rangle \geq 0$. ■

2.4 Estimates for $[\mathrm{S}(\mathrm{q}, \boldsymbol{\Phi})]^\infty$

In order to obtain bounds-estimates for the asymptotic cones of the solution sets to (MCP), which will allow us to investigate the boundedness of solutions, we introduce the following sets:

$$\mathrm{U}_q(\Phi) \doteq \{v \geq 0 : w \in \Phi(v),\ \langle w, v\rangle = 0, \langle q, v\rangle \leq 0\}\,;$$

$$\mathrm{V}_q(\Phi) \doteq \mathbb{R}^n_+ \cap [-\Phi(\mathbb{R}^n_+) - q]^*;$$

$$\mathrm{W}_q(\Phi) \doteq \{v \geq 0 : w \in \Phi(v), \langle w, v\rangle = 0, w \geq 0, \langle q, v\rangle \leq 0\}.$$

One immediately obtains

$$q \in [\mathrm{U}_0(\Phi)]^\# \Longleftrightarrow \mathrm{U}_q(\Phi) = \{0\};$$

$$q \in [\mathcal{S}(0,\Phi)]^\# \Longleftrightarrow \mathrm{W}_q(\Phi) = \{0\}.$$

Proposition 2.4.1 *Let $d > 0$, $c \in \mathcal{C}$, $q \in \mathbb{R}^n$, and $\Phi, \Psi \in \mathcal{X}$.*

(a) $[\mathrm{S}(q, \Phi+\Psi)]^\infty_d \subseteq \mathcal{S}(0,\Phi) \cap \Delta_d$ *if Φ is c-subhomogeneous and Ψ is uniformly bounded. Moreover, if $\Phi \in \tilde{\mathbf{T}}(d) \cup \mathbf{T}(d)$ and Ψ is copositive as well,* $[\mathcal{S}(q, \Phi+\Psi)]^\infty_d \subseteq \mathrm{W}_q(\Phi) \cap \Delta_d$;

(b) $[\mathcal{S}(q, \Phi+\Psi)]^\infty_d \subseteq \mathrm{U}_q(\Phi) \cap \Delta_d$ *if Φ is copositive c-Moré and Ψ is copositive uniformly bounded;*

(c) $[\mathcal{S}(q, \Phi+\Psi)]^\infty_d \subseteq \mathrm{V}_q(\Phi) \cap \Delta_d$ *if Φ is monotone and Ψ is copositive uniformly bounded.*

If Φ is q-pseudomonotone and $\mathcal{S}(q,\Phi) \neq \emptyset$, then $[\mathcal{S}(q,\Phi)]^\infty = \mathrm{V}_q(\Phi)$.

Proof. (a): Let $v \in [\mathrm{S}(q, \Phi+\Psi)]^\infty_d$. Then, there exists $x^k \in \mathcal{S}(q, \Phi+\Psi)$ such that $||x^k||_d \to +\infty$ and $\frac{x^k}{||x^k||_d} \to v$. Moreover, there exist $y^k \in \Phi(x^k)$ and $r^k \in \Psi(x^k)$ such that $y^k + r^k + q \geq 0$ and $\langle y^k + r^k + q, x^k\rangle = 0$ for all k. Clearly, $\sigma_k = \langle d, x^k\rangle \to +\infty$ and $\frac{x^k}{\sigma_k} \to v$ as $k \to +\infty$. Consequently, the Basic Lemma (for $\Phi^k = \Phi$, $\Psi^k = \Psi$, and $q^k = q$ for all k) implies the existence of $w \in \Phi(v)$ and $\emptyset \neq J_v \subseteq I$, such that (a)-(d) of that lemma hold. Dividing $y^k + r^k + q \geq 0$ (resp. $\langle y^k + r^k + q, x^k\rangle = 0$) by $c(\sigma_k)$ (resp. $c(\sigma_k)\sigma_k$) and taking the limit we obtain $w \geq 0$, $\langle w, v\rangle = 0$, and $w_{J_v} = 0$. Thus, in particular $v \in \mathcal{S}(0,\Phi)$.
Let $\Phi \in \mathbf{T}(d) \cup \tilde{\mathbf{T}}(d)$ and Ψ be copositive uniformly bounded, by proceeding exactly as in Lemma 2.3.6 we obtain that $\langle q, v\rangle \leq 0$. Thus, $v \in \mathrm{W}_q(\Phi)$.
(b): By proceeding as above and in Lemma 2.3.6, we obtain that $v \in \mathrm{U}_q(\Phi)$.

(c): If Φ is monotone and Ψ is uniformly bounded, by proceeding as above, (g) of the Basic Lemma implies that $v \in \mathrm{V}_q(\Phi)$.
If Φ is q-pseudomonotone, it is well known that (see [16] for instance)

$$\mathcal{S}(q,\Phi) = \bigcap_{x\geq 0}\bigcap_{y\in\Phi(x)} \{\bar{x} \geq 0 : \langle y+q, x-\bar{x}\rangle \geq 0\}.$$

Since the sets involved in the intersection are closed and convex and $\mathcal{S}(q,\Phi)$ is nonempty, applying a property of asymptotic cones we conclude that

$$[\mathcal{S}(q,\Phi)]^{\infty} = \bigcap_{x\geq 0}\bigcap_{y\in\Phi(x)} \Big\{\bar{x} \geq 0 :\ \langle y+q, x-\bar{x}\rangle \geq 0\Big\}^{\infty} = \mathrm{V}_q(\Phi),$$

since $\Big\{\bar{x} \geq 0 :\ \langle y+q, x-\bar{x}\rangle \geq 0\Big\}^{\infty} = \Big\{v \geq 0 :\ \langle y+q, v\rangle \leq 0\Big\}$. ■

Example 2.4.2 *The inclusions in the preceding proposition may be strict.*
1. *Let* $\Phi(x_1,x_2) = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ *be in* $\mathbf{T}(d) \cap \tilde{\mathbf{T}}(d)$ *(for all* $d > 0$*),* $\Psi = 0$*, and* $q = (-\frac{1}{2}, -1)^{\mathrm{T}}$*. The inclusions in* (a) *are strict since* $\mathcal{S}(q,\Phi) = \{(0,1)^{\mathrm{T}}\}$ *and* $\mathcal{S}(0,\Phi) = \mathrm{W}_q(\Phi) = \{(v_1,0)^{\mathrm{T}} : v_1 \geq 0\}$.
2. *Let* $\Phi(x_1,x_2) = [x_1, 2x_1] \times \{0\}$ *be copositive* $\frac{\lambda}{2}$*-Moré on* Δ_d *for* $d = (1,1)^{\mathrm{T}}$*,* $\Psi = 0$*, and* $q = (-1,1)^{\mathrm{T}}$*. The inclusion in* (b) *is strict since* $\mathcal{S}(q,\Phi) = [\frac{1}{2}, 1] \times \{0\}$ *and* $\mathrm{U}_0(\Phi) = \Big\{(0,v_2)^{\mathrm{T}} :\ v_2 \geq 0\Big\}$.

2.5 Main existence results

In this section we present new existence results, which generalize and unify several ones found in the literature. This is carried out by using the classes of mappings introduced in Sections 2.1 and 2.3, and applying mostly Theorem 2.2.1. Actually, our main results of this section establishes sufficient conditions implying a kind of robustness property for some classes mappings with respect to certain perturbation.

Lemma 2.5.1 *Let $d > 0$, $c \in \mathcal{C}$, $q \in \mathbb{R}^n$, $\Phi \in \mathcal{X}$, and $\Psi \in \mathcal{X}$ be copositive uniformly bounded. The set $\mathcal{D}(\Phi + \Psi)$ is closed at q under any of the following circumstances:*

(a) *Φ is c-subhomogeneous $\mathbf{GT}(d)$ and $q \in [\mathcal{S}(0, \Phi)]^{\#}$;*

(b) *Φ is c-subhomogeneous $\mathbf{G\tilde{T}}(d)$ and $q \in [\mathcal{S}(0, \Phi)]^{\#}$;*

(c) *Φ is copositive c-Moré and $q \in [\mathrm{U}_0(\Phi)]^{\#}$;*

(d) *Φ is monotone and $V_q(\Phi) = \{0\}$.*

Proof. Let $\{q^k\} \subseteq \mathcal{D}(\Phi + \Psi)$ be a sequence converging to q. There exist $x^k \geq 0$, $y^k \in \Phi(x^k)$, and $r^k \in \Psi(x^k)$ such that $y^k + r^k + q^k \geq 0$ and $\langle y^k + r^k + q^k, x^k\rangle = 0$. If the sequence $\{x^k\}$ is bounded, each of its limit points is in $\mathcal{S}(q, \Phi + \Psi)$ since $\Phi, \Psi \in \mathcal{X}$. Thus $q \in \mathcal{D}(\Phi + \Psi)$.
If the sequence $\{x^k\}$ is unbounded, setting $\sigma_k = \langle d, x^k\rangle \to +\infty$, we may consider that there exists v such that, up to subsequences, $\frac{x^k}{\sigma_k} \to v$ and $\{(x^k, y^k, r^k)\}$ are solutions to $(\mathrm{PMVIP_k})$ for $\Phi^k = \Phi$, $\Psi^k = \Psi$ for all k. By the Basic Lemma (for $\Phi^k = \Phi$, $\Psi^k = \Psi$ for all k) and proceeding as in Lemma 2.3.6 it follows that:
(a): there exist $\{x^{k_m}\}$, r, and $\emptyset \neq J_v \subseteq I$, such that $0 \neq v \in \mathcal{S}(0, \Phi)$, $w_{J_v} = 0$, and by (2.5) $v \in [\Phi(\mathrm{pos}^+\Delta_{J_v})]^*$, which in turn implies $\langle y^{k_m}, v\rangle \geq 0$. Moreover, from (c) of the Basic Lemma (for $z = v$) we get $\langle y^{k_m} + r^{k_m} + q^{k_m}, v\rangle = \langle y^{k_m} + r^{k_m} + q^{k_m}, \frac{x^{k_m}}{\sigma_{k_m}}\rangle = 0$, thus $\langle r^{k_m} + q^{k_m}, v\rangle \leq 0$, then $\langle q, v\rangle \leq \langle r + q, v\rangle \leq 0$, contradicting the choice of q.
(b): there exist $\{x^{k_m}\}$ and $\emptyset \neq J_v \subseteq I$, such that $0 \neq v \in \mathcal{S}(0, \Phi)$, $w_{J_v} = 0$, and $\langle y, x\rangle \geq 0$ for all $x \in \mathrm{pos}^+\Delta_{J_v}$, $y \in \Phi(x)$, which in turn implies $\langle y^{k_m}, x^{k_m}\rangle \geq 0$. Moreover, from $\langle y^{k_m} + r^{k_m} + q^{k_m}, x^{k_m}\rangle = 0$, we get $\langle q^{k_m}, x^{k_m}\rangle \leq 0$, then $\langle q, v\rangle \leq 0$, contradicting the choice of q.
(c): there exists $w^k \in \Phi(\frac{x^k}{\sigma_k})$ such that $w^k \to w \in \Phi(v)$ and $\langle w, v\rangle = 0$, thus $0 \neq v \in \mathrm{U}_0(\Phi)$. Moreover, from $0 = \langle y^k + r^k + q^k, x^k\rangle \geq \langle q^k, x^k\rangle$, we get $\langle q, v\rangle \leq 0$, contradicting the choice of q.
(d): $0 \neq v \in V_q(\Phi)$ a contradiction. ■

We first obtain existence theorems for problem (MCP) for mappings of the form $\Phi + \Psi$, and Φ respectively. In this way, we generalize some results from [41, 64, 65, 66] as will be shown in the example below.

Theorem 2.5.2 *Let $d > 0$, $c \in \mathcal{C}$, $\Phi \in \mathcal{X}$ be $\mathbf{GT}(d)$ or $\mathbf{G\tilde{T}}(d)$ c-subhomogeneous, and $\Psi \in \mathcal{X}$ be copositive uniformly bounded:*

(a) *if $q \in [\mathcal{S}(0, \Phi)]^*$ and $\mathcal{D}(\Phi + \Psi)$ is closed at q, then $\mathcal{S}(q, \Phi + \Psi)$ is nonempty;*

(b) *if $q \in [\mathcal{S}(0, \Phi)]^\#$, then $\mathcal{S}(q, \Phi + \Psi)$ is nonempty and compact.*

Proof. (a): Let $\{\sigma_k\}$ be an increasing sequence of positive numbers converging to $+\infty$, $d > 0$, and $\{(x^k, y^k + r^k)\} \in \mathcal{W}$ for $\Phi + \Psi$. Since $\langle d, \frac{x^k}{\sigma_k}\rangle = 1$, up to subsequences, there exists $0 \neq v \geq 0$ such that $\frac{x^k}{\sigma_k} \to v$. Thus,

$$x^k \in D_k,\ y^k \in \Phi(x^k),\ r^k \in \Psi(x^k),\ \langle y^k + r^k + q, x - x^k\rangle \geq 0\ \forall x \in D_k. \tag{2.9}$$

By the Basic Lemma (for $\Phi^k = \Phi$, $\Psi^k = \Psi$, and $q^k = q$ for all k) and Lemma 2.3.6 for Φ in $\mathbf{G\tilde{T}}(d)$ (resp. in $\mathbf{GT}(d)$) and Ψ copositive uniformly bounded there exist $w \in \Phi(v)$, r, $\emptyset \neq J_v \subseteq I$, and $\{x^{k_m}\}$ such that $w \geq 0$, $\langle w, v\rangle = 0$, $w_{J_v} = 0$, $\langle q, v\rangle \leq 0$, $\langle r, v\rangle \geq 0$, and $\langle y, x\rangle \geq 0$ for all $x \in \mathrm{pos}^+\Delta_{J_v}$, $y \in \Phi(x)$ (resp. $v \in [\Phi(\mathrm{pos}^+\Delta_{J_v})]^*$, which in turn implies $\langle y^{k_m}, x^{k_m}\rangle \geq 0$ (resp. $\langle y^{k_m}, v\rangle \geq 0$). Moreover, $v \in \mathcal{S}(0, \Phi)$ implies $\langle q, v\rangle = 0$. From (2.4) for $\Phi + \Psi$, we get

$$\theta_{k_m} = -\langle y^{k_m} + r^{k_m} + q, \frac{x^{k_m}}{\sigma_{k_m}}\rangle = -\langle y^{k_m} + r^{k_m} + q, v\rangle. \tag{2.10}$$

then $0 \leq \theta_{k_m} \leq -\langle q, \frac{x^{k_m}}{\sigma_{k_m}}\rangle$ (resp. $0 \leq \theta_{k_m} \leq -\langle r^{k_m}, v\rangle$). Thus $\liminf\limits_{k\to+\infty} \theta_k = 0$, and the result follows from Theorem 2.2.1.
(b): By Lemma 2.5.1 the set $\mathcal{D}(\Phi + \Psi)$ is closed at q and by (a) we obtain that $\mathcal{S}(q, \Phi + \Psi)$ is nonempty. Its boundedness follows from Proposition 2.4.1 since by the choice of q, we get $\mathrm{W}_q(\Phi) = \{0\}$. ■

If $\Psi = 0$ in the previous theorem, the closedness of $\mathcal{D}(\Phi)$ is not needed for $\mathbf{T}$-mappings, as is shown in the next corollary.

Corollary 2.5.3 *Let $d > 0$, $c \in \mathcal{C}$, $q \in \mathbb{R}^n$, and $\Phi \in \mathcal{X}$ be c-subhomogeneous:*

(a) *if $\Phi \in \mathbf{GT}(d)$ and $q \in [\mathcal{S}(0, \Phi)]^*$, then $\mathcal{S}(q, \Phi)$ is nonempty;*

(b) *if $\Phi \in \mathbf{G\tilde{T}}(d)$, $\mathcal{D}(\Phi)$ is closed at q, and $q \in [\mathcal{S}(0, \Phi)]^*$, then $\mathcal{S}(q, \Phi)$ is nonempty;*

(c) *if Φ is $\mathbf{GT}(d)$ or $\mathbf{G\tilde{T}}(d)$ and $q \in [\mathcal{S}(0, \Phi)]^{\#}$, then $\mathcal{S}(q, \Phi)$ is nonempty and compact.*

Proof. By setting $\Psi = 0$ in the above theorem we obtain (b) and (c).
(a): By proceeding as in the above theorem with $\Psi = 0$, from (2.10) we obtain that $0 \leq \theta_{k_m} \leq -\langle y^{k_m} + q, v \rangle \leq 0$, thus $\theta_{k_m} = 0$ and $x^{k_m} \in \mathcal{S}(q, \Phi)$. ■

Remark 2.5.4 *Since copositive mappings are $\mathbf{G\tilde{T}}(d)$ for each $d > 0$, the above result contains Corollary 2 of* [41].

The next two results do not require c-subhomogeneity.

Theorem 2.5.5 *Let $d > 0$, $c \in \mathcal{C}$, $q \in \mathbb{R}^n$, $\Phi \in \mathcal{X}$ be copositive c-Moré, and $\Psi \in \mathcal{X}$ be copositive uniformly bounded:*

(a) *if $q \in [\mathrm{U}_0(\Phi)]^*$ and $\mathcal{D}(\Phi + \Psi)$ is closed at q, then $\mathcal{S}(q, \Phi + \Psi)$ is nonempty;*

(b) *if $q \in [\mathrm{U}_0(\Phi)]^{\#}$, then $\mathcal{S}(q, \Phi + \Psi)$ is nonempty and compact.*

Proof. (a): Let $\{\sigma_k\}$ be an increasing sequence of positive numbers converging to $+\infty$, $d > 0$, and $\left\{(x^k, y^k)\right\} \in \mathcal{W}$ for $\Phi + \Psi$. Since $\langle d, \frac{x^k}{\sigma_k} \rangle = 1$, up to subsequences, there exists $0 \neq v \geq 0$ such that $\frac{x^k}{\sigma_k} \to v$. By the Basic Lemma (for $\Phi^k = \Phi$, $\Psi^k = \Psi$, and $q^k = q$ for all k) and Lemma 2.3.6 there exist w, r, $\left\{x^{k_m}\right\}$, and $\emptyset \neq J_v \subseteq I$ such that $w \in \Phi(v)$ and $\langle w, v \rangle = 0$, thus $v \in \mathrm{U}_0(\Phi)$, and then $\langle q, v \rangle \geq 0$. From (2.10) and copositivity of Φ and Ψ we get that $0 \leq \theta_{k_m} = -\langle y^{k_m} + r^{k_m} + q, \frac{x^{k_m}}{\sigma_{k_m}} \rangle \leq -\langle q, \frac{x^{k_m}}{\sigma_{k_m}} \rangle$, thus $\liminf\limits_{k \to \infty} \theta_k = 0$ and the result follows from Theorem 2.2.1.

(b): By Lemma 2.5.1 the set $\mathcal{D}(\Phi+\Psi)$ is closed at q, and by (a) we conclude that $\mathcal{S}(q,\Phi+\Psi)$ is nonempty. Its boundedness follows from Proposition 2.4.1, since by the choice of q, we get $\mathrm{U}_q(\Phi)=\{0\}$. ■

The previous theorem allows us to recover Theorem 3.1 from [69], where Φ is assumed to admit contractible images.

Corollary 2.5.6 *Let $\Phi \in \mathcal{X}$ be such that $0 \in \Phi(0)$. If Φ is strongly copositive, then $\mathcal{S}(q,\Phi)$ is nonempty and compact for all $q \in \mathbb{R}^n$.*

Proof. By Proposition 2.1.5, Φ is c-Moré for some $c \in \mathcal{C}$, moreover $\mathrm{U}_0(\Phi)=\{0\}$. The result follows from the above theorem. ■

Theorem 2.5.7 *Let $d>0$, $q \in \mathbb{R}^n$, $\Phi \in \mathcal{X}$ be monotone copositive, and $\Psi \in \mathcal{X}$ be copositive uniformly bounded:*

(a) *if the following implication holds $[v \in \mathrm{V}_q(\Phi) \Longrightarrow \langle q,v\rangle=0]$ and $\mathcal{D}(\Phi+\Psi)$ is closed at q, then $\mathcal{S}(q,\Phi+\Psi)$ is nonempty.*

(b) *if $\mathrm{V}_q(\Phi)=\{0\}$, then $\mathcal{S}(q,\Phi+\Psi)$ is nonempty and compact.*

Proof. (a): Let $\{\sigma_k\}$ be an increasing sequence of positive numbers converging to $+\infty$, $d>0$, and $\{(x^k,y^k+r^k)\} \in \mathcal{W}$ for $\Phi+\Psi$. Since $\langle d,\frac{x^k}{\sigma_k}\rangle=1$, up to subsequences, there exists $0 \neq v \geq 0$ such that $\frac{x^k}{\sigma_k} \to v$. By (g) of the Basic Lemma (for $\Phi^k=\Phi$, $\Psi^k=\Psi$, and $q^k=q$ for all k), $v \in \mathrm{V}_q(\Phi)$ and by hypothesis $\langle q,v\rangle=0$. From (2.10) since Φ and Ψ are copositive we get $0 \leq \theta_{k_m}=-\langle y^{k_m}+r^{k_m}+q,\frac{x^{k_m}}{\sigma_{k_m}}\rangle \leq -\langle q,\frac{x^{k_m}}{\sigma_{k_m}}\rangle$. Therefore $\liminf\limits_{k\to+\infty}\theta_k=0$ and the result follows from Theorem 2.2.1.
(b): By Lemma 2.5.1 the set $\mathcal{D}(\Phi+\Psi)$ is closed at q and by (a) we conclude that $\mathcal{S}(q,\Phi+\Psi)$ is nonempty. Its boundedness follows from Proposition 2.4.1, since $\mathrm{V}_q(\Phi)=\{0\}$. ■

It is worth mentioning that a monotone mapping is copositive if $\Phi(0)\cap\mathbb{R}^n_+ \neq \emptyset$ (in particular if $0 \in \Phi(0)$).

We now revise the pseudomonotone case. Part of the next theorem was first observed in [16].

Theorem 2.5.8 *Let $q \in \mathbb{R}^n$ and $\Phi \in \mathcal{X}$ be q-pseudomonotone. Consider the statements*

(a) *$\mathcal{F}_s(q, \Phi)$ is nonempty;*

(b) $\mathrm{V}_q(\Phi) = \{0\}$;

(c) *$\mathcal{S}(q, \Phi)$ is nonempty and compact;*

(d) *There exists a compact convex set $K \subseteq \mathbb{R}^n_+$ such that*

$$\forall\, x \in \mathbb{R}^n_+ \setminus K\ \forall\, y \in \Phi(x)\ \exists\, z \in K:\ \langle y + q, z - x\rangle < 0.$$

The following implications hold: $\quad$(a) $\Longrightarrow$ (b) $\Longleftrightarrow$ (c) $\Longleftrightarrow$ (d).
Moreover, if $\Phi(\mathbb{R}^n_+)$ is convex, then all the statements are equivalent.

Proof. (a)⇒(b): Let $x^0 \geq 0$ and $y^0 \in \Phi(x^0)$ such that $y^0 + q > 0$, and let $v \in \mathrm{V}_q(\Phi)$, thus $\langle y^0 + q, v\rangle \leq 0$ a contradiction if $v \neq 0$.
(b)⇒(c): Let $\{\sigma_k\}$ be an increasing sequence of positive numbers converging to $+\infty$, $d > 0$, and $\{(x^k, y^k, \theta_k)\}$ a sequence which solves $(\mathrm{MCP_k})$ for all k. If there exists k such that $\langle d, x^k\rangle < \sigma_k$, then $\theta_k = 0$ and therefore $x^k \in \mathcal{S}(q, \Phi)$. If $\langle d, x^k\rangle = \sigma_k$ for all k, then up to subsequences $\frac{x^k}{\sigma_k} \to v \neq 0$. By (f) of the Basic Lemma (for $\Phi^k = \Phi$, $\Psi^k = 0$, and $q^k = q$ for all k) we obtain $0 \neq v \in \mathrm{V}_q(\Phi)$ a contradiction. The boundedness of the solution set follows from Proposition 2.4.1.
(c)⇒(b): It follows from (c) of Proposition 2.4.1.
(c)⇒(d): See [22].
(d)⇒(c): See [16] (this implication holds without the q-pseudomonotonicity assumption).

(b)⇒(a): On the contrary suppose that $(\Phi(\mathbb{R}^n_+) + q) \cap \operatorname{int} \mathbb{R}^n_+ = \emptyset$. By using standard separation arguments, we obtain the existence of $0 \neq v \in \mathrm{V}_q(\Phi)$ a contradiction. ■

Remark 2.5.9 *In* [22] *is shown that for $\Phi \in \mathcal{X}$ to be q-pseudomonotone, condition (d) known as* Karamardian's condition *is equivalent to the following ones:*
(d')*: there exists a compact convex set $K \subseteq \mathbb{R}^n_+$ such that*

$$\forall x \in \mathbb{R}^n_+ \setminus K \; \exists z \in K \; \forall y \in \Phi(x): \; \langle y + q, z - x \rangle < 0;$$

(d")*: there exists a compact convex set $K \subseteq \mathbb{R}^n_+$ such that*

$$\forall x \in \mathbb{R}^n_+ \setminus K \; \exists z \in K \; \exists y \in \Phi(x): \; \langle y + q, z - x \rangle < 0.$$

Theorem 2.5.10 *Let $q \in \mathbb{R}^n$, $c \in \mathcal{C}$, $\Phi \in \mathcal{X}$ be q-pseudomonotone c-homogeneous and lsc. If $\mathcal{F}(q, \Phi)$ is nonempty and $\mathcal{D}(\Phi)$ is closed at q, then $\mathcal{S}(q, \Phi)$ is nonempty.*

Proof. Let $\{\sigma_k\}$ be an increasing sequence of positive numbers converging to $+\infty$, $d > 0$, and $\{(x^k, y^k)\} \in \mathcal{W}$. Since $\langle d, \frac{x^k}{\sigma_k} \rangle = 1$, up to subsequences, there exists $0 \neq v \geq 0$ such that $\frac{x^k}{\sigma_k} \to v$. By (f) of the Basic Lemma (for $\Phi^k = \Phi$, $\Psi^k = 0$, and $q^k = q$ for all k) we get $0 \leq v \in -[\Phi(\mathbb{R}^n_+)]^*$ and $\langle q, v \rangle \leq 0$, which in turn implies $\langle q, v \rangle = 0$ by (2.7), and by Proposition 2.3.5(b), Φ is copositive, thus $\langle y^k, x^k \rangle \geq 0$, and from (2.4) we get $0 \leq \theta_k = -\langle y^k + q, \frac{x^k}{\sigma_k} \rangle \leq -\langle q, \frac{x^k}{\sigma_k} \rangle$. Hence $\liminf_{k \to \infty} \theta_k = 0$, and the result follows from Theorem 2.2.1. ■

Theorem 2.5.11 *Let $d > 0$, $c \in \mathcal{C}$, $q \in \mathbb{R}^n$, and $\Phi \in \mathcal{X}$ be q-pseudomonotone c-homogeneous and superadditive. Consider the statements:*

(a) $\mathcal{F}(q, \Phi) \neq \emptyset$*;*

(b) $v \geq 0, \; v \in -[\Phi(\mathbb{R}^n_+)]^* \implies \langle q, v \rangle \geq 0$*;*

(c) $\mathcal{S}(q, \Phi) \neq \emptyset$*.*

The following implications hold: (c) $\Longleftrightarrow$ (a) $\Longrightarrow$ (b).
Moreover, if $\mathbb{R}^n_+ - \Phi(\mathbb{R}^n_+)$ *is convex and closed together with* $\Phi(0) = \{0\}$, *then all the statements are equivalent.*

Proof. (a)$\Rightarrow$(b): It follows from (2.7).
(a)$\Rightarrow$(c): Let $\{\sigma_k\}$ be an increasing sequence of positive numbers converging to $+\infty$, $d > 0$, and $\{(x^k, y^k)\} \in \mathcal{W}$. Since $\langle d, \frac{x^k}{\sigma_k}\rangle = 1$, up to subsequences, there exists $0 \neq v \geq 0$ such that $\frac{x^k}{\sigma_k} \to v$. By (f) of the Basic Lemma (for $\Phi^k = \Phi$, $\Psi^k = 0$, and $q^k = q$ for all k) we get $0 \leq v \in -[\Phi(\mathbb{R}^n_+)]^*$ and $\langle q, v\rangle \leq 0$, which in turn imply $\langle q, v\rangle = 0$ by the above implication. By (d) of the same lemma and (2.8) we conclude that $v \in [\Phi(\Delta_{J_v})]^*$, which in turn implies $\langle y^{k_m}, v\rangle \geq 0$, therefore by (2.4), $0 \leq \theta_{k_m} = -\langle y^{k_m} + q, v\rangle = -\langle y^{k_m}, v\rangle \leq 0$, thus $\theta_{k_m} = 0$ and then $x^{k_m} \in \mathcal{S}(q, \Phi)$.
(c)$\Rightarrow$(a): It is obvious.
(b)$\Rightarrow$(a): It follows from the remark made after (2.7). ■

In what follows we give a variety of results existing in the literature which are direct consequences of our theorems.

Example 2.5.12 **1.** [41, Cor. 4,5] *Let* $\Phi(x) = Mx$, *where* $M \in \mathbb{R}^{n\times n}$ *and* $\Psi(x) = \partial h(x)$, *where* h *is a nonnegative on* $\mathbb{R}^n_+$ *support function of a nonempty compact convex set* C (*see Example* 2.1.2). *By applying Theorem 2.5.2 we obtain that:*

• *if* M *is copositive,* $q \in [\mathcal{S}(0, M)]^*$, *and* $\mathcal{D}(M + \partial h)$ *is closed at* q, *then* $\mathcal{S}(q, M + \partial h)$ *is nonempty. Moreover, if* C *is a polyhedral set the closedness condition is clearly satisfied (see Chapter 3);*

• *if* M *is copositive-star and there exists a vector* $x^0 \geq 0$ *such that* $Mx^0 + q > 0$, *then* $\mathcal{S}(q, M + \partial h)$ *is nonempty and compact. Since the existence of such an* x^0 *implies that* $q \in [\mathcal{S}(0, M)]^{\#}$ *(see Chapter 4);*

• *if* M *is regular, then* $\mathcal{S}(q, M + \partial h)$ *is nonempty and compact for all* $q \in \mathbb{R}^n$.

2. *Let* $\Phi(x) = F(x)$, *where* $F : \mathbb{R}^n_+ \to \mathbb{R}^n$ *is a continuous homogeneous of degree* $\gamma > 0$ *function. Let* $\Psi : \mathbb{R}^n_+ \rightrightarrows \mathbb{R}^n$ *be a multifunction.*

• [66, Th. 7] *if F is regular and Ψ be an usc convex-valued uniformly bounded multifunction, then $\mathcal{S}(q, F+\Psi)$ is nonempty and compact for all $q \in \mathbb{R}^n$. This follows from Theorem 2.5.2;*

• [65, Th. 3.3] *if F is monotone, $\partial h(x)$ as above, and there exist $u \geq 0$, $\tilde{y} \in \partial h(u)$ such that $F(u)+\tilde{y}+q > 0$, then $\mathcal{S}(q, F+\partial h)$ is nonempty and compact. This follows from Theorem 2.5.8 since $F+\partial h$ is q-pseudomonotone and $\mathcal{F}_s(q, F+\partial h)$ is nonempty.*

Remark 2.5.13 *The results we obtained allow us to find Karush-Kuhn-Tucker stationary points for the following mathematical programming problem:*

$$\begin{array}{ll} \text{minimize} & F(x) + h(x) \\ \text{subject to} & x \geq 0,\ g(x) \geq 0. \end{array}$$

where $F : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}^m$ are differentiable functions and h is the support function of a nonempty compact convex set in $\mathbb{R}^n$, since its corresponding Karush-Kuhn-Tucker stationary point problem can be expressed as a multivalued complementarity problem [65].

Given $d > 0$, the system

$$v \geq 0,\ \langle d, v\rangle = 1,\ w \in \Phi(v),\ \langle w, v\rangle \leq 0,\ w - \langle w, v\rangle d \geq 0. \tag{2.11}$$

found in the Basic Lemma (for $\Phi^k = \Phi$, $\Psi^k = \Psi$, and $q^k = q$ for all k), plays a fundamental role in characterizing the nonemptiness and boundedness of $\mathcal{S}(q, \Phi)$ for all $q \in \mathbb{R}^n$. When Φ is c-subhomogeneous the inconsistency of (2.11) is equivalent to the inconsistency of the following system

$$0 \neq v \geq 0,\ z \in \Phi(v),\ \tau \geq 0,\ z + \tau d \geq 0,\ \langle z + \tau d, v\rangle = 0. \tag{2.12}$$

This system has its origin in [47] where the case $\Phi(x) = Mx$ with M being a matrix and d to be the vector of ones is treated. It was further developed in [49] for Φ having single-values and nonlinear. Afterwards, the set-valued version was introduced in [66, 41].

The next theorem generalizes Corollary 2 of [41] and provides, in this setting, new characterizations of regular mappings. In particular, it shows the existence of some kind of robustness property with respect to certain classes of perturbations.

Theorem 2.5.14 *Let $d > 0$, $c \in \mathcal{C}$, and $\Phi \in \mathcal{X}$ be c-subhomogeneous. Consider the statements*

(a) *the system* (2.11) *is inconsistent;*

(b) $\Phi \in \mathbf{G}(d)$ *and $\mathcal{S}(q, \Phi + \Psi)$ is nonempty and compact for all $q \in \mathbb{R}^n$ and all $\Psi \in \mathcal{X}$ copositive uniformly bounded;*

(c) $\Phi \in \mathbf{G}(d)$ *and $\mathcal{S}(q, \Phi + \Psi)$ is nonempty and compact for all $q \in \mathbb{R}^n$ and all $\Psi \in \mathcal{X}$ copositive zero-subhomogeneous;*

(d) $\Phi \in \mathbf{G}(d)$ *and $\mathcal{S}(q, \Phi)$ is nonempty and compact for all $q \in \mathbb{R}^n$;*

(e) $\Phi \in \mathbf{R}(d)$.

The following implications hold: (e)$\Longleftrightarrow$(a)$\Longrightarrow$(b)$\Longrightarrow$ (c)$\Longrightarrow$(d). *Moreover, if Φ is c-homogeneous, then all the statements are equivalent.*

Proof. (a)$\Rightarrow$(b): We first prove that Φ is $\mathbf{G}(d)$. Let $\tau > 0$ and $x \in \mathcal{S}(\tau d, \Phi)$. Then there is $y \in \Phi(x)$ such that $y + \tau d \geq 0$ and $\langle y + \tau d, x\rangle = 0$. If $\langle y, x\rangle = 0$ then $\langle d, x\rangle = 0$, which implies $x = 0$. If $\langle y, x\rangle < 0$ then for $v = x/\|x\|_d$ we get $w = y/c(\|x\|_d) \in \Phi(v)$ and since $\tau \|x\|_d = -\langle y, x\rangle$, clearly (2.11) holds, a contradiction. The previous reasoning also shows that $\mathcal{S}(0, \Phi) = \{0\}$, and thus $\Phi \in \tilde{\mathbf{T}}(d)$. Hence $\Phi \in \mathbf{G}\tilde{\mathbf{T}}(d)$, and by Theorem 2.5.2 we conclude that $\mathcal{S}(q, \Phi + \Psi)$ is nonempty and compact for all $q \in \mathbb{R}^n$ and all $\Psi \in \mathcal{X}$ copositive uniformly bounded.
(a)$\Leftrightarrow$(e): It follows from the equivalence between (2.11) and (2.12).
(b)$\Rightarrow$(c): It follows from Proposition 2.1.3(b).

(c)⇒(d): It is obvious.
(d)⇒(e): If there exists $v \in \mathcal{S}(0,\Phi)$, $v \neq 0$, then by c-homogeneity, $tv \in \mathcal{S}(0,\Phi)$ for all $t > 0$, contradicting the boundedness of $\mathcal{S}(0,\Phi)$. ■

We rewrite the previous theorem to get the next corollary which is new in the literature, even in the case when $\Phi(x) = Mx$ with M being a real matrix. Our corollary gives more information than the existing ones, e.g. [41].

Corollary 2.5.15 *Let $d > 0$, $c \in \mathcal{C}$, and $\Phi \in \mathcal{X}$ be c-homogeneous. Assume in addition that $\Phi \in \mathbf{G}(d)$. The following assertions are equivalent:*

(a) *$\mathcal{S}(q,\Phi)$ is nonempty and compact for all $q \in \mathbb{R}^n$;*

(b) *$\mathcal{S}(q,\Phi+\Psi)$ is nonempty and compact for all $q \in \mathbb{R}^n$ and all $\Psi \in \mathcal{X}$ copositive uniformly bounded;*

(c) *$\mathcal{S}(q,\Phi+\Psi)$ is nonempty and compact for all $q \in \mathbb{R}^n$ and all $\Psi \in \mathcal{X}$ copositive zero-subhomogeneous;*

(d) $\mathcal{S}(0,\Phi) = \{0\}$.

2.6 Sensitivity and approximable multifunctions

In this section we give sensitivity results for problem (MCP), whose data are small perturbations of a given pair (q^0, Φ^0); prove some continuity properties of its solution-set multifunction, and establish further existence results for mappings which are approximable in some sense.

Proposition 2.6.1 *Let $d > 0$, $c \in \mathcal{C}$, $q^0 \in \mathbb{R}^n$, and $\Phi^0 \in \mathcal{X}$. If $q^0 \in [\mathcal{S}(0,\Phi^0)]^\#$ (resp. $\mathrm{V}_{q^0}(\Phi^0) = \{0\}$), then there exists $\varepsilon > 0$ such that for all $q \in \mathbb{R}^n$ and all $\Phi \in \mathcal{X}$ which are c-subhomogeneous (resp. simply cuscos) satisfying*

$$\|q - q^0\| + \mathrm{d\!I}(\Phi,\Phi^0) < \varepsilon,$$

one has $q \in [\mathcal{S}(0,\Phi)]^\#$ (resp. $\mathrm{V}_q(\Phi) = \{0\}$).

Proof. We first consider the case $q^0 \in [\mathrm{S}(0,\Phi^0)]^{\#}$. Suppose on the contrary, that there exist sequences $\{q^k,\Phi^k,v^k\}$ satisfying $q^k \to q^0$, $\mathrm{dI}(\Phi^k,\Phi^0) \to 0$, $0 \neq v^k \in \mathcal{S}(0,\Phi^k)$, and $\langle q^k, v^k\rangle \leq 0$ with $\Phi^k \in \mathcal{X}$ being c-subhomogeneous. By c-subhomogeneity we may assume that $\|v^k\|_d = 1$, therefore up to subsequences $v^k \to v$ and $\|v\|_d = 1$. Moreover, for all k , there exist $w^k \in \Phi^k(v^k)$ such that $w^k \geq 0$ and $\langle v^k, w^k\rangle = 0$. As $\Phi^k \xrightarrow{g} \Phi^0$, by Theorem 1.3.2 we may also assume that $w^k \to w$. From (a) of Theorem 1.3.3, it follows in particular that $w \in \Phi^0(v)$. Furthermore, $w \geq 0$ and $\langle w, v\rangle = 0$. Hence $0 \neq v \in \mathcal{S}(0,\Phi^0)$ and $\langle q^0, v\rangle \leq 0$, contradicting the choice of q^0.
We now consider the case $\mathrm{V}_{q^0}(\Phi^0) = \{0\}$. Suppose on the contrary that there exist sequences $\{q^k,\Phi^k,v^k\}$ satisfying $q^k \to q^0$, $\mathrm{dI}(\Phi^k,\Phi^0) \to 0$, and $0 \neq v^k \in \mathrm{V}_{q^k}(\Phi^k)$, with $\Phi^k \in \mathcal{X}$. We may assume that $\|v^k\|_d = 1$, therefore up to subsequences $v^k \to v$ and $\|v\|_d = 1$. Let us fix $x \geq 0$ and $y \in \Phi^0(x)$. Since Φ^k, Φ^0 are closed-valued and $\Phi^k \xrightarrow{g} \Phi^0$, we invoke again Theorem 1.3.3 to obtain x as the limit of a sequence $\{a^j\}$, corresponding to some $\{b^j\}$ satisfying $b^j \in \Phi^j(a^j)$ and $b^j \to y$. By the choice of v^j, we obtain $\langle b^j + q^j, v^j\rangle \leq 0$. Thus $\langle y+q, v\rangle \leq 0$, and therefore $0 \neq v \in \mathrm{V}_{q^0}(\Phi^0)$ a contradiction. ■

Theorem 2.6.2 *Let $d > 0$, $c \in \mathcal{C}$, $q^0 \in \mathbb{R}^n$, and $\Phi^0 \in \mathcal{X}$. If $q^0 \in [\mathcal{S}(0,\Phi^0)]^{\#}$ (resp. $\mathrm{V}_{q^0}(\Phi^0) = \{0\}$), then there exists $\varepsilon > 0$ such that for all $q \in \mathbb{R}^n$ and all $\Phi \in \mathcal{X}$ which are c-subhomogeneous and from $\mathbf{GT}(d) \cup \mathbf{G\tilde{T}}(d)$ (resp. simply q-pseudomonotone) satisfying $\|q - q^0\| + \mathrm{dI}(\Phi,\Phi^0) < \varepsilon$, the set $\mathcal{S}(q,\Phi)$ is nonempty and compact.*

Proof. This follows from the above proposition and Corollary 2.5.3 and Theorem 2.5.8. ■

The next theorem may be considered as a stability result for $\mathrm{MCP}(q^0,\Phi^0)$ under a copositivity c-subhomogeneous or a q-pseudomonotone perturbation. Notice it is only required that $\Phi^0 \in \mathcal{X}$. This theorem extends Theorem 7.5.1 of [15], where only the copositive linear case is considered.

Theorem 2.6.3 *Let $d > 0$, $c \in \mathcal{C}$, $q^0 \in \mathbb{R}^n$, and $\Phi^0 \in \mathcal{X}$. If $q^0 \in [\mathcal{S}(0, \Phi^0)]^{\#}$ (resp. $\mathrm{V}_{q^0}(\Phi^0) = \{0\}$), then there exist $\varepsilon > 0$ and $r > 0$ such that for all $q \in \mathbb{R}^n$ and all $\Phi \in \mathcal{X}$ which are copositive c-subhomogeneous (resp. simply q-pseudomonotone) the following implication holds*

$$\|q - q^0\| + \mathrm{d\!I}(\Phi, \Phi^0) < \varepsilon \implies \|\mathcal{S}(q, \Phi)\| \le r,$$

where $\|\mathcal{S}(q, \Phi)\| \doteq \sup\{\|x\| : x \in \mathcal{S}(q, \Phi)\}$.

Proof. By Theorem 2.6.2 for such q and Φ the set $\mathcal{S}(q, \Phi)$ is nonempty and compact.
Suppose on the contrary that there exist sequences $\{q^k, \Phi^k, x^k\}$ satisfying $q^k \to q^0$, $\mathrm{d\!I}(\Phi^k, \Phi^0) \to 0$, $x^k \in \mathcal{S}(q^k, \Phi^k)$, and $\langle d, x^k\rangle \to +\infty$. There exists a sequence $\{y^k\}$ such that for all k

$$y^k \in \Phi^k(x^k), \quad y^k + q^k \ge 0, \quad \text{and} \quad \langle y^k + q^k, x^k\rangle = 0. \tag{2.13}$$

Setting $\sigma_k = \langle d, x^k\rangle$, up to subsequences, $\frac{x^k}{\sigma_k} \to v \neq 0$. Clearly (x^k, y^k) is a solution of problem $(\mathrm{PMVIP_k})$ for $\Phi = \Phi^0$, $\Psi^k = 0$, and $q = q^0$, so we can apply the Basic Lemma (for $\Phi = \Phi^0$, $\Psi^k = 0$, and $q = q^0$ for all k).
If Φ^k is copositive c-subhomogeneous for all k, by (d) of the Basic Lemma and (2.13) we obtain that $0 \neq v \in \mathcal{S}(0, \Phi^0)$ and $\langle q^k, x^k\rangle \le 0$, thus $\langle q^0, v\rangle \le 0$ contradicting the choice of q^0.
If Φ^k is q^k-pseudomonotone for all k. By (f) of the Basic Lemma $0 \neq v \in \mathrm{V}_{q^0}(\Phi^0)$ a contradiction. ■

In what follows, we recall another type of continuity for multifunctions. Let X, Y be two metric spaces and $M \subseteq X$. The mapping $\mathcal{F} : X \rightrightarrows Y$ is said to be *outer semicontinuous* (osc) *at $\bar{x}$ relative to M* if,

$$\limsup_{M \ni x \to \bar{x}} \mathcal{F}(x) \subseteq \mathcal{F}(\bar{x}),$$

where

$$\limsup_{M \ni x \to \bar{x}} \mathcal{F}(x) = \left\{ z : \liminf_{M \ni x \to \bar{x}} d_{\mathcal{F}(x)}(z) = 0 \right\}.$$

Given $c \in \mathcal{C}$ and $d > 0$. Let us consider the following sets

$$\mathcal{Z}_{GT} \doteq \{(q, \Phi) \in \mathbb{R}^n \times \mathcal{X} : \ \Phi \text{ is } c\text{-subhomogeneous and } \mathbf{GT}(d)\}.$$

$$\mathcal{Z}_{G\tilde{T}} \doteq \left\{(q, \Phi) \in \mathbb{R}^n \times \mathcal{X} : \ \Phi \text{ is } c\text{-subhomogeneous and } \mathbf{G\tilde{T}}(d)\right\}.$$

$$\mathcal{Z}_{Ps} \doteq \{(q, \Phi) \in \mathbb{R}^n \times \mathcal{X} : \ \Phi \text{ is } q\text{-pseudomonotone}\}.$$

On $\mathbb{R}^n \times \mathcal{X}$ we introduce the metric

$$D((q_1, \Phi_1)(q_2, \Phi_2)) \doteq \|q_1 - q_2\| + \mathrm{d\!I}(\Phi_1, \Phi_2).$$

Obviously $(\mathbb{R}^n \times \mathcal{X}, D)$ is a metric space. We now investigate continuity properties of the solution-set mapping $\mathcal{S} : \mathbb{R}^n \times \mathcal{X} \rightrightarrows \mathbb{R}^n$.

Theorem 2.6.4 *Let $\mathcal{S}$ be the solution-set mapping associated to problem* (MCP). *Then*

(a) *$\mathcal{S}$ is* osc *relative to $\mathcal{Z}_{GT}$ or $\mathcal{Z}_{G\tilde{T}}$ at (q^0, Φ^0) provided $q^0 \in [\mathcal{S}(0, \Phi^0)]^{\#}$;*

(b) *$\mathcal{S}$ is* osc *relative to $\mathcal{Z}_{Ps}$ at (q^0, Φ^0) provided $\mathrm{V}_{q^0}(\Phi^0) = \{0\}$.*

Proof. We only consider the case (a) relative to $\mathcal{Z}_{GT}$, the other being entirely similar. By Theorem 2.6.2, the mapping $\mathcal{S}$ has nonempty compact values in a neighborhood of (q^0, Φ^0), thus it is well defined. We have to prove

$$\limsup_{\mathcal{Z}_{GT} \ni (q,\Phi) \to (q^0, \Phi^0)} \mathcal{S}(q, \Phi) \subseteq \mathcal{S}(q^0, \Phi^0). \tag{2.14}$$

Let x be in the left-hand set of (2.14), then there exist a sequence $(q^k, \Phi^k) \in \mathcal{Z}_{GT}$ such that $q^k \to q^0$, $\mathrm{d\!I}(\Phi^k, \Phi^0) \to 0$, and a sequence $x^k \to x$ with $x^k \in \mathcal{S}(q^k, \Phi^k)$. Thus, $x^k \geq 0$ and there is $y^k \in \Phi^k(x^k)$ such that $y^k + q^k \geq 0$ and $\langle y^k + q^k, x^k \rangle = 0$. By Theorem 1.3.2, we conclude that $\{y^k\}$ is bounded, and so, up to subsequences, we may assume $y^k \to y$. From Theorem 1.3.3 it follows that $y \in \Phi^0(x)$. Taking the limit we obtain $x \geq 0$, $y + q^0 \geq 0$, and $\langle y + q^0, x \rangle = 0$, that is, $x \in \mathcal{S}(q^0, \Phi^0)$. ■

By using the above continuity properties and the Basic Lemma, we can provide existence results to other classes of multifunctions, which admit some kind of approximating mappings.

Definition 2.6.5 *Let $\mathcal{Y}$ be any class of multifunctions, the mapping $\Phi \in \mathcal{X}$ is said to be* approximable *by $\mathcal{Y}$ if there exists a sequence $\{\Phi^k\} \subseteq \mathcal{Y} \cap \mathcal{X}$ such that $\Phi^k \xrightarrow{g} \Phi$.*

Theorem 2.6.6 *Let $d > 0$, $c \in \mathcal{C}$, $q \in \mathbb{R}^n$, and $\Phi \in \mathcal{X}$.*

(a) *If Φ is approximable by copositive c-subhomogeneous mappings and $q \in [\mathcal{S}(0, \Phi)]^{\#}$, then $\mathcal{S}(q, \Phi)$ is nonempty;*

(b) *If Φ is approximable by q-pseudomonotone mappings and $\mathrm{V}_q(\Phi) = \{0\}$, then $\mathcal{S}(q, \Phi)$ is nonempty.*

Proof. It follows from Theorems 2.6.2 to 2.6.4. ■

Theorem 2.6.7 *Let $d > 0$, $c \in \mathcal{C}$, $q \in \mathbb{R}^n$, and $\Phi = \Phi^0 + \Psi^0$ where $\Phi^0 \in \mathcal{X}$ and $\Psi^0 \in \mathcal{X}$ is approximable by copositive uniformly bounded with respect to the same set mappings.*

(a) *If $\Phi^0 \in \mathbf{GT}(d) \cup \mathbf{G\tilde{T}}(d)$ is approximable by c-subhomogeneous mappings and $q \in [\mathcal{S}(0, \Phi^0)]^{\#}$, then $\mathcal{S}(q, \Phi)$ is nonempty;*

(b) *If Φ^0 is approximable by copositive c-Moré mappings and $q \in [\mathrm{U}_0(\Phi^0)]^{\#}$, then $\mathcal{S}(q, \Phi)$ is nonempty.*

Proof. Let $\{\Phi^k\}$ and $\{\Psi^k\}$ be the sequence of mappings that approximate Φ^0 and Ψ^0 respectively. Let $\{\sigma_k\}$ be an increasing sequence of positive numbers converging to $+\infty$, $d > 0$, and $\{(x^k, y^k, r^k)\}$ be a sequence of solutions to $(\mathrm{PMVIP_k})$ for $\Phi = \Phi^0$, $\Psi = \Psi^0$, and $q^k = q$ for all k.
If $\{x^k\}$ is bounded, by Theorems 1.3.2 and 1.3.3, any limit point of such a sequence belongs to $\mathcal{S}(q, \Phi)$.

Otherwise, we may consider (by redefining σ_k if necessary) that $\sigma_k = \langle d, x^k \rangle$. Since $\langle d, \frac{x^k}{\sigma_k} \rangle = 1$, up to subsequences, there exists $0 \neq v \geq 0$ such that $\frac{x^k}{\sigma_k} \to v$. By applying the Basic Lemma (for $\Phi = \Phi^0$, $\Psi = \Psi^0$, and $q^k = q$ for all k) we obtain that:

(a): there exist $w \in \Phi^0(v)$, $\{r^{k_m}\}$, r, and $\emptyset \neq J_v \subset I$, such that $r^{k_m} \to r$, $0 \neq v \in \mathcal{S}(-\langle w, v \rangle d, \Phi^0)$, $\langle w, v \rangle \leq 0$, $\langle r, v \rangle \geq 0$, and $\langle w, z \rangle = \langle w, v \rangle$ for all $z \in \Delta_{J_v}$. As $\Phi^0 \in \mathbf{G}(d)$ we conclude that $\langle w, v \rangle = 0$ and $w_{J_v} = 0$. If $\Phi \in \mathbf{T}(d)$ (resp. $\Phi \in \tilde{\mathbf{T}}(d)$), by (2.5) (resp. (2.6)) $v \in [\Phi^0(\mathrm{pos}^+\Delta_{J_v})]^*$ (resp. $\langle y, x \rangle \geq 0$ for $x \in \mathrm{pos}^+\Delta_{J_v}$, $y \in \Phi(x)$), which in turn implies $\langle y^{k_m}, v \rangle \geq 0$ (resp. $\langle y^{k_m}, x^{k_m} \rangle \geq 0$). By setting $x = 0$ in $(\mathrm{PMVIP_k})$ and by (c) of the Basic Lemma we get

$$\langle y^{k_m} + r^{k_m} + q, v \rangle = \langle y^{k_m} + r^{k_m} + q, \frac{x^{k_m}}{\sigma_{k_m}} \rangle \leq 0. \tag{2.15}$$

then $\langle r^{k_m} + q, v \rangle \leq 0$ (resp. $\langle q, \frac{x^{k_m}}{\sigma_{k_m}} \rangle \leq 0$) thus $\langle q, v \rangle \leq 0$ contradicting the choice of q.

(b): there exist w and $w^k \in \Phi^k(\frac{x^k}{\sigma_k})$ such that $w^{k_m} \to w \in \Phi^0(v)$ and $\langle w, v \rangle \leq 0$. Since each Φ^k is copositive we get $\langle w, v \rangle = 0$, thus $0 \neq v \in \mathrm{U}_0(\Phi^0)$. Moreover, from (2.15) and copositivity we get $\langle q, x^{k_m} \rangle \leq 0$, thus $\langle q, v \rangle \leq 0$ contradicting the choice of q. ■

Notice that under the assumptions of Theorem 2.6.7 we actually prove that any limit point of every approximate sequence $\{x^k\}$ constructed through $(\mathrm{PMVIP_k})$ (which is bounded) is a solution to (MCP).

2.7 Asymptotic analysis via the outer norm

Let $\Phi : \mathbb{R}^n_+ \rightrightarrows \mathbb{R}^n$ be an homogeneous multifunction of degree 1 (or simply homogeneous), in [68, p. 365] is defined the *outer norm* $|\Phi|^+ \in [0, +\infty]$

$$|\Phi|^+ \doteq \sup \{\|y\| : \, x \in \mathbb{B}, \, y \in \Phi(x)\}. \tag{2.16}$$

This is the infimum over all constants $k \geq 0$ such that $\|y\| \leq k\,\|x\|$ for all $(x, y) \in \operatorname{gph}\Phi$. When Φ is actually a linear mapping, i.e. $\Phi(x) = Mx$ for $M \in \mathbb{R}^{n\times n}$, then $|\Phi|^+ = \|M\|$.
In general, however, homogeneous multifunctions, don't even form a vector space under addition and scalar multiplication, so $|\cdot|^+$ is not truly a ´´norm``. Nevertheless, elementary rules are valid like

$$|\lambda\Phi|^+ = |\lambda|\,|\Phi|^+,\ |\Phi^1 + \Phi^2|^+ \leq |\Phi^1|^+ + |\Phi^2|^+,\ |\Phi^2 \circ \Phi^1|^+ \leq |\Phi^2|^+|\Phi^1|^+.$$

We shall define a similar object for c-subhomogeneous multifunctions.

Definition 2.7.1 *Let $d > 0$, $c \in \mathcal{C}$, and $\Phi : \mathbb{R}^n_+ \rightrightarrows \mathbb{R}^n$ be a c-subhomogeneous multifunction. The d-*outer norm *of Φ is by definition*

$$|\Phi|^+_d \doteq \sup\left\{\|y\|_d : x \in \Delta_d,\ y \in \Phi(x)\right\}. \tag{2.17}$$

Let $\Phi \in \mathcal{X}$ be c-subhomogeneous on Δ_d such that $\Phi(0) = \{0\}$, by Proposition 1.3.1 the set $\Phi(\Delta_d)$ is compact, therefore the supremum in (2.17) is attained and we may write $\max$ instead of $\sup$, and $|\Phi|^+_d < +\infty$. Moreover

$$\|y\|_d \leq |\Phi|^+_d\, c(\|x\|_d) \text{ for all } x \geq 0,\ y \in \Phi(x). \tag{2.18}$$

As we already know $|\cdot|^+_d$ is not a norm. However, the mapping defined by $(\Phi^1, \Phi^2) \mapsto |\Phi^1 - \Phi^2|^+_d$ is a metric on the set of c-subhomogeneous on Δ_d cuscos such that $\Phi(0) = \{0\}$. If Φ is a linear mapping, i.e., $\Phi(x) = Mx$ for $M \in \mathbb{R}^{n\times n}$, by (1.1) we obtain that $|\Phi|^+_d = \|M\|_d$.

Now we present an alternative of the asymptotic analysis by means of the d-outer norm.

Lemma 2.7.2 *Let $d > 0$, $c \in \mathcal{C}$, $\{\sigma_k\}$ be an increasing sequence of positive numbers converging to $+\infty$; $q, q^k \in \mathbb{R}^n$; and $\Phi, \Psi, \Phi^k, \Psi^k \in \mathcal{X}$ be such that $|\Phi^k - \Phi|^+_d \to 0$, $\Psi^k \xrightarrow{g} \Psi$, $q^k \to q$, and $\left\{(x^k, y^k, r^k)\right\}$ be a sequence of solutions to $(\mathrm{PMVIP_k})$ such that $\langle d, x^k\rangle = \sigma_k$ and $\frac{x^k}{\sigma_k} \to v$. Then, there exist subsequences $\left\{(x^{k_m}, y^{k_m}, r^{k_m})\right\}$, $\{\sigma_{k_m}\}$, numbers k_0, $m_0 \in \mathbb{N}$, and an index set $\emptyset \neq J_v \subseteq I$ such that in addition to the properties (a)-(c) of the Basic Lemma we also obtain*

(d′) *if Φ, Φ^k are c-subhomogeneous such that $\Phi(0) = \Phi^k(0) = \{0\}$ and Ψ^k are uniformly bounded with respect to the same set, then the subsequences $\{y^{k_m}\}$, $\{r^{k_m}\}$, $\{\sigma_{k_m}\}$ may be chosen in such a way that there are vectors w and r such that $\frac{1}{c(\sigma_{k_m})} y^{k_m} \to w \in \Phi(v)$, $r^{k_m} \to r$, $\langle w, v\rangle \leq 0$, $\langle w, y\rangle \geq \langle d, y\rangle\langle w, v\rangle$ for all $y \geq 0$, and $\langle w, z\rangle = \langle w, v\rangle$, for all $z \in \Delta_{J_v}$.*

Proof. (d′): By assumption $\frac{y^k}{c(\sigma_k)} \in \Phi^k(\frac{x^k}{\sigma_k})$. Suppose that $w^k \in \Phi(\frac{x^k}{\sigma_k})$, then $\|\frac{y^k}{c(\sigma_k)} - w^k\| \leq |\Phi^k - \Phi|_d^+$. As $\left\{\frac{x^k}{\sigma_k}\right\}$ is bounded, by Proposition 1.3.1 we may assume that $w^{k_m} \to w \in \Phi(v)$ and by the above inequality we get $\frac{y^{k_m}}{c(\sigma_{k_m})} \to w$. Moreover, from $r^k \in \Psi^k(x^k)$, since Ψ^k is uniformly bounded with respect to the same set, the sequence $\{r^k\}$ is bounded and $r^{k_m} \to r$.
Proceeding exactly as in the last part of the proof of (d) from the Basic Lemma we obtain the properties for v and w. ■

Now we give sensitivity results for problem (MCP) by using the d-outer norm. As consequences of these results we shall obtain in the next chapters continuity properties for the solution set mapping.

Proposition 2.7.3 *Let $d > 0$, $c \in \mathcal{C}$, $q^0 \in \mathbb{R}^n$, and $\Phi^0 \in \mathcal{X}$ be c-subhomogeneous such that $\Phi^0(0) = \{0\}$. If $q^0 \in [\mathcal{S}(0, \Phi^0)]^\#$, then there exists $\varepsilon > 0$ such that for all $q \in \mathbb{R}^n$ and all $\Phi \in \mathcal{X}$ which are c-subhomogeneous such that $\Phi(0) = \{0\}$ satisfying*

$$\|q - q^0\| + |\Phi - \Phi^0|_d^+ < \varepsilon,$$

one has $q \in [\mathcal{S}(0, \Phi)]^\#$.

Proof. Suppose on the contrary, that there exist sequences $\{q^k, \Phi^k, v^k\}$ satisfying $q^k \to q^0$, $|\Phi^k - \Phi^0| \to 0$, $0 \neq v^k \in \mathcal{S}(0, \Phi^k)$, and $\langle q^k, v^k\rangle \leq 0$ with $\Phi^k \in \mathcal{X}$ being c-subhomogeneous such that $\Phi^k(0) = \{0\}$. By c-subhomogeneity we may assume that $\|v^k\|_d = 1$, therefore up to subsequences $v^k \to v$ and $\|v\|_d = 1$. Moreover, for all k , there exist $w^k \in \Phi^k(v^k)$ such that $w^k \geq 0$ and $\langle v^k, w^k\rangle = 0$. Let $u^k \in \Phi^0(v^k)$ then $\|w^k - u^k\| \leq |\Phi^k - \Phi^0|_d^+$. Since $v^k \to v$ by

Proposition 1.3.1 we may consider that $u^k \to w \in \Phi^0(v)$ for some w. From the above inequality it follows that $w^k \to w$. Furthermore, $w \geq 0$ and $\langle w, v\rangle = 0$. Hence $0 \neq v \in \mathcal{S}(0, \Phi^0)$ and $\langle q^0, v\rangle \leq 0$, contradicting the choice of q^0. ■

Theorem 2.7.4 *Let $d > 0$, $c \in \mathcal{C}$, $q^0 \in \mathbb{R}^n$, and $\Phi^0 \in \mathcal{X}$ be c-subhomogeneous such that $\Phi^0(0) = \{0\}$. If $q^0 \in [\mathcal{S}(0, \Phi^0)]^{\#}$, then there exists $\varepsilon > 0$ such that for all $q \in \mathbb{R}^n$ and all $\Phi \in \mathcal{X}$ which are c-subhomogeneous such that $\Phi(0) = \{0\}$ and from $\mathbf{GT}(d) \cup \mathbf{G\tilde{T}}(d)$ satisfying $\|q - q^0\| + |\Phi - \Phi^0|_d^+ < \varepsilon$, the set $\mathcal{S}(q, \Phi)$ is nonempty and compact.*

Proof. This follows from the above proposition and Corollary 2.5.3 and Theorem 2.5.8. ■

Theorem 2.7.5 *Let $d > 0$, $c \in \mathcal{C}$, $q^0 \in \mathbb{R}^n$, and $\Phi^0 \in \mathcal{X}$ be c-subhomogeneous such that $\Phi^0(0) = \{0\}$. If $q^0 \in [\mathcal{S}(0, \Phi^0)]^{\#}$, then there exist $\varepsilon > 0$ and $r > 0$ such that for all $q \in \mathbb{R}^n$ and all $\Phi \in \mathcal{X}$ which are copositive c-subhomogeneous such that $\Phi(0) = \{0\}$, the following implication holds*

$$\|q - q^0\| + |\Phi - \Phi^0|_d^+ < \varepsilon \implies \|\mathcal{S}(q, \Phi)\| \leq r,$$

Proof. By the above theorem for such q and Φ the set $\mathcal{S}(q, \Phi)$ is nonempty and compact.
Suppose on the contrary that there exist sequences $\{q^k, \Phi^k, x^k\}$ satisfying $q^k \to q^0$, $|\Phi^k - \Phi^0|_d^+ \to 0$, $x^k \in \mathcal{S}(q^k, \Phi^k)$, and $\langle d, x^k\rangle \to +\infty$. There exists a sequence $\{y^k\}$ such that (2.13) holds for all k. Setting $\sigma_k = \langle d, x^k\rangle$, up to subsequences, $\frac{x^k}{\sigma_k} \to v \neq 0$. Clearly (x^k, y^k) is a solution of problem $(\mathrm{PMVIP_k})$ for $\Phi = \Phi^0$, $\Psi^k = 0$, and $q = q^0$, so we can apply Lemma2.7.2 (for $\Phi = \Phi^0$, $\Psi^k = 0$, and $q = q^0$ for all k). If Φ^k is copositive c-subhomogeneous such that $\Phi^k(0) = \{0\}$ for all k, by (d) of such a Lemma and (2.13) we obtain that $0 \neq v \in \mathcal{S}(0, \Phi^0)$ and $\langle q^k, x^k\rangle \leq 0$, thus $\langle q^0, v\rangle \leq 0$ contradicting the choice of q^0. ■

2.8 Estimates for the solution set

In this section we extend and generalize some results from [55], where the monotone linear complementarity problem is studied; [43], where the numerical range of an operator is used to obtain bounds for the linear complementarity problem in Hilbert spaces; and [65], where a bound for the solution set of a quasidifferentiable convex programming problem is obtained. Indeed, we consider the set-valued case and obtain bounds for the solution set to problem (MCP) under subhomogeneity, Moré and pseudomonotonicity properties.

If $\Phi \in \mathcal{X}$ is c-subhomogeneous (on Δ_d), we infer that

$$m_\Phi \left\|x\right\|_d \, c(\|x\|_d) \leq \langle y, x\rangle \leq M_\Phi \left\|x\right\|_d \, c(\|x\|_d) \ \forall x \geq 0,\ y \in \Phi(x). \tag{2.19}$$

In the following we denote $d_0 \doteq \min_{1\leq i\leq n} d_i > 0$. We point out that Ψ is uniformly bounded if and only if $||\Psi|| < +\infty$, where

$$||\Psi|| \doteq \sup\left\{||y|| :\ y \in \Psi(x),\ x \geq 0\right\}.$$

Theorem 2.8.1 *Let $d > 0$, $c \in \mathcal{C}$, $q \in \mathbb{R}^n$, and $\Phi, \Psi \in \mathcal{X}$.*

(a) *Assume Φ is c-subhomogeneous and Ψ is uniformly bounded:*

- *if $M_\Phi < 0$, then $\mathcal{S}(q, \Phi + \Psi) \subseteq \left\{x \geq 0 : c(\|x\|_d) \leq \max\left(c(0), \frac{||\Psi||+\|q\|}{d_0|M_\Phi|}\right)\right\}$;*
- *if $m_\Phi > 0$, then $\mathcal{S}(q, \Phi + \Psi) \subseteq \left\{x \geq 0 : c(\|x\|_d) \leq \max\left(c(0), \frac{||\Psi||+\|q\|}{d_0\ m_\Phi}\right)\right\}$;*

(b) *Assume Φ is c-Moré, Ψ is uniformly bounded, and $m_\Phi > 0$, then*

$$\mathcal{S}(q, \Phi + \Psi) \cap \mathrm{cl}(\mathbb{R}^n_+ \setminus \mathbb{B}_d) \subseteq \left\{x \geq 0 : c(\|x\|_d) \leq \max\left(c(0), \frac{||\Psi|| + \|q\|}{d_0\ m_\Phi}\right)\right\};$$

(c) *Assume Φ is monotone, Ψ is copositive uniformly bounded, there exist $0 \neq x^0 \geq 0$, $y^0 \in \Phi(x^0)$ such that $y^0 + q > 0$, then*

$$\mathcal{S}(q, \Phi + \Psi) \subseteq \left\{x \geq 0 : \|x\|_1 \leq \frac{\langle y^0 + q, x^0\rangle + \|\Psi\|\ \|x^0\|}{\min\limits_{1\leq i\leq n} (y^0 + q)_i}\right\}.$$

Proof. (a): First we notice that for $u \geq 0$ and $w \in \Psi(u)$, $\|w\| \leq ||\Psi||$. Assume $M_\Phi < 0$, and let $0 \neq x \geq 0$, $y \in \Phi(x)$, and $r \in \Psi(x)$. By (2.19) we obtain

$$\begin{aligned}\langle y + r + q, x\rangle &\leq M_\Phi \|x\|_d \, c(\|x\|_d) + (\|r\| + \|q\|) \|x\| \\ &\leq \|x\|_d \left(M_\Phi \, c(\|x\|_d) + \frac{1}{d_0}(||\Psi|| + \|q\|) \right).\end{aligned}$$

It follows that $M_\Phi \, c(\|x\|_d) + \frac{1}{d_0}(||\Psi|| + \|q\|) \geq 0$ if $0 \neq x \in \mathcal{S}(q, \Phi + \Psi)$.

Let $m_\Phi > 0$, $0 \neq x \geq 0$, $y \in \Phi(x)$, and $r \in \Psi(x)$. By (2.19) again

$$\begin{aligned}\langle y + r + q, x\rangle &\geq m_\Phi \|x\|_d \, c(\|x\|_d) - (\|r\| + \|q\|) \|x\| \\ &\geq \|x\|_d \left(m_\Phi \, c(\|x\|_d) - \frac{1}{d_0}(||\Psi|| + \|q\|) \right).\end{aligned}$$

It follows that $m_\Phi \, c(\|x\|_d) - \frac{1}{d_0}(||\Psi|| + \|q\|) \leq 0$ if $0 \neq x \in \mathcal{S}(q, \Phi + \Psi)$.
(b): Let $m_\Phi > 0$, $x \geq 0$ such that $\|x\|_d \geq 1$, $y \in \Phi(x)$, and $r \in \Psi(x)$. Since $y \in \Phi(\|x\|_d \frac{x}{\|x\|_d})$ by definition there exists $z \in \Phi(\frac{x}{\|x\|_d})$ such that

$$\langle y, x\rangle \geq c(\|x\|_d) \, \langle z, x\rangle = \|x\|_d \, c(\|x\|_d) \langle z, \frac{x}{\|x\|_d}\rangle \geq \|x\|_d \, c(\|x\|_d) m_\Phi$$

thus we obtain that

$$\begin{aligned}\langle y + r + q, x\rangle &\geq m_\Phi \|x\|_d \, c(\|x\|_d) - (\|r\| + \|q\|) \|x\| \\ &\geq \|x\|_d \left(m_\Phi \, c(\|x\|_d) - \frac{1}{d_0}(||\Psi|| + \|q\|) \right).\end{aligned}$$

It follows that $m_\Phi \, c(\|x\|_d) - \frac{1}{d_0}(||\Psi|| + \|q\|) \leq 0$ if $x \in \mathcal{S}(q, \Phi + \Psi)$ and $\|x\|_d \geq 1$.
(c): Let $x \in \mathcal{S}(q, \Phi + \Psi)$, there exist $y \in \Phi(x)$ and $r \in \Psi(x)$ such that $\langle y + r + q, u - x\rangle \geq 0$ for all $u \geq 0$, since Φ is $(r + q)$-pseudomonotone we get $\langle y^0 + r + q, x^0 - x\rangle \geq 0$ and since Ψ is copositive

$$\langle y^0 + r + q, x^0\rangle \geq \langle y^0 + r + q, x\rangle \geq \langle y^0 + q, x\rangle \geq \min_{1 \leq i \leq n} (y^0 + q)_i \, \|x\|_1 .$$

Since Ψ is uniformly bounded, $\langle r, x^0\rangle \leq \|\Psi\| \; \|x^0\|$. Thus

$$\langle y^0 + q, x^0 \rangle + \|\Psi\| \, \|x^0\| \geq \min_{1 \leq i \leq n} (y^0 + q)_i \, \|x\|_1. \blacksquare$$

Remark 2.8.2 *One can check that hypothesis $m_\Phi > 0$ in (a) implies that $\Phi \in \mathbf{G\tilde{T}}(d)$ and $\mathcal{S}(0, \Phi) = \{0\}$, which in turn implies that $\mathcal{S}(q, \Phi + \Psi)$ is nonempty and compact for all $q \in \mathbb{R}^n$ provided Ψ is copositive (Theorem 2.5.2). The hypothesis $b_\Phi \doteq \min\{\langle y, x \rangle : x \geq 0, \|x\|_d \leq 1, y \in \Phi(x)\} > 0$ in (b) implies that $m_\Phi > 0$, Φ is copositive, and $\mathrm{U}(0, \Phi) = \{0\}$, which in turn implies that $\mathcal{S}(q, \Phi + \Psi)$ is nonempty and compact for all $q \in \mathbb{R}^n$ provided Ψ is copositive (Theorem 2.5.5). Similarly, the hypothesis in (c) implies that $\mathcal{S}(q, \Phi + \Psi)$ is nonempty and compact provided Φ is copositive as well (Theorem 2.5.8 implication $(a) \Rightarrow (b)$ and Theorem 2.5.7).*

Corollary 2.8.3 *Let $d > 0$, $c \in \mathcal{C}$, $q \in \mathbb{R}^n$, and $\Phi \in \mathcal{X}$.*

(a) *Assume Φ is c-subhomogeneous:*

- *if $M_\Phi < 0$, then $\mathcal{S}(q, \Phi) \subseteq \left\{x \geq 0 : c(\|x\|_d) \leq \max\left(c(0), \frac{\|q\|}{d_0 |M_\Phi|}\right)\right\}$;*
- *if $m_\Phi > 0$, then $\mathcal{S}(q, \Phi) \subseteq \left\{x \geq 0 : c(\|x\|_d) \leq \max\left(c(0), \frac{\|q\|}{d_0 \, m_\Phi}\right)\right\}$;*

(b) *Assume Φ is c-Moré and $m_\Phi > 0$, then*

$$\mathcal{S}(q, \Phi) \cap \mathrm{cl}(\mathbb{R}^n_+ \setminus \mathbb{B}_d) \subseteq \left\{x \geq 0 : c(\|x\|_d) \leq \max\left(c(0), \frac{\|q\|}{d_0 \, m_\Phi}\right)\right\};$$

(c) *Assume Φ is q-pseudomonotone, and there exist $0 \neq x^0 \geq 0$, $y^0 \in \Phi(x^0)$ such that $y^0 + q > 0$, then*

$$\mathcal{S}(q, \Phi) \subseteq \left\{ x \geq 0 : \|x\|_1 \leq \frac{\langle y^0 + q, x^0 \rangle}{\min\limits_{1 \leq i \leq n} (y^0 + q)_i} \right\}.$$

Proof. (a)-(b): We set $\Psi = 0$ in (a)-(b) of the above theorem.
(c): We proceed as in (c) of the above theorem with $\Psi = 0$, and taking into account that Φ is q-pseudomonotone. $\blacksquare$

Remark 2.8.4 *The hypothesis in (c) implies that $\mathcal{S}(q, \Phi)$ is nonempty and compact (Theorem 2.5.8).*

Chapter 3

The polyhedral complementarity problem

In this chapter our main concern is the study of the *polyhedral complementarity problem*, i.e., the problem $\mathrm{MCP}(q, \Phi)$ where $\Phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a piecewise polyhedral multifunction. For such a problem we have some additional information which can facilitate the analysis. For instance the set $\mathcal{D}(\Phi)$ is closed, piecewise polyhedral multifunctions are locally $UL(\lambda)$ for some λ and the solutions set mapping $q \mapsto \mathcal{S}(q) \doteq \mathcal{S}(q, \Phi)$ for Φ being a piecewise polyhedral multifunction is piecewise polyhedral as well. Due to this fact it makes sense to try to approximate complementarity problems by polyhedral ones, we give an example of this approximation in the last section. Our research on this topic is in progress.

3.1 Piecewise polyhedral multifunctions

Definition 3.1.1 *A multifunction* $\Phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ *is said to be* piecewise polyhedral *if* $\mathrm{gph}\,\Phi$ *is piecewise polyhedral i.e. is expressible as the union of finitely many polyhedral sets, called components. If* Φ *has one component i.e* $\mathrm{gph}\,\Phi$ *is a polyhedron, then it* *is said to be a* polyhedral *multifunction.*

Let Φ be a piecewise polyhedral multifunction then

$$\mathrm{gph}\,\Phi = \bigcup_{\mathrm{i}=1}^{\mathrm{m}} \mathrm{P_i}$$

where each $P_i \subseteq \mathbb{R}^n \times \mathbb{R}^n$ is a polyhedral set. Let $(v, w) \in \mathrm{gph}\,\Phi$ we denote by $P_{v,w} \doteq \bigcup_{1 \le i \le m} \{P_i : (v, w) \in P_i\}$ the union of polyhedral sets P_i containing (v, w).

The class of piecewise polyhedral multifunctions can be shown to be closed under (finite) addition, scalar multiplication, and (finite) composition. The inverse mapping of a piecewise polyhedral multifunction is piecewise polyhedral as well. Moreover, a graph-convex piecewise polyhedral multifunction is polyhedral by Lemma 1.1.1.

Example 3.1.2 **1.** *The multifunction defined by $\Phi(x) = Mx$ where $M \in \mathbb{R}^{n\times n}$, is a polyhedral mapping.*

2. *The multifunction $\Phi(x) = \{y : Ax + Qy \le b\}$ where $A \in \mathbb{R}^{l\times n}$, $Q \in \mathbb{R}^{l\times n}$ and $b \in \mathbb{R}^l$, is polyhedral. In fact, every polyhedral multifunctions has this form.*

3. *Let $\mathbb{R}^n_+ = \cup_{1\le i\le k} C_i$ where each subset C_i is a polyhedral set. The multifunction $\Phi(x) = \{y : A^i x + Q^i y \le b^i\}$ if $x \in C_i$, where $A^i \in \mathbb{R}^{l_i\times n}$, $Q \in \mathbb{R}^{l_i\times n}$ and $b \in \mathbb{R}^l_i$, is piecewise polyhedral.*

4. *The multifunction Φ_7 from Example 2.1.2 is piecewise polyhedral and $\bar{c}$-subhomogeneous on Δ_1 but not c-homogeneous for any $c \in \mathcal{C}$.*

5. *The multifunction defined by $\Phi(1) = [0, 1]$ and $\Phi(x) = 0$ if $x \ne 1$, is piecewise polyhedral and λ^γ-subhomogeneous on Δ_1 for all $\gamma > 0$.*

6. *A function $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ is called* piecewise linear-quadratic *if $\mathrm{dom}\, f$ can be represented as the union of finitely many polyhedral sets, relative to each of which $f(x)$ is given by an expression of the form $\frac{1}{2}\langle x, Ax\rangle + \langle a, x\rangle + \alpha$, for $\alpha \in \mathbb{R}$, $a \in \mathbb{R}^n$, and $A \in \mathbb{R}^{n\times n}$ a symmetric matrix. If $A = 0$ the term* piecewise linear *is used.*

If $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ is a proper lsc convex piecewise linear-quadratic function, then the subgradient mapping ∂f is piecewise polyhedral [68, Prop. 12.30]. *For instance, if $C \subset \mathbb{R}^n$ is a nonempty polyhedral set, the multifunctions $\Phi_1 = \partial\sigma_C$ and $\Phi_2 = \partial d_C^2$ are piecewise polyhedral, since σ_C is piecewise linear* [68, Prop. 8.29] *and d_C^2 is piecewise linear-quadratic* [68, Ex. 11.28].

7. *A norm $\|\cdot\|_P$ on $\mathbb{R}^n$ is said to be a* polyhedral norm *if the corresponding unit ball $\{x \in \mathbb{R}^n : \|x\|_P \leq 1\}$ is polyhedral. A norm $\|\cdot\|_P$ on $\mathbb{R}^n$ is polyhedral if and only if there exist vectors $c^1, \ldots, c^r \in \mathbb{R}^n$ such that $\|x\|_P = \max\limits_{1\leq i\leq r}\langle c^i, x\rangle$. The* metric projection *from $(\mathbb{R}^n, \|\cdot\|_P)$ to a polyhedral subset $K \subseteq \mathbb{R}^n$ is the multifunction defined by $\Pi_{K,P}(x) \doteq \arg\min\limits_{y\in K}\|y - x\|_P$. This multifunction is piecewise polyhedral* [26, Th. 18].

8. *The multifunction $\Phi(x) = Mx + \partial\sigma_C(x)$ where $M \in \mathbb{R}^{n\times n}$ and $C \subseteq \mathbb{R}^n$ is a polyhedron, is polyhedral.*

The following result shows that for polyhedral multifunctions it makes sense work only with $\mathbf{T}(d)$-mappings.

Proposition 3.1.3 *Let $d > 0$ and $\Phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a polyhedral multifunction.*

(a) *If Φ is a $\tilde{\mathbf{T}}(d)$-mapping, then it is a $\mathbf{T}(d)$-mapping;*

(b) *If $0 \in \Phi(0)$, then Φ is λ-Moré and $\Phi(\lambda x) \subseteq \lambda\Phi(x)$ for all $x \geq 0$ and $\lambda \geq 1$, whereas $\Phi(\lambda x) \supseteq \lambda\Phi(x)$ for all $x \geq 0$ and $0 < \lambda \leq 1$;*

(c) *Φ is λ-subhomogeneous if and only if it is λ-homogeneous;*

(d) *If Φ is λ-homogeneous and $\Phi(0) = \{0\}$, then Φ is superadditive and homogeneous of degree $\gamma = 1$.*

Proof. (a): Let v, w, J such that the premiss of (2.5) holds. Let $x \in \mathrm{pos}^+\Delta_J$, $y \in \Phi(x)$, clearly $tv + (1-t)x \in \mathrm{pos}^+\Delta_J$ for all $t \in]0, 1[$. Since $\mathrm{gph}\,\Phi$ is convex, we get $tw + (1-t)y \in \Phi(tv + (1-t)x)$ and by (2.6) we obtain that $\langle tw +$

$(1-t)y, tv+(1-t)x\rangle \geq 0$ for all $t \in]0,1[$. By the choice of x, v, w, we get $t\langle y, v\rangle + (1-t)\langle y, x\rangle \geq 0$ for all $t \in]0,1[$, thus $\langle y, v\rangle \geq 0$.
(b): By Proposition 2.1.5(d) Φ is λ-Moré. Since Φ is polyhedral, there exist $A \in \mathbb{R}^{l\times n}$, $Q \in \mathbb{R}^{l\times n}$, and $b \in \mathbb{R}^l$ such that $\Phi(x) = \{y : \ Ax + Qy \leq b\}$. From $0 \in \Phi(0)$ we obtain $b \geq 0$ and if $x \geq 0$, $\lambda \geq 1$, and $y \in \Phi(\lambda x)$, then $Ax + \frac{1}{\lambda}Qy \leq \frac{1}{\lambda}b \leq b$, thus $y \in \lambda\Phi(x)$. The remaining part can be obtained similarly.
(c): It follows from (b).
(d): It follows from (c) and the graph-convexity of Φ. ■

We need the following continuity [67, Prop. 1] and closedness [41, Prop. 3] properties of piecewise polyhedral multifunctions.

Proposition 3.1.4 *Let $\mathcal{F} : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ be a piecewise polyhedral multifunction, Then there exists a constant λ such that $\mathcal{F}$ is locally* UL(λ) *at each $x^0 \in \mathbb{R}^m$.*

Proposition 3.1.5 *Let $\Phi : \mathbb{R}^n_+ \rightrightarrows \mathbb{R}^n$ be a piecewise polyhedral multifunction. Then the sets $\mathcal{D}(\Phi)$ and $\mathbb{R}^n_+ - \Phi(\mathbb{R}^n_+)$ are closed.*

Proof. The closedness of the former set is proved in [41], by proceeding analogously we can prove that property of the latter set. ■

3.2 New classes of multifunctions

When dealing with piecewise polyhedral multifunctions, we can weak the notion of $\mathbf{T}(d)$-mapping, in order to obtain finer existence results.

Definition 3.2.1 *Let $d > 0$, $\Phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a piecewise polyhedral multifunction ($\{P_i\}_{i=1}^m$ as above) such that $0 \in \Phi(0)$. We say that Φ is a*

- $\mathbf{T}_p(d)$-mapping, *if, for any index subset $J \subseteq I$, one has*

$$\left.\begin{array}{l} v \geq 0, w \geq 0, w \in \Phi(v) \\ w_J = 0, \emptyset \neq \mathrm{supp}\{v\} \subseteq J \end{array}\right\} \Longrightarrow \begin{array}{l} \langle y, v\rangle \geq 0, \\ \forall x \in \mathrm{pos}^+\Delta_J, (x,y) \in P_{v,w}. \end{array} \tag{3.1}$$

- $\tilde{\mathbf{T}}_p(d)$-mapping *if, for any index subset* $J \subseteq I$,

$$\left.\begin{array}{l} v \geq 0, w \geq 0, w \in \Phi(v) \\ w_J = 0, \emptyset \neq \operatorname{supp}\{v\} \subseteq J \end{array}\right\} \Longrightarrow \begin{array}{l} \langle y, x\rangle \geq 0, \\ \forall x \in \operatorname{pos}^+\Delta_J, (x, y) \in P_{v,w}. \end{array} \tag{3.2}$$

- $\mathbf{GT}_p(d)$-mapping *if, it is* $\mathbf{G}(d)$ *and* $\mathbf{T}_p(d)$.

Proposition 3.2.2 *Let $d > 0$ and $\Phi : \mathbb{R}^n_+ \rightrightarrows \mathbb{R}^n$ be a piecewise polyhedral multifunction. If Φ is a $\tilde{\mathbf{T}}_p(d)$-mapping, then it is a $\mathbf{T}_p(d)$-mapping. Moreover, for polyhedral multifunctions the classes $\mathbf{T}(d)$ and $\mathbf{T}_p(d)$ coincide.*

Proof. We proceed exactly as in the proof of Proposition 3.1.3(a), instead of the convexity of $\operatorname{gph}\Phi$ we use the fact that $P_{v,w}$ is star-shaped relative to the point (v, w). The last assertion holds since for Φ to be polyhedral $P_{v,w} = \operatorname{gph}\Phi$. ■

Henceforth, for the piecewise polyhedral complementarity problem we shall deal only with $\mathbf{T}_p(d)$-mappings.

Remark 3.2.3 *From the above proposition follows that a copositive piecewise polyhedral multifunction is $\mathbf{GT}_p(d)$ for any $d > 0$ (without any superadditivity or homogeneity assumption as in Proposition* 2.3.3(c)).

We introduce the following notation

$$\mathcal{X}_0 \doteq \{\Phi : \mathbb{R}^n_+ \rightrightarrows \mathbb{R}^n : \ \Phi \text{ is compact convex valued}\}$$

By Proposition 3.1.4 we obtain in particular that if $\Phi \in \mathcal{X}_0$ is piecewise polyhedral then $\Phi \in \mathcal{X}$. Thus, for piecewise polyhedral multifunctions in $\mathcal{X}_0$ the Basic Lemma, Lemma 2.3.6 and Proposition 2.4.1 hold. As a consequence we obtain the following result for $\mathbf{T}_p(d)$-mappings.

Lemma 3.2.4 *Let $d > 0$, $c \in \mathcal{C}$; $\Phi, \Psi \in \mathcal{X}_0$ be piecewise polyhedral, and $\{\sigma_k\}$ be an increasing sequence of positive numbers converging to $+\infty$. Assume there exist a sequence $\left\{(x^k, y^k + r^k)\right\} \in \mathcal{W}$ for $\Phi + \Psi$ such that $\frac{x^k}{\sigma_k} \to v$. Then, in addition to the* *existence of w, r, $\{x^{k_m}\}$, $\{w^k\}$, and subindex set $\emptyset \neq J_v \subseteq I$ satisfying the properties established in the Basic Lemma (for $\Phi^k = \Phi$, $\Psi^k = \Psi$, and $q^k = q$ for all k), we also obtain that if Φ is $\mathbf{GT}_p(d)$ c-subhomogeneous and Ψ uniformly bounded, then $w \geq 0$, $w_{J_v} = 0$ (hence $\langle w, v\rangle = 0$), $\langle q, v\rangle \leq 0$, $\langle r, v\rangle \geq 0$, and $\langle y, v\rangle \geq 0$ for all $x \in \mathrm{pos}^+\Delta_J$, $(x, y) \in P_{v,w}$. Moreover, $(\frac{x^{k_m}}{\sigma_{k_m}}, \frac{y^{k_m}}{c(\sigma_{k_m})}) \in P_{v,w}$.*

Proof. We set $\Phi^k = \Phi$, $\Psi^k = \Psi$ and $q^k = q$ for all k in the Basic Lemma. Since $\Phi \in \mathbf{GT}_p(d)$, then (a.1) of Proposition 2.3.6 holds, and by (3.1) we obtain $\langle y, v\rangle \geq 0$ for all $x \in \mathrm{pos}^+\Delta_{J_v}$, $(x, y) \in P_{v,w}$, which in turn implies $\langle y^{k_m}, v\rangle \geq 0$ since $\frac{x^{k_m}}{\sigma_{k_m}} \in \Delta_{J_v}$ and $(\frac{x^{k_m}}{\sigma_{k_m}}, \frac{y^{k_m}}{c(\sigma_{k_m})}) \in P_{v,w}$. Indeed, as $(\frac{x^{k_m}}{\sigma_{k_m}}, \frac{y^{k_m}}{c(\sigma_{k_m})}) \in \mathrm{gph}\,\Phi$, there exists $i_0 \in \{1, 2, \ldots, m\}$ such that up to subsequences $(\frac{x^{k_m}}{\sigma_{k_m}}, \frac{y^{k_m}}{c(\sigma_{k_m})}) \in P_{i_0}$, thus $(v, w) \in P_{i_0}$ and $P_{i_0} \subseteq P_{v,w}$.
From (c) of the Basic Lemma (for $z = v$), and setting $x = 0$ in $(\mathrm{MVIP_k})$ for $\Phi + \Psi$ we get $\langle r^{k_m} + q, v\rangle \leq \langle y^{k_m} + r^{k_m} + q, v\rangle = \langle y^{k_m} + r^{k_m} + q, \frac{x^{k_m}}{\sigma_{k_m}}\rangle \leq 0$, thus $\langle r + q, v\rangle \leq 0$. Since Ψ is copositive from $\langle r^k, x^k\rangle \geq 0$ we get $\langle r, v\rangle \geq 0$ and then $\langle q, v\rangle \leq 0$. ■

Proposition 3.2.5 *Let $d > 0$, $c \in \mathcal{C}$, $q \in \mathbb{R}^n$, and $\Phi, \Psi \in \mathcal{X}_0$ be piecewise polyhedral. If Φ is $\mathbf{T}_p(d)$ c-subhomogeneous and Ψ is copositive uniformly bounded, then $[\mathcal{S}(q, \Phi + \Psi)]^\infty_d \subseteq \mathrm{W}_q(\Phi) \cap \Delta_d$.*

Proof. We proceed exactly as in the proof of Proposition 2.4.1(a) taking into account the above lemma. ■

3.3 Main existence results

Theorem 3.3.1 *Let $d > 0$, $c \in \mathcal{C}$, $\Phi \in \mathcal{X}_0$ be $\mathbf{GT}_p(d)$ c-subhomogeneous piecewise polyhedral, and $\Psi \in \mathcal{X}_0$ be copositive uniformly bounded piecewise polyhedral:*

(a) *if $q \in [\mathcal{S}(0, \Phi)]^*$, then $\mathcal{S}(q, \Phi + \Psi)$ is nonempty;*

(b) *if $q \in [\mathcal{S}(0, \Phi)]^\#$, then $\mathcal{S}(q, \Phi + \Psi)$ is nonempty and compact.*

Proof. (a): Let $\{\sigma_k\}$ be an increasing sequence of positive numbers converging to $+\infty$, $d > 0$, and $\{(x^k, y^k + r^k)\} \in \mathcal{W}$ for $\Phi + \Psi$. Since $\langle d, \frac{x^k}{\sigma_k}\rangle = 1$, up to subsequences, there exists $0 \neq v \geq 0$ such that $\frac{x^k}{\sigma_k} \to v$. Thus,

$$x^k \in D_k,\ y^k \in \Phi(x^k),\ r^k \in \Psi(x^k),\ \langle y^k + r^k + q, x - x^k\rangle \geq 0\ \forall x \in D_k. \tag{3.3}$$

By the Basic Lemma (for $\Phi^k = \Phi$, $\Psi^k = \Psi$, and $q^k = q$ for all k) and Lemma 3.2.4 there exist $w \in \Phi(v)$, r, $\emptyset \neq J_v \subseteq I$, and $\{x^{k_m}\}$ such that $w \geq 0$, $\langle w, v\rangle = 0$, $w_{J_v} = 0$, $\langle q, v\rangle \leq 0$, $\langle r, v\rangle \geq 0$, and $\langle y, v\rangle \geq 0$ for all $x \in \mathrm{pos}^+\Delta_{J_v}$, $(x, y) \in P_{v,w}$, which in turn implies $\langle y^{k_m}, v\rangle \geq 0$ since $(\frac{x^{k_m}}{\sigma_{k_m}}, \frac{y^{k_m}}{c(\sigma_{k_m})}) \in P_{v,w}$. Moreover, $v \in \mathcal{S}(0, \Phi)$ implies $\langle q, v\rangle = 0$. From (2.4) for $\Phi + \Psi$, we get

$$\theta_{k_m} = -\langle y^{k_m} + r^{k_m} + q, \frac{x^{k_m}}{\sigma_{k_m}}\rangle = -\langle y^{k_m} + r^{k_m} + q, v\rangle. \tag{3.4}$$

then $0 \leq \theta_{k_m} \leq -\langle r^{k_m}, v\rangle$. Thus $\liminf_{k\to+\infty} \theta_k = 0$, and the result follows from Theorem 2.2.1 since by Proposition 3.1.5 the set $\mathcal{D}(\Phi + \Psi)$ is closed.
(b): By (a) we obtain that $\mathcal{S}(q, \Phi + \Psi)$ is nonempty. Its boundedness follows from Proposition 3.2.5 since by the choice of q, we get $\mathrm{W}_q(\Phi) = \{0\}$. ■

Remark 3.3.2 *The above theorem generalizes Corollary 3(a) from* [41]*, which is an analogue result valid for Φ being a copositive homogeneous of degree $\gamma > 0$ mapping and $\Psi = 0$.*

Theorem 3.3.3 *Let $d > 0$, $c \in \mathcal{C}$, $q \in \mathbb{R}^n$, $\Phi \in \mathcal{X}_0$ be copositive c-Moré piecewise polyhedral, and $\Psi \in \mathcal{X}_0$ be copositive uniformly bounded piecewise polyhedral:*

(a) *if $q \in [\mathrm{U}_0(\Phi)]^*$, then $\mathcal{S}(q, \Phi + \Psi)$ is nonempty;*

(b) *if $q \in [\mathrm{U}_0(\Phi)]^\#$, then $\mathcal{S}(q, \Phi + \Psi)$ is nonempty and compact.*

Proof. It is a consequence of Theorem 2.5.5. since by Proposition 3.1.5 the set $\mathcal{D}(\Phi + \Psi)$ is closed. ■

For polyhedral multifunctions we can drop the c-subhomogeneous and d-Moré hypotheses and obtain the following existence result.

Corollary 3.3.4 *Let $d > 0$, $c \in \mathcal{C}$, $\Phi \in \mathcal{X}_0$ be polyhedral such that $0 \in \Phi(0)$, and $\Psi \in \mathcal{X}_0$ be copositive uniformly bounded piecewise polyhedral.*

(a) *Assume Φ is $\mathbf{GT}_p(d)$: if $q \in [\mathcal{S}(0,\Phi)]^*$, then $\mathcal{S}(q, \Phi + \Psi)$ is nonempty; if $q \in [\mathcal{S}(0,\Phi)]^\#$, then $\mathcal{S}(q, \Phi + \Psi)$ is nonempty and compact;*

(b) *Assume Φ is copositive: if $q \in [\mathrm{U}_0(\Phi)]^*$, then $\mathcal{S}(q, \Phi + \Psi)$ is nonempty; if $q \in [\mathrm{U}_0(\Phi)]^\#$, then $\mathcal{S}(q, \Phi + \Psi)$ is nonempty and compact;.*

Proof. It is a consequence of Theorems 3.3.1, 3.3.3, Proposition 3.1.4(b) and the reasoning after the definition of c-(sub)homogeneous mappings. ■

We now revise the pseudomonotone case and obtain the following corollaries of Theorems 2.5.8 and 2.5.11 by applying Propositions 3.1.3 and 3.1.4.

Corollary 3.3.5 *Let $q \in \mathbb{R}^n$ and $\Phi \in \mathcal{X}_0$ be q-pseudomonotone piecewise polyhedral. Consider the statements*

(a) *$\mathcal{F}_s(q, \Phi)$ is nonempty;*

(b) *$\mathrm{V}_q(\Phi) = \{0\}$;*

(c) *$\mathcal{S}(q, \Phi)$ is nonempty and compact;*

(d) *There exists a compact convex set $K \subseteq \mathbb{R}^n_+$ such that*

$$\forall\, x \in \mathbb{R}^n_+ \setminus K\ \forall\, y \in \Phi(x)\ \exists\, z \in K :\ \langle y + q, z - x\rangle < 0.$$

The following implications hold: $\quad \text{(a)} \Longrightarrow \text{(b)} \Longleftrightarrow \text{(c)} \Longleftrightarrow \text{(d)}.$
Moreover, if $\Phi(\mathbb{R}^n_+)$ is convex (in particular if Φ is polyhedral), then all the statements are equivalent.

Corollary 3.3.6 *Let $d > 0$, $c \in \mathcal{C}$, $q \in \mathbb{R}^n$, and $\Phi \in \mathcal{X}_0$ be q-pseudomonotone. Consider the statements:*

(a) *$\mathcal{F}(q, \Phi) \neq \emptyset$;*

(b) $v \geq 0,\ v \in -[\Phi(\mathbb{R}^n_+)]^* \implies \langle q, v\rangle \geq 0$;

(c) $\mathcal{S}(q, \Phi) \neq \emptyset$.

The following implications hold: $\quad$ (c) $\Longleftrightarrow$ (a) $\Longrightarrow$ (b)

(i) *if Φ is piecewise polyhedral c-homogeneous and superadditive. Moreover, if $\mathbb{R}^n_+ - \Phi(\mathbb{R}^n_+)$ is convex and $\Phi(0) = \{0\}$ as well, then all the statements are equivalent;*

(ii) *if Φ is polyhedral λ-homogeneous. Moreover, if $\Phi(0) = \{0\}$ as well, then all the statements are equivalent.*

3.4 Sensibility and approximable multifunctions

Let us fix a piecewise polyhedral mapping $\Phi : \mathbb{R}^n_+ \rightrightarrows \mathbb{R}^n$, we define the *solution set mapping* $\mathcal{S} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ by $\mathcal{S}(q) \doteq \mathcal{S}(q, \Phi)$. For such a mapping we obtain the following Lipschitzian property.

Proposition 3.4.1 *There exists a constant λ such that the solution set mapping $\mathcal{S}$ is locally UL(λ) at each $q^0 \in \mathbb{R}^n$, i.e. there exists a neighborhood U of q^0 such that*

$$\mathcal{S}(q) \subseteq \mathcal{S}(q^0) + \lambda \left\| q - q^0 \right\| \mathbb{B} \quad \textit{for all } q \in U.$$

Proof. It is sufficient to prove that $\mathcal{S}(\cdot)$ is a piecewise polyhedral multifunction and the result follows by applying Proposition 3.1.4. Indeed, by definition $\mathrm{gph}\,\Phi = \bigcup_{\mathrm{i=1}}^{\mathrm{m}} \mathrm{P_i}$ where each $P_i \subseteq \mathbb{R}^n \times \mathbb{R}^n$ is a polyhedral set. We define the set

$$\Sigma \doteq \left\{(q, x, y) \in \mathbb{R}^{3n} :\ x \geq 0,\ y \in \Phi(x),\ y + q \geq 0,\ \langle y + q, x\rangle = 0\right\}.$$

In a standard way we can write $\Sigma = \bigcup_{i=1}^{m} \bigcup_{\emptyset \neq \alpha \subset I} X_{i,\alpha}$ where

$$X_{i,\alpha} \doteq \{(q, x, y) :\ (x, y) \in P_i,\ (y + q)_\alpha = 0,\ (y + q)_{\bar\alpha} \geq 0,\ x_{\bar\alpha} = 0\}$$

is a polyhedral set, then Σ is piecewise polyhedral. Let $\Pi : \mathbb{R}^{3n} \to \mathbb{R}^{2n}$ be the orthogonal projection defined by $\Pi(q, x, y) \doteq (q, x)$, then $\mathrm{gph}\,\mathcal{S} = \Pi(\Sigma)$, thus $\mathrm{gph}\,\mathcal{S}$ is a finite union of polyhedral sets. ■

Now we investigate the continuity properties of the solution set mapping $\mathcal{S} : \mathbb{R}^n \times \mathcal{X}_0 \rightrightarrows \mathbb{R}^n$ defined by $(q, \Phi) \mapsto \mathcal{S}(q, \Phi)$.

Theorem 3.4.2 *Let $d > 0$, $c \in \mathcal{C}$ be nondecreasing, $q^0 \in \mathbb{R}^n$, and $\Phi^0 \in \mathcal{X}_0$ be piecewise polyhedral c-subhomogeneous such that $\Phi^0(0) = \{0\}$. If $q^0 \in [\mathcal{S}(0, \Phi^0)]^\#$, then there exist positive scalars ε and L such that for all $q \in \mathbb{R}^n$ and all $\Phi \in \mathcal{X}_0$ which are copositive piecewise polyhedral c-subhomogeneous such that $\Phi(0) = \{0\}$ satisfying $\|q - q^0\| + |\Phi - \Phi^0|_d^+ < \varepsilon$, the following inclusion holds*

$$\mathcal{S}(q, \Phi) \subseteq \mathcal{S}(q^0, \Phi^0) + L\left(\left\|q - q^0\right\| + |\Phi - \Phi^0|_d^+\right)\mathbb{B}.$$

Proof. We proceed similarly as in [15, Th. 7.5.1]. Let $x \in \mathcal{S}(q, \Phi)$, there exists $y \in \Phi(x)$ such that $y + q \geq 0$ and $\langle y + q, x\rangle = 0$. Let $y^0 \in \Phi^0(x)$ be arbitrary and let $\bar{q} = q + (y - y^0)$, then $x \in \mathcal{S}(\bar{q}, \Phi^0)$. From $y - y^0 \in (\Phi - \Phi^0)(x)$ we get that if $x = 0$ then $\bar{q} = q$, otherwise by c-subhomogeneity

$$y - y^0 \in (\Phi - \Phi^0)(x) \subseteq c(\|x\|_d)(\Phi - \Phi^0)(\frac{x}{\|x\|_d}), \tag{3.5}$$

and by taking $\varepsilon > 0$ sufficiently small, from Theorem 2.7.5 the elements of $\mathcal{S}(q, \Phi)$ are uniformly bounded for all (q, Φ) as given, i.e. there exists $r > 0$ such that if $x \in \mathcal{S}(q, \Phi)$, then $\|x\|_d \leq r$ and from (3.5) and (2.18) we obtain

$$\left\|y - y^0\right\|_d \leq |\Phi - \Phi^0|_d^+\, c(\|x\|_d) \leq |\Phi - \Phi^0|_d^+\, c(r),$$

since c is nondecreasing, therefore $\bar{q}$ can be made arbitrarily close to q by restricting ε if necessary. Hence by the above theorem there exists a constant $\lambda > 0$ such that $\mathcal{S}(\bar{q}, \Phi^0) \subseteq \mathcal{S}(q^0, \Phi^0) + \lambda\,\|\bar{q} - q^0\|\,\mathbb{B}$. Moreover, as we proved $\mathcal{S}(q, \Phi) \subseteq \mathcal{S}(\bar{q}, \Phi^0)$ and by replacing $\bar{q}$ we obtain

$$\begin{aligned} \mathcal{S}(q, \Phi) &\subseteq \mathcal{S}(q^0, \Phi^0) + k\left(\|q - q^0\| + \|y - y^0\|_d\right)\mathbb{B} \\ &\subseteq \mathcal{S}(q^0, \Phi^0) + k\left(\|q - q^0\| + c(r)|\Phi - \Phi^0|_d^+\right)\mathbb{B} \end{aligned}$$

the result follows by taking $L = k\max\{1, c(r)\}$. ■

We now reformulate Theorem 2.6.7 which is concerned about approximable mappings in terms of piecewise polyhedral approximations.

Corollary 3.4.3 *Let $d > 0$, $c \in \mathcal{C}$, $q \in \mathbb{R}^n$, and $\Phi = \Phi^0 + \Psi^0$ where $\Phi^0 \in \mathcal{X}$ and $\Psi^0 \in \mathcal{X}$ is approximable by copositive piecewise polyhedral uniformly bounded with respect to the same set mappings from $\mathcal{X}_0$.*

(a) *If $\Phi^0 \in \mathbf{GT}(d) \cup \mathbf{G\tilde{T}}(d)$ is approximable by piecewise polyhedral c-subhomogeneous mappings from $\mathcal{X}_0$ and $q \in [\mathcal{S}(0, \Phi^0)]^\#$, then $\mathcal{S}(q, \Phi)$ is nonempty;*

(b) *If Φ^0 is approximable by copositive piecewise polyhedral c-Moré mappings from $\mathcal{X}_0$ and $q \in [\mathrm{U}_0(\Phi^0)]^\#$, then $\mathcal{S}(q, \Phi)$ is nonempty.*

We now exhibit an instance where the above corollary is applicable. To do this we recall the example given just after Remark 2.2.4. Let us consider $\Phi = \Phi^0 + \Psi^0$ where $\Phi^0(x) = Mx$ is a polyhedral multifunction and $\Psi^0(x) = \partial\sigma_C(x)$, where $C \subseteq \mathbb{R}^n$ is a nonempty compact convex set with nonempty interior. As we now from Chapter 1, there exists a sequence $\{C^k\}$ of polytopes such that $C^k \to C$, and if $\Psi^k(x) = \partial\sigma_{C^k}(x)$ then Ψ^k is piecewise polyhedral (see Example 3.1.2(6)) and $\Psi^k \xrightarrow{g} \Psi^0$. Each Ψ^k is uniformly bounded with respect to the same set. In addition, if $0 \in C^k$, then Ψ^k is copositive.

Chapter 4

The linear complementarity problem

In this chapter our main concern is the study of the *linear complementarity problem*, denoted by $\mathrm{LCP}(q, M)$, and which reads as follow

$$\text{find } \bar{x} \geq 0 \text{ such that } M\bar{x} + q \geq 0, \ \ \langle M\bar{x} + q, \bar{x}\rangle = 0, \tag{LCP}$$

where $M \in \mathbb{R}^{n\times n}$ and $q \in \mathbb{R}^n$ are given. One denotes by $\mathcal{S}(q, M)$ the solution set to $\mathrm{LCP}(q, M)$ and by $\mathcal{F}(q, M)$ its feasible set, i.e., the set of all vector $x \geq 0$ such that $Mx + q \geq 0$. As we know problem (LCP) is equivalent to the variational inequality problem $\mathrm{VIP}(\mathbb{R}^n_+, T_0)$ with $T_0(x) = Mx + q$ (see [48])

$$\text{find } \bar{x} \geq 0 \text{ such that } \langle M\bar{x} + q, y - \bar{x}\rangle \geq 0, \text{ for all } y \geq 0. \tag{VIP}$$

We proceed as in (MCP) and approximate the latter problem by the sequence of problems $\mathrm{VIP}(D_k, T_0)$

$$\text{find } x^k \in D_k \text{ such that } \langle Mx^k + q, y - x^k\rangle \geq 0, \text{ for all } y \in D_k. \tag{VIP$_\mathrm{k}$}$$

Where $d > 0$ is a fixed positive vector of $\mathbb{R}^n$, $\{\sigma_k\}$ is an increasing sequence of positive numbers converging to $+\infty$, and

$$D_k \doteq \Big\{x \geq 0 : \ \langle d, x\rangle \leq \sigma_k\Big\}.$$

Since D_k is compact, convex and nonempty such a solution x^k exists for each k by Theorem 1.4.1. As in the preceding chapters we shall use asymptotical properties of the sequence $\{x^k\}$ to obtain information on $\mathcal{S}(q, M)$.

It is clear that x^k is a solution of $(\mathrm{VIP_k})$ if and only if $x^k \in D_k$ is an optimal solution of the linear program

$$\inf_x [\langle Mx^k + q, x\rangle : \ x \geq 0, \ \langle d, x\rangle \leq \sigma_k]. \tag{P}$$

Applying optimality conditions , we obtain that x^k is a solution of $(\mathrm{VIP_k})$ if and only if there exists $\theta_k \in \mathbb{R}$ such that (x^k, θ_k) is a solution of the problem

$$\begin{gathered} \text{find } x^k \geq 0 \text{ and } \ \theta_k \geq 0, \ \text{ such that} \\ Mx^k + q + \theta_k d \geq 0, \ \ \sigma_k \geq \langle d, x^k\rangle \\ \langle Mx^k + q + \theta_k d, x^k\rangle = 0 \ \text{ and } \ \theta_k(\sigma_k - \langle d, x^k\rangle) = 0, \end{gathered} \tag{$\mathrm{LCP_k}$}$$

or, equivalently, if (x^k, θ_k) is a solution to the augmented linear complementarity problem $\mathrm{LCP}(\tilde{q}^k, \tilde{M})$ in $\mathbb{R}^{n+1}$, with

$$\tilde{M} = \begin{pmatrix} M & d \\ -d^{\mathrm{T}} & 0 \end{pmatrix}, \quad \tilde{q}^k = \begin{pmatrix} q \\ \sigma_k \end{pmatrix}.$$

Observe that $\langle d, x^k\rangle < \sigma_k$ implies $\theta_k = 0$. Furthermore, if $\theta_k = 0$, then $x^k \in \mathcal{S}(q, M)$.

As in Chapter 2, we denote by $\mathcal{W}$ the set of sequences $\{x^k\}$ in $\mathbb{R}^n_+$ satisfying: x^k solves $(\mathrm{VIP_k})$ for each k and

$$\langle d, x^k\rangle = \sigma_k \ \text{ for all } k. \tag{4.1}$$

The condition

$$\liminf_k \theta_k = 0$$

is used in [40] to derive various existence results suitable for **G**-matrices. We recall that M is a **G**-matrix if for some $d > 0$, $\mathcal{S}(d, M) = \{0\}$.

One of the main goals of this chapter is to provide various equivalent conditions in order to have $\theta_i = 0$ for some $i \in \mathbb{N}$ whenever (4.1) holds. Surprisingly, such conditions, given in terms of the asymptotic behavior of the normalized approximate solutions, are also equivalent to $\liminf_k \theta_k = 0$.

Our approach, allows us to deal with copositive, semimonotone, q-pseudomonotone, **G** matrices among others (including a new class introduced in this thesis), in a unified framework.

A different approach to the existence theory in LCP is applied in [17, 18], it involves the investigation of the quadratic programming problem associated to $\mathrm{LCP}(q, M)$. In this case, the problem reduces to finding conditions implying that $\mathcal{S}(q, M)$ coincides with the set of KKT-points of such a quadratic problem (the former set is always contained in the latter). An example of a $\mathrm{LCP}(q, M)$ having KKT-points without being in $\mathcal{S}(q, M)$ to which our theory is applicable, is exhibited in Section 4.6. In particular, our results neither contain nor are contained in that of [18].

4.1 Asymptotic analysis and the general existence theorem

A carefully description of the asymptotic behavior of the approximate normalized solutions $\{x^k\}$ is given in the next lemma.

Lemma 4.1.1 (Basic Lemma for the LCP) *Let $d > 0$ and $\{\sigma_k\}$ be an increasing sequence of positive numbers converging to $+\infty$, and $\{x^k\}$ a sequence of solutions to $\mathrm{VIP_k}$ such that $\langle d, x^k \rangle = \sigma_k$ for all k and $\frac{x^k}{\sigma_k} \to v$. Then*

(a) *$v \geq 0$, $\langle d, v \rangle = 1$, $\langle Mv, v \rangle \leq 0$, $\langle Mv, y \rangle \geq ||y||_d \langle Mv, v \rangle$ for all $y \geq 0$. Consequently $v \in \mathcal{S}(-\langle Mv, v \rangle d, M)$;*

(b) *there exists* $k_0 \in \mathbb{N}$ *such that for all* $k \geq k_0$, $x^k - \frac{\sigma_k}{2} v \geq 0$ *and*

$$0 < ||x^k - \frac{\sigma_k}{2} v||_d < ||x^k||_d.$$

Moreover, there exist $\emptyset \neq J_v \subseteq I$*, a subsequence* $\{x^{k_m}\}$ *and* $m_0 \in \mathbb{N}$ *such that*

(c) *for all* $m \geq m_0$, $\frac{1}{\sigma_{k_m}} x^{k_m} \in \mathrm{ri}(\Delta_{J_v})$*. Thus* $\mathrm{supp}\{x^{k_m}\} = J_v$ *for all* $m \geq m_0$ *(hence* $\mathrm{supp}\{v\} \subseteq J_v$*);*

(d) $\langle M x^{k_m} + q, \sigma_{k_m} z - x^{k_m} \rangle = 0$ *for all* $z \in \Delta_{J_v}$ *and all* $m \geq m_0$;

(e) $\langle Mv, z \rangle = \langle Mv, v \rangle$*, for all* $z \in \Delta_{J_v}$.

Proof. This lemma is a consequence of (a)-(d) of the Basic Lemma from Chapter 2, since setting $\Phi^k(x) = \Phi(x) \doteq Mx$ and $\Psi^k = \Psi = 0$, the former mappings are usc and linear (homogeneous of degree $\gamma = 1$) whereas the latter are usc and uniformly bounded respect to the same set. ■

Remark 4.1.2 *In general, one cannot expect in the previous theorem that*

$$\langle Mv, v \rangle = 0. \tag{4.2}$$

Matrices having this property will play an important role in the existence theory for problem (LCP)*. Indeed, Theorem 4.1.6 below presents various equivalent conditions each one implying* (4.2) *and the non-emptiness of* $\mathcal{S}(q, M)$*. A suitable class of matrices are those due to García* [33].

It is known that any linear complementarity problem $\mathrm{LCP}(q, M)$ always admits a solution which is an extreme point of the polyhedron $Mx + q \geq 0, x \geq 0$, provided a solution exists. Using this fact for $\mathrm{LCP}(\tilde{q}^k, \tilde{M})$, it is not difficult to prove the following theorem.

Theorem 4.1.3 [15, Th. 3.7.9] *Let* $\{\sigma_k\}$ *be an increasing sequence of positive numbers converging to* $+\infty$ *and let* $\{(x^k, \theta_k)\}$ *be a sequence of solutions to problem* $\mathrm{LCP_k}$

such that $\langle d, x^k\rangle = \sigma_k$ *for all* $k \in \mathbb{N}$*. Then, there exist a subsequence* $\{\sigma_{k_m}\}$*, along with (possibly different) corresponding solutions* $\{(\bar{x}^{k_m}, \bar{\theta}_{k_m})\}$ *to* $\mathrm{LCP}(\tilde{q}^{k_m}, \tilde{M})$ *and two vectors* u *and* w *such that*

$$\langle d, u\rangle = 0,\ w \geq 0,\ \langle d, w\rangle = 1,\ \text{and}\ \bar{x}^{k_m} = u + \sigma_{k_m} w\ \ \forall\, m. \tag{4.3}$$

Remark 4.1.4 *We have to point out that the sequence* $\{(\bar{x}^{k_m}, \bar{\theta}_{k_m})\}$ *may has nothing to do with the original sequence* $\{(x^{k_m}, \theta_{k_m})\}$*. However, we may consider that the subsequence found in the Basic Lemma also satisfies* (4.3)*. Otherwise, we first apply Theorem 4.1.3 to obtain a further sequence and afterwards use the Basic Lemma to get a subsequence of the latter.*

With this observation, it follows from (d) of the Basic Lemma that

$$\langle Mx^{k_m} + q, x^{k_m}\rangle = \langle Mx^{k_m} + q, \sigma_{k_m} v\rangle = \langle Mx^{k_m} + q, \sigma_{k_m} z\rangle\ \ \forall\, z \in \Delta_{J_v}. \tag{4.4}$$

By using the second equality of $(\mathrm{VIP_k})$, (4.4) gives

$$\begin{aligned} \theta_{k_m} &= -\langle Mx^{k_m} + q, \frac{x^{k_m}}{\sigma_{k_m}}\rangle = -\langle Mx^{k_m} + q, v\rangle \\ &= -\langle x^{k_m} - \frac{\sigma_{k_m}}{2} v, (M + M^{\mathrm{T}})v\rangle - \langle q, v\rangle. \end{aligned} \tag{4.5}$$

Moreover, (4.4) also implies

$$(Mx^{k_m} + q)_{J_v} = \langle Mx^{k_m} + q, v\rangle d_{J_v}. \tag{4.6}$$

On the other hand, by (a) and (e) of the Basic Lemma we obtain

$$\langle Mv, v\rangle \leq 0,\ (Mv)_{\overline{J}_v} \geq \langle Mv, v\rangle d_{\overline{J}_v}\ \text{and}\ (Mv)_{J_v} = \langle Mv, v\rangle d_{J_v}. \tag{4.7}$$

Furthermore, by substituting (4.3) (notice that $w = v$) in (4.4) and (4.5), and using (e) of the Basic Lemma, we get for all m sufficiently large

$$\sigma_{k_m}[\langle Mu + q, z - v\rangle - \langle Mv, u\rangle] = \langle Mu + q, u\rangle\ \ \forall\, z \in \Delta_{J_v},$$

$$\theta_{k_m} = -\langle Mu + q, v\rangle - \sigma_{k_m}\langle Mv, v\rangle. \tag{4.8}$$

It follows for all $z \in \Delta_{J_v}$

$$\langle Mu + q, z - v\rangle = \langle Mv, u\rangle \text{ and } \langle Mu + q, u\rangle = 0.$$

Hence

$$\begin{aligned} \langle Mv, u\rangle = 0,\ \langle Mu + q, u\rangle = 0, \\ (Mu + q)_{J_v} = \langle Mu + q, v\rangle d_{J_v}. \end{aligned} \tag{4.9}$$

We also obtain

$$\langle u, d\rangle = 0,\ u_{\overline{J}_v} = 0,\ u_{J_v\setminus J} > 0,\ J = \mathrm{supp}\{v\} \subseteq J_v. \tag{4.10}$$

Taking into account (4.9)–(4.10), the proof of the following lemma is straightforward.

Lemma 4.1.5 *Let $u \in \mathbb{R}^n$, $v \in \mathbb{R}^n_+$ to be satisfying* (4.9) *and* (4.10) *and set* $J = \mathrm{supp}\{v\}$*. Then, the following assertions are equivalent:*

(a) $(Mu + q)_{i_0} = 0\ (> 0)$ *for some* $i_0 \in J$;

(b) $(Mu + q)_i = 0\ (> 0)$ *for all* $i \in J$;

(c) $\langle Mu + q, v\rangle = 0\ (> 0)$;

(d) $\max\{v_i(Mu + q)_i :\ i \in J\} = 0\ (> 0)$;

(e) $\min\{v_i(Mu + q)_i :\ i \in J\} = 0\ (> 0)$.

The next theorem provides various equivalent conditions in order to have $\theta_i = 0$ for some $i \in \mathbb{N}$ whenever $\langle d, x^k\rangle = \sigma_k\ \forall\ k$.

Theorem 4.1.6 (Main Existence Theorem) *Assume that there exists* $\{(x^k, \theta_k)\}$ *a sequence of solutions to* $\mathrm{LCP_k}$ *such that* $\langle d, x^k\rangle = \sigma_k$ *for all* k, $\sigma_k \to +\infty$ *and* (4.3) *holds for* $w = v$*. Then, the following assertions are equivalent each other:*

(a) $\liminf_{k\to+\infty} \theta_k = 0$;

(b) $\exists\, m_0$, $\exists\, \{\theta_{k_m}\}$, $\theta_{k_m} = 0\ \forall\, m \geq m_0$;

(c) $\exists\, m_0$, $\exists\, \{x^{k_m}\}$, $\langle Mx^{k_m} + q, v\rangle \geq 0\ \forall\, m \geq m_0$;

(d) $\exists\, m_0$, $\exists\, \{x^{k_m}\}$, *such that* $\forall\, m \geq m_0\ \exists\, u^{k_m} \geq 0$, $0 < \langle d, u^{k_m}\rangle < \langle d, x^{k_m}\rangle$ *and* $\langle Mx^{k_m} + q, u^{k_m} - x^{k_m}\rangle \leq 0$.

(e) $\exists\, m_0$, $\exists\, \{x^{k_m}\}$, $x^{k_m} \in \mathcal{S}(q, M)\ \forall\, m \geq m_0$.

Under any of the conditions (a) to (e), one has

$$Mv \geq 0,\ (Mv)_{J_v} = 0 \textit{ for some } \emptyset \neq J_v \supseteq \operatorname{supp}\{v\}\ (\textit{ thus } \langle Mv, v\rangle = 0). \tag{4.11}$$

Proof. (a)⇒(b): We may suppose, up to a subsequence, that $\theta_k \to 0$. By assumption, for some vector u, $\langle d, u\rangle = 0$, $v \geq 0$, $\langle d, v\rangle = 1$, $x^k = u + \sigma_k v$. From the analysis carried out in Remark 4.1.4 , we get by (4.8)

$$\theta_{k_m} = -\langle Mu + q, v\rangle - \sigma_{k_m}\langle Mv, v\rangle,$$

and therefore $\langle Mv, v\rangle = 0$ (hence (4.11) holds), which implies $\theta_{k_m} = -\langle Mu + q, v\rangle = 0$, proving that (b) holds.
$(b) \Rightarrow (a)$ It is obvious.
$(b) \Leftrightarrow (c)$ It is straightforward since $\theta_{k_m} = -\langle Mx^{k_m} + q, v\rangle$ by (4.5).
$(c) \Rightarrow (d)$ For all $k \geq k_0$, (b) of the Basic Lemma implies $u^k = x^k - \frac{\sigma_k}{2}v \geq 0$ and $0 < \langle d, u^k\rangle < \langle d, x^k\rangle$. By hypothesis, for all $m \geq m_0$ such that $k_m \geq k_0$ we get

$$\langle Mx^{k_m} + q, u^{k_m} - x^{k_m}\rangle = -\frac{\sigma_{k_m}}{2}\langle Mx^{k_m} + q, v\rangle = 0.$$

$(d) \Rightarrow (e)$ We assert that $x^{k_m} \in \mathcal{S}(q, M)$ for all $m \geq m_0$. Suppose to the contrary, there exist $m \geq m_0$ and $y_0 \in \mathbb{R}^n_+ \setminus D_{k_m}$ such that $\langle Mx^{k_m} + q, y_0 - x^{k_m}\rangle < 0$. As $0 < \langle d, u^{k_m}\rangle < \langle d, x^{k_m}\rangle$, there is $t \in\ ]0, 1[$ such that $z_t \doteq tu^{k_m} + (1 - t)y_0 \in D_{k_m}$. Thus $\langle Mx^{k_m} + q, z_t - x^{k_m}\rangle \geq 0$, and then $\langle Mx^{k_m} + q, y_0 - x^{k_m}\rangle \geq 0$, leading to a contradiction.

$(e) \Rightarrow (a)$ Suppose there exist m_0 and a subsequence $\{x^{k_m}\}$ of $\{x^k\}$ such that $x^{k_m} \in \mathcal{S}(q, M)$ for all $m \geq m_0$. Then, for all $y \geq 0$, and $m \geq m_0$, $\langle Mx^{k_m} + q, y - x^{k_m}\rangle \geq 0$. Taking $y = x^{k_m} + v \geq 0$ (resp. $y = x^{k_m} - \frac{\sigma_{k_m}}{2} v \geq 0$ for k_m such that $k_m \geq k_0$) we obtain $\langle Mx^{k_m} + q, v\rangle \geq 0$ (resp. $\langle Mx^{k_m} + q, v\rangle \leq 0$). Hence $\theta_{k_m} = -\langle Mx^{k_m} + q, v\rangle = 0$. ■

It is worth mentioning that for the multivalued complementarity problem (for the polyhedral case in particular) we do not have the above equivalence even if (4.3) holds for the approximate solutions, see Example 2.2.2.

4.2 The class of García matrices

In the spirit of using Theorem 4.1.6 we deduce that in order to $\{x^{k_m}\}$ yields a solution to (LCP), we have to exclude $\langle Mv, v\rangle < 0$ in Lemma 4.1.1. A general class of matrices having that property is that due to García. This results from (4.7) and Proposition 4.2.2 below.

Definition 4.2.1 *It is said that $M \in \mathbb{R}^{n\times n}$ is a $\mathbf{G}$-matrix if there is $p > 0$ such that*

$$\mathcal{S}(p, M) = \{0\}. \tag{4.12}$$

In this case we say that M is a $\mathbf{G}$-matrix with respect to p, or simply $M \in \mathbf{G}(p)$.

Proposition 4.2.2 *Let $d > 0$ and $M \in \mathbb{R}^{n\times n}$. The following assertions are equivalent each other:*

(a) $M \in \mathbf{G}(p)$;

(b) *for any three index sets α, β and γ which partition $I \doteq \{1, \ldots, n\}$, the following* *implication holds:*

$$\left.\begin{array}{r} v \geq 0, \alpha = \operatorname{supp}\{v\}, \langle d, v\rangle = 1, \\ M_{\alpha\alpha}v_\alpha - \langle Mv, v\rangle d_\alpha = 0, \\ M_{\beta\alpha}v_\alpha - \langle Mv, v\rangle d_\beta = 0, \\ M_{\gamma\alpha}v_\alpha - \langle Mv, v\rangle d_\gamma > 0. \end{array}\right\} \Longrightarrow \langle Mv, v\rangle \geq 0; \tag{4.13}$$

(c) *the following implication holds*

$$[v \geq 0, \langle d, v\rangle = 1, \langle Mv, v\rangle < 0] \Rightarrow \exists i \in I : (Mv)_i < \langle Mv, v\rangle d_i. \tag{4.14}$$

Proof. $(a) \Rightarrow (b)$ If on the contrary $\langle Mv, v\rangle < 0$, from the left hand side of (4.13), we have $(Mv)_\alpha = \langle Mv, v\rangle d_\alpha$, and $(Mv)_{\bar\alpha} \geq \langle Mv, v\rangle d_{\bar\alpha}$. It follows that $v \in \mathcal{S}(\tau d, M)$ for $\tau = -\langle Mv, v\rangle > 0$. Then $\frac{v}{\tau} \in \mathrm{S}(d, M)$, which implies $v = 0$ if (a) holds, a contradiction.
$(b) \Rightarrow (c)$: Suppose that

$$v \geq 0,\ \langle d, v\rangle = 1,\ \langle Mv, v\rangle < 0,\ \text{and } Mv \geq \langle Mv, v\rangle d.$$

Then $0 \neq v \in \mathcal{S}(-\langle Mv, v\rangle d, M)$. Set

$$\alpha \doteq \operatorname{supp}\{v\},\ \beta \doteq \{i \in I \setminus \alpha :\ (Mv)_i - \langle Mv, v\rangle d_i = 0\},\ \gamma = I \setminus (\alpha \cup \beta).$$

Thus, we can apply (4.13) to conclude that $\langle Mv, v\rangle \geq 0$, a contradiction.
$(c) \Rightarrow (a)$: It is straightforward. ■

4.3 Classes of matrices

In the study of the existence of solutions to $\mathrm{LCP}(q, M)$ different classes of matrices arise, some of the most important ones existing in the literature (see [15] for instance), are listed in what follows.
We say that $M \in \mathbb{R}^{n\times n}$ is

- *copositive* if $\langle Mx, x\rangle \geq 0 \ \ \forall\, x \geq 0$;
- *copositive-star* if M is copositive and $[x \in \mathcal{S}(0, M) \Longrightarrow M^{\mathrm{T}}x \leq 0]$;
- *copositive-plus* if M is copositive and
$$[x \geq 0,\, \langle Mx, x\rangle = 0 \Longrightarrow (M + M^{\mathrm{T}})x = 0];$$
- *semimonotone* if $\mathcal{S}(p, M) = \{0\}$ for all $p > 0$, or equivalently,
$$x \neq 0,\ x \geq 0 \Longrightarrow \exists\, i :\ x_i > 0,\ (Mx)_i \geq 0;$$
- *regular* if there exists $p > 0$ such that $\mathrm{S}(\tau p, M) = \{0\}$ for all $\tau \geq 0$, in this case we say that M is a regular matrix with respect to p;
- *positive subdefinite* if $[\langle Mx, x\rangle < 0 \Longrightarrow \textit{either } M^{\mathrm{T}}x \leq 0 \text{ or } M^{\mathrm{T}}x \geq 0]$;
- (given $q \in \mathbb{R}^n$) *q-pseudomonotone* if given $x, y \geq 0$,
$$\langle Mx + q, y - x\rangle \geq 0 \Longrightarrow \langle My + q, x - y\rangle \leq 0;$$
- (given $q \in \mathbb{R}^n$) *q-quasimonotone* if given $x, y \geq 0$,
$$\langle Mx + q, y - x\rangle > 0 \Longrightarrow \langle My + q, x - y\rangle \leq 0;$$

  • a $\#$-matrix (condition (17) in [40]) if
$$x \in \mathcal{S}(0, M) \Longrightarrow (M + M^{\mathrm{T}})x \geq 0;$$
- a $\mathbf{G}^{\#}$-matrix [40] if it is a $\#$ and $\mathbf{G}$-matrix. Like in Definition 4.2.1, we may also introduce the notion of a $\mathbf{G}^{\#}$-matrix with respect to $p > 0$. In this case we write $M \in \mathbf{G}^{\#}(p)$.

The asymptotic analysis carefully described in the Basic Lemma motivates the following new class of matrices, which coincides with the class $\mathbf{T}(d)$ (for each $d > 0$) from Chapter 2.

Definition 4.3.1 *Given $d > 0$, $M \in \mathbb{R}^{n\times n}$ is said to be a $\mathbf{T}$-matrix if one has*

$$\left.\begin{array}{r} 0 \neq v \geq 0, \;\; Mv \geq 0, \\ (Mv)_\alpha = 0, \; \mathrm{supp}\{v\} \subseteq \alpha \end{array}\right\} \Longrightarrow (M^{\mathrm{T}}v)_\alpha \geq 0. \tag{4.15}$$

We also say that M is a $\mathbf{GT}$-matrix if it is a $\mathbf{T}$-matrix and $M \in \mathbf{G}(d)$ for some $d > 0$, in this case we write $M \in \mathbf{GT}(d)$.

Remark 4.3.2 **(i)** *It is clear that every copositive-plus matrix is copositive-star, and both classes do not coincide. The class of copositive matrices is contained strictly in the class of semimonotone matrices, and the latter class is obviously a proper subset of the* $\mathbf{G}$*-matrices.*
(ii) *A q-pseudomonotone matrix is also q-quasimonotone, and in general, the reverse implication is not true. We also have that every q-quasimonotone matrix is positive subdefinite* [17]. *On the other hand, we have the following result:*

• *if M is q-pseudomonotone and $\mathcal{F}(q, M) \neq \emptyset$, then M is copositive. In fact, for $x \in \mathcal{F}(q, M)$, we have $\langle Mx + q, x + tu - x\rangle \geq 0 \;\; \forall\, t > 0, \; \forall\, u \geq 0$. Then*

$$0 \leq \langle M(x + tu) + q, x + tu - x\rangle = t\langle Mx + q, u\rangle + t^2\langle Mu, u\rangle \;\; \forall\, t > 0 \;\; \forall\, u \geq 0.$$

From which the copositivity of M follows.
Moreover, it is not difficult to prove the following (*see also* [17]):

• *if M is q-quasimonotone and $q \neq 0$, then M is q-pseudomonotone.*

• *if M is q-quasimonotone and there exists $x \geq 0$ so that $Mx + q \geq 0$ and $Mx + q \neq 0$, then M is copositive.*

Remark 4.3.3 *Matrices which are symmetric, copositive* (*because of* (b) *in Lemma 4.3.5 below*) *and those satisfying $\mathcal{S}(0, M) = \{0\}$, are #-matrices. The new class of* $\mathbf{T}$*-matrices contains properly the #-matrices, as shown by*

$$M = \begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix}.$$

It is also semimonotone and thus $M \in \mathbf{G}(d)$ for all $d > 0$. Whence, the class of $\mathbf{G}^{\#}$-matrices is also contained properly in the class of $\mathbf{GT}$-matrices. Notice that M is not copositive.

Example 4.3.4 *We have to point out that there is no relationship between semimonotone and either $\mathbf{T}$ or $\#$-matrices. In fact, the matrix*

$$M_1 = \begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix},$$

is non semimonotone but it is a $\#$-matrix and hence M_1 is a $\mathbf{T}$-matrix, whereas the matrix

$$M_2 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

is semimonotone without being a $\mathbf{T}$-matrix. Obviously $M_1 \in \mathbf{G}(\mathbb{1})$ and $M_2 \in \mathbf{G}(d)$ for all $d > 0$.
However, the matrix

$$M = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

is not a $\mathbf{G}$-matrix (and thus it is not semimonotone) but it is a $\#$-matrix, and hence is a $\mathbf{T}$-matrix.

The next two results describe the asymptotic behavior of the approximate solutions to $\mathrm{LCP}(q, M)$ for $\mathbf{G}$, copositive, and q-pseudomonotone matrices. Part of such theorems are direct consequences of Basic Lemma(g) from Chapter 2 and Lemma 2.3.6.

Lemma 4.3.5 *Let $d > 0$, $\{\sigma_k\}$ be an increasing sequence of positive numbers converging to $+\infty$, and $\{x^k\} \in \mathcal{W}$ be a sequence such that $\frac{x^k}{\sigma_k} \to v$. Then, in addition to the properties established in the Basic Lemma for some $\{x^{k_m}\}$ and $J_v \supseteq \operatorname{supp}\{v\} \neq \emptyset$, we also obtain*

(a) *$Mv \geq 0$ and $(Mv)_{J_v} = 0$ (hence $\langle Mv, v\rangle = 0$), provided $M \in \mathbf{G}(d)$;*

(b) *$Mv \geq 0$, $(Mv)_{J_v} = 0$, $\langle q, v\rangle \leq 0$, and $(M+M^{\mathrm{T}})v \geq 0$, provided M is copositive.*

Proof. (a): It follows from (4.13) and (a) of Lemma 4.1.1.
(b) Since every copositive matrix is a $\mathbf{G}(d)$-matrix, it only remains to prove that $\langle q, v\rangle \leq 0$. By the choice of $\{x^k\}$, we have $\langle Mx^k + q, y - x^k\rangle \geq 0$ for all $y \in D_k$. By taking $y = 0 \in D_k$ in $(\mathrm{VIP_k})$, dividing it by σ_k, and letting k goes $+\infty$, we get $\langle q, v\rangle \leq 0$. By splitting the inequality $\langle M(x+tv), x+tv\rangle \geq 0$ for fixed $x \geq 0$, and letting $t \to +\infty$, we obtain $(M + M^{\mathrm{T}})v \geq 0$. ■

Lemma 4.3.6 *Let $d > 0$, $\{\sigma_k\}$ be an increasing sequence of positive numbers converging to $+\infty$, $q \in \mathbb{R}^n$, M be q-pseudomonotone, and $\{x^k\} \in \mathcal{W}$ such that $\frac{x^k}{\sigma_k} \to v$. Then, in addition to the properties established in the Basic Lemma for some $\{x^{k_m}\}$ and $J_v \supseteq \mathrm{supp}\{v\} \neq \emptyset$, we also obtain*

(a) *$M^{\mathrm{T}}v \leq 0$ and $\langle q, v\rangle \leq 0$;*

(b) *$\langle M(z - \frac{x^{k_m}}{\sigma_{k_m}}), z - \frac{x^{k_m}}{\sigma_{k_m}}\rangle \geq 0$ and $\langle q, v\rangle \geq \langle q, \frac{x^{k_m}}{\sigma_{k_m}}\rangle$ for all $z \in \Delta_{J_v}$, $m \geq m_0$.*

Proof. (a): By the choice of x^k, the q-pseudomonotonicity implies

$$\langle My + q, x^k - y\rangle \leq 0 \quad \forall\, y \in D_k. \tag{4.16}$$

For any $y \geq 0$ there exists k_y such that for all $k \geq k_y$, $y \in D_k$ and $\langle My + q, x^k - y\rangle \leq 0$. If we divide it by σ_k and take the limit, we obtain $\langle My + q, v\rangle \leq 0$ for all $y \geq 0$. This implies that $M^{\mathrm{T}}v \leq 0$ and $\langle q, v\rangle \leq 0$.
In the Basic Lemma, we proved the existence of subsequences $\{x^{k_m}\}$, $\{\sigma_{k_m}\}$, m_0 and $J_v \subseteq I$ such that $\frac{1}{\sigma_{k_m}}x^{k_m} \in \mathrm{ri}(\Delta_{J_v})$ with $v \in \Delta_{J_v}$ for all $m \geq m_0$. We now prove (b). In case Δ_{J_v} is a singleton, $\Delta_{J_v} = \{v\}$. Then $\frac{1}{\sigma_{k_m}}x^{k_m} = v$ for all $m \geq m_0$, and so all the assertions hold. We analyze the case when Δ_{j_v} is not a singleton. For $m \geq m_0$ and $z \in \Delta_{J_v}$, we proceed as in the proof of the Basic Lemma, to get the existence of $\varepsilon_z > 0$, such that for all t, $|t| < \varepsilon_z$:

$$y_t \doteq \frac{x^{k_m}}{\sigma_{k_m}} + t\left(z - \frac{x^{k_m}}{\sigma_{k_m}}\right) \in \Delta_{J_v}.$$

We substitute $y = \sigma_{k_m} y_t$ in (4.16), and obtain

$$-t\sigma_{k_m}\langle Mx^{k_m} + q, z - \frac{x^{k_m}}{\sigma_{k_m}}\rangle - t^2(\sigma_{k_m})^2\langle M(z - \frac{x^{k_m}}{\sigma_{k_m}}), z - \frac{x^{k_m}}{\sigma_{k_m}}\rangle \le 0, \;\; \forall\, |t| < \varepsilon_z.$$

We use (d) of the same Lemma to conclude that $-t^2\sigma_{k_m}\langle M(z-\frac{x^{k_m}}{\sigma_{k_m}}), z-\frac{x^{k_m}}{\sigma_{k_m}}\rangle \le 0$ for all $m \ge m_0$, which proves the first part of (b). By the q-pseudomonotonicity again

$$\langle M(\sigma_{k_m} v) + q, x^{k_m} - \sigma_{k_m} v\rangle \le 0.$$

We use (e) of the same Lemma to conclude $\langle q, x^{k_m} - \sigma_{k_m} v\rangle \le 0.$ ■

4.4 Estimates for $[\mathcal{S}(\mathbf{q},\mathbf{M})]^\infty$

We introduce the following closed cones which are not necessarily convex (except $V_q(M)$):

$$\mathrm{W}_q(M) \doteq \Big\{v \ge 0 :\; Mv \ge 0,\; \langle Mv, v\rangle = 0,\; \langle q, v\rangle \le 0\Big\};$$

$$\mathrm{V}_0(M) \doteq \Big\{v \ge 0 :\; \langle Mv, v\rangle = 0,\; M^{\mathrm{T}}v \le 0,\; \langle q, v\rangle \le 0,\; Mv \ge 0\Big\};$$

$$\mathrm{V}_q(M) \doteq \Big\{v \ge 0 :\; M^{\mathrm{T}}v \le 0,\; \langle q, v\rangle \le 0\Big\};$$

$$\mathcal{A}_0(M) \doteq \Big\{v \ge 0 :\; Mv \ge 0,\; \langle Mv, v\rangle = 0,\; (M + M^{\mathrm{T}})v \ge 0,\; \langle q, v\rangle \le 0\Big\}.$$

Clearly $\mathrm{W}_0(M) = \mathcal{S}(0, M)$ and $\mathrm{V}_0(M) \subseteq \mathrm{V}_q(M)$.

Theorem 4.4.1 *Let $q \in \mathbb{R}^n$ and $M \in \mathbb{R}^{n\times n}$. Then*

(a) $[\mathcal{S}(q, M)]^\infty \subseteq \mathcal{S}(0, M)$;

(b) $[\mathcal{S}(q, M)]^\infty \subseteq \mathrm{W}_q(M)$ *if M is a $\mathbf{T}$-matrix*;

(c) $[\mathcal{S}(q, M)]^\infty \subseteq \mathcal{A}_0(M)$ *if M is copositive*;

(d) $\mathcal{S}(q, M) + \lambda \mathrm{W}_q(M) = \mathcal{S}(q, M)\; \forall \lambda \ge 0$ *if M is copositive-plus and $\mathcal{S}(q, M) \ne \emptyset$. Moreover, $[\mathcal{S}(q, M)]^\infty = \mathcal{S}_q(M) = [\mathcal{S}(q, M)]^\infty \cap q^\perp$;*

(e) $[\mathcal{S}(q,M)]^\infty = \mathrm{V}_q(M) = \mathrm{V}_0(M)$ *if M is q-pseudomonotone, $\mathcal{S}(q,M) \neq \emptyset$.*

Proof. (a)-(b) are consequences of Proposition 2.4.1(a)-(b) since $\mathcal{S}(0,M)$ and $\mathcal{W}_q(M)$ are cones and if M is a $\mathbf{T}$-matrix, then $\Phi(x) = Mx$ is $\mathbf{T}(d)$ for each $d > 0$.
(c): If $v \in [\mathcal{S}(q,M)]^\infty$ then $v \in \mathrm{W}_q(M)$ since a copositive matrix is a $\mathbf{T}$-matrix. Moreover, by copositivity again $\langle M(x+tv), x+tv\rangle \geq 0$ for all $x \geq 0$, $t \geq 0$, and letting $t \to +\infty$, we obtain $(M + M^{\mathrm{T}})v \geq 0$.
(d): Let $v \in \mathrm{W}_q(M)\backslash\{0\}$ and let $\bar{x} \in \mathcal{S}(q,M)$. Then $M^{\mathrm{T}}v \leq 0$ by the copositivity-plus of M and $\langle M\bar{x}+q, y-\bar{x}\rangle \geq 0$ for all $y \geq 0$. By taking $y = v+\bar{x} \geq 0$ we obtain $\langle M\bar{x} + q, v\rangle \geq 0$, which together with $M^{\mathrm{T}}v \leq 0$ imply $\langle M\bar{x}, v\rangle = 0 = \langle q, v\rangle$. Again by copositivity-plus of M we have $\langle Mv, \bar{x}\rangle = 0$. Using these facts it is not difficult to check that $\bar{x} + \lambda v \in \mathcal{S}(q,M)$ for all $\lambda \geq 0$, proving one inclusion of the first equality; the other inclusion always holds since $0 \in \mathrm{W}_q(M)$. It also implies that $(\mathcal{S}(q,M))^\infty \cap \{q\}^\perp = \mathrm{W}_q(M)$.
(e): The first equality is a consequence of Proposition 2.4.1. The other equality results from (a) since M is copositive by Remark 4.3.2. ∎

It is worth noting that here we estimate the asymptotic cone of the solution set instead of the d-asymptotic cone, due to the linearity of the mapping.

Remark 4.4.2 *According to (a) of the previous theorem, the strongest condition implying the non-existence of solution rays for $\mathrm{LCP}(q,M)$, is $\mathcal{S}(0,M) = \{0\}$. The same theorem provides weaker conditions when some specializations on M are made.*

Example 4.4.3 *The inclusions in the preceding theorem may be strict.*
(i) *Take for instance $M = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$, and $q = \begin{pmatrix} -\frac{1}{2} \\ -1 \end{pmatrix}$. Clearly $\mathcal{S}(q,M) = \{(0,1)^{\mathrm{T}}\}$, so $[\mathcal{S}(q,M)]^\infty = \{0\}$, and $\mathcal{S}(0,M) = \mathrm{W}_q(M) = \{(v_1,0)^{\mathrm{T}} : v_1 \geq 0\}$. Notice that M is copositive, and therefore* (4.15) *is satisfied (see (b) in Lemma* 4.3.5). *This instance also shows that the inclusion $\mathcal{S}(q,M) \subseteq \mathcal{S}(q,M) + \lambda\mathrm{W}_q(M)$ may be strict if M is not copositive-plus.*

(ii) *The first equality in* (d) *of the above theorem may induce us to conclude that* $\mathcal{S}(q, M)$ *is convex if* M *is copositive-plus. This is not so. Indeed, let us consider*

$$M = \begin{pmatrix} 1 & 4 \\ 4 & 1 \end{pmatrix}, \quad q = \begin{pmatrix} -1 \\ -1 \end{pmatrix}.$$

Simple computations show that M *is copositive-plus, and*

$$\mathcal{S}(0, M) = \{0\} = \mathrm{W}_q(M),\ \mathcal{S}(q, M) = \{(1,0)^{\mathrm{T}}, \frac{3}{15}(1,1)^{\mathrm{T}}, (0,1)^{\mathrm{T}}\}.$$

4.5 Main existence and sensitivity results

In this section we shall present general existence results for larger classes of matrices than those considered in the literature of Linear Complementarity. In particular, extensions and generalizations of the main results in [40] will be established. Moreover, new characterizations of regular matrices are provided, and sensivity results for $\mathbf{GT}$ and q-pseudomonotone matrices are proved as well.

Motivated by Remark 4.1.4, given d, q, M, we introduce the following definition which is a refinement of a *homogeneous pair* introduced in [40].

Definition 4.5.1 *We say that* (u, v) *is a* (d, q, M)-homogeneous pair *if the following conditions in* (4.17) *are satisfied:* $(J = \mathrm{supp}\{v\})$

$$\left.\begin{aligned} & v \geq 0,\ Mv \geq 0,\ (Mv)_\alpha = 0,\ J \subseteq \alpha, \\ & \langle Mv, u\rangle = 0,\ \langle Mu + q, u\rangle = 0,\ \langle u, d\rangle = 0, \\ & u + \lambda v \geq 0,\ \text{for some } \lambda > 0, \\ & \langle d, v\rangle (Mu + q)_\alpha = \langle Mu + q, v\rangle d_\alpha. \end{aligned}\right\} \tag{4.17}$$

By Lemma 4.1.5, the next theorem strengthens Theorem 7 in [40].

Theorem 4.5.2 *Let* $d > 0$, $q \in \mathbb{R}^n$ *and* $M \in \mathbb{R}^{n\times n}$ *be a* $\mathbf{G}(d)$*-matrix.*

(a) *Assume that any* $(J \doteq \mathrm{supp}\{v\})$ (d,q,M)*-homogeneous pair*

$$(u,v), v \neq 0, \textit{ satisfies } (Mu+q)_{i_0} \geq 0 \textit{ for some } i_0 \in J, \tag{4.18}$$

then $\mathcal{S}(q,M) \neq \emptyset$ *(possibly unbounded);*

(b) *If (4.18) holds with strict inequality "$>$", then* $\mathcal{S}(q,M)$ *is nonempty and compact.*

Proof.(a): For a given sequence $\sigma_k \to +\infty$, we consider the sequence $\{x^k\}$ of solutions to $(\mathrm{VIP_k})$. If there exists $k \in \mathbb{N}$ such that $||x^k||_d < \sigma_k$, we already know that $x^k \in \mathcal{S}(q,M)$. If, on the contrary, for all $k \in \mathbb{N}$, $||x^k||_d = \sigma_k$, we may suppose that $\{x^k\}$ satisfies (4.3) and the assumptions of Lemma 4.3.5. Since M is a $\mathbf{G}(d)$-matrix, we obtain with the notations of the same lemma, that $\langle Mv, v\rangle = 0$, $Mv \geq 0$, $||v||_d = 1$, $(Mv)_{J_v} = 0$, where

$$J \doteq \mathrm{supp}\{v\} \subseteq J_v = \mathrm{supp}\{x^{k_m}\} \ \forall\, m \geq m_0.$$

By (4.7), (4.9) and (4.10), (u,v) is a (d,q,M)-homogeneous pair. Moreover, from (4.8) it follows $\theta_{k_m} = -\langle Mu+q, v\rangle \geq 0$. Thus by Lemma 4.1.5 and (4.17) with $\alpha = J_v$, we obtain $\theta_{k_m} = 0$ thus $x^{k_m} \in \mathcal{S}(q,M)$.
(b): The nonemptiness follows from (a) and the boundedness is a consequence of the reasoning in (a) along with Lemma 4.1.5. ■

Remark 4.5.3 **(i)** *Condition (4.18) holds vacuously if* $\mathcal{S}(0,M) = \{0\}$*. Thus the assumptions of Theorem 4.5.2 hold if* M *is a regular matrix with respect to* d*.*
(ii) *Equivalent conditions satisfying the requirement* $(Mu+q)_{i_0} \geq 0$ *for some* i_0 *in (4.18) for any* (d,q,M)*-homogeneous pair, are exhibited in Lemma 4.1.5.*

We observe that (see Chapter 1)

$$\mathrm{int}\,[\mathcal{S}(0,M)]^* = [\mathcal{S}(0,M)]^\# \tag{4.19}$$

The next theorem, which applies to the example exhibited in Remark 4.3.3, extends Theorem 9 in [40]. Such a theorem has its origin in a result due to Lemke [52, p. 114] valid for copositive matrices.

Theorem 4.5.4 *Let $d > 0$ and $M \in \mathbf{GT}(d)$:*

(a) *if $q \in [\mathcal{S}(0, M)]^*$, then $\mathcal{S}(q, M)$ is nonempty (possibly unbounded);*

(b) *if $q \in \operatorname{int}[\mathcal{S}(0, M)]^*$ or equivalently, when $\mathrm{W}_q(M) = \{0\}$, then $\mathcal{S}(q, M)$ is nonempty and compact.*

Proof. It is a consequence of Theorem 3.3.1 for $\Phi(x) = Mx$ and $\Psi = 0$, since by hypothesis $\Phi \in \mathbf{GT}_p(d)$ and it is linear (homogeneous of degree $\gamma = 1$). ■

We now exhibit three instances showing that (a) in the previous theorem may be false if either M is not a $\mathbf{T}$-matrix, or $q \in (\mathcal{S}(0, M))^*$ or M is not a $\mathbf{G}$-matrix. The fourth instance shows that condition $q \in \operatorname{int}(\mathrm{S}(0, M))^*$ is not necessary for the boundedness of $\mathrm{S}(q, M)$.

Example 4.5.5 **(i)** *Let us consider the semimonotone (hence a $\mathbf{G}$-) matrix M_2 given in Example 4.3.4 which is not a $\mathbf{T}$-matrix, and $q = (1, -1, 1)^{\mathrm{T}}$. Clearly $\mathcal{S}(0, M_2) = \{(x_1, 0, x_3)^{\mathrm{T}} : x_1 \geq 0, x_3 \geq 0, x_1 x_3 = 0\}$. Thus, $q \in [\mathcal{S}(0, M_2)]^*$ but $\mathcal{S}(q, M_2) = \emptyset$.*
(ii) *We now consider the $\mathbf{GT}(d)$-matrix (for all $d > 0$) of Remark 4.3.3 and $q = (-1, 1)^{\mathrm{T}}$. Clearly $\mathcal{S}(0, M) = \{(x_1, 0)^{\mathrm{T}} :\ x_1 \geq 0\}$, $q \notin [\mathcal{S}(0, M)]^*$ and $\mathcal{S}(q, M) = \emptyset$.*
(iii) *Take the $\mathbf{T}$-matrix $M \in \mathbb{R}^{2\times 2}$ given in Example 4.3.4, which is not a $\mathbf{G}$-matrix, and consider $q = (-1, 0)^{\mathrm{T}}$. Clearly $\mathcal{S}(0, M) = \{0\}$ and $\mathcal{S}(q, M) = \emptyset$.*
(iv) *Let us consider the matrix M considered in* (i) *of Example 4.4.3 which is copositive (hence a $\mathbf{GT}(d)$-matrix for all $d > 0$) and $\mathcal{S}(0, M) = \{(v_1, 0)^{\mathrm{T}} : v_1 \geq 0\}$. Take $q = (0, -1)^{\mathrm{T}} \notin \operatorname{int}[\mathcal{S}(0, M)]^*$. Clearly $\mathcal{S}(q, M) = \{(0, 1)^{\mathrm{T}}\}$. Notice that M is not copositive-star.*

We say that $\mathrm{LCP}(q, M)$ is *feasible* if $\mathcal{F}(q, M) \neq \emptyset$, and is *strictly feasible* if the interior of $\mathcal{F}(q, M)$ is nonempty, that is, if

$$\Big\{x > 0 :\ Mx + q > 0\Big\} \neq \emptyset, \tag{4.20}$$

or equivalently, $q \in \operatorname{int}(\mathbb{R}^n_+) - M(\operatorname{int}(\mathbb{R}^n_+))$. As a consequence of Farkas Lemma

$$\mathcal{F}(q, M) \neq \emptyset \Longleftrightarrow [x \geq 0,\ M^{\mathrm{T}} x \leq 0 \Longrightarrow \langle q, x\rangle \geq 0]. \tag{4.21}$$

Given $d > 0$, analogously as in Section 2.4, the system

$$v \geq 0,\ \langle d, v\rangle = 1,\ \langle Mv, v\rangle \leq 0,\ v \in \mathcal{S}(-\langle Mv, v\rangle d, M), \tag{4.22}$$

found in the Basic Lemma, will play a fundamental role in characterizing the nonemptiness and boundedness of $\mathcal{S}(q, M)$ for all $q \in \mathbb{R}^n$. It is easy to check that the inconsistency of (4.22) is equivalent to the inconsistency of the follwing system:

$$\begin{aligned}(Mv)_i + td_i &= 0 \ \ i \in \operatorname{supp}\{v\},\\ (Mv)_i + td_i &\geq 0 \ \ i \notin \operatorname{supp}\{v\},\\ 0 \neq v &\geq 0,\ \ t \geq 0.\end{aligned}$$

The next theorem provides new characterizations of regular matrices.

Theorem 4.5.6 *Let $d > 0$ and $M \in \mathbb{R}^{n\times n}$. The following assertions are equivalent:*

(a) *the system* (4.22) *is inconsistent*;

(b) *M is a $\mathbf{G}(d)$-matrix and $\mathcal{S}(q, M)$ is nonempty and compact for all $q \in \mathbb{R}^n$;*

(c) *M is regular with respect to d.*

Proof. It follows from the equivalence (a)-(d)-(e) of Theorem 2.5.14 for $\Phi(x) = Mx$, which is linear (homogeneous of degree $\gamma = 1$). ■

By rewriting the latter theorem we get the next corollary which extends an earlier result valid for copositive-star matrices [35].

Corollary 4.5.7 *Let $d > 0$ and $M \in \mathbb{R}^{n\times n}$ be a $\mathbf{G}(d)$-matrix. The following assertions are equivalent:*

(a) *the system* (4.22) *is inconsistent*;

(b) *$\mathcal{S}(q, M)$ is nonempty and compact for all $q \in \mathbb{R}^n$;*

(c) $\mathcal{S}(0, M) = \{0\}$.

We now present some new equivalences to the inconsistency of (4.22) within the class of positive subdefinite matrices.

Theorem 4.5.8 *Let $d > 0$ and $M \in \mathbb{R}^{n\times n}$ be a positive subdefinite matrix. The following assertions are equivalent:*

(a) *the system* (4.22) *is inconsistent*;

(b) $\mathcal{S}(q, M)$ *is nonempty and compact for all* $q \in \mathbb{R}^n$;

(c) $\mathcal{S}(0, M) = \{0\}$ *and* $\mathcal{F}(q, M) \neq \emptyset$ *for all* $q \in \mathbb{R}^n$.

Proof. $(a) \Rightarrow (b)$ is part of the preceding theorem and $(b) \Rightarrow (c)$ is obvious. The remaining implication is a consequence of the definition of positive subdefiniteness and (4.21) together with (c). ■

Proposition 4.5.9 *Let $M \in \mathbb{R}^{n\times n}$, $q \in \mathbb{R}^n$. Then,*

$$\operatorname{int}(\mathcal{F}(q, M)) \neq \emptyset \Longleftrightarrow q \in \operatorname{int}(\mathbb{R}^n_+ - M(\mathbb{R}^n_+)).$$

Proof. It results from (4.20) and $\operatorname{int}(\mathbb{R}^n_+ - M(\mathbb{R}^n_+)) = \mathbb{R}^n_{++} - M(\mathbb{R}^n_{++})$ (see [68, Prop. 2.44]). ■

In [35], Gowda shows that for a copositive matrix M, $[\mathcal{S}(0, M)]^* = \mathbb{R}^n_+ - M(\mathbb{R}^n_+)$ if and only if $[x \in \mathcal{S}(0, M) \Longrightarrow M^{\mathrm{T}}x \leq 0]$. In other words, when M is copositive: $[\mathcal{S}(0, M)]^* = \mathbb{R}^n_+ - M(\mathbb{R}^n_+)$ if and only if M is copositive-star. In [40, Theorem 10], the same result is extended to $\mathbf{G}^{\#}$-matrices. We further extend it to the larger class of $\mathbf{GT}$-matrices, by establishing a necessary and sufficient condition for the cone $[\mathcal{S}(0, M)]^*$ to equal the set of feasible vectors q, i.e., the set $\mathbb{R}^n_+ - M(\mathbb{R}^n_+)$. The proof of such a result is essentially the same as in [40].

Theorem 4.5.10 *Let $d > 0$, $M \in \mathbb{R}^{n\times n}$, and $q \in \mathbb{R}^n$. Assume $M \in \mathbf{GT}(d)$. Then the following assertions are equivalent:*

(a) $[\mathcal{S}(0, M)]^* = \mathbb{R}^n_+ - M(\mathbb{R}^n_+)$;

(b) $x \in \mathcal{S}(0, M) \Longrightarrow M^{\mathrm{T}} x \leq 0$.

Proof. Set $N = \{x \geq 0 : M^{\mathrm{T}} x \leq 0\}$. Then $N^* = \mathbb{R}^n_+ - M(\mathbb{R}^n_+)$ and hence the result amounts to saying that $[\mathcal{S}(0, M)]^* = N^* \Longleftrightarrow \mathcal{S}(0, M) \subseteq N]$. If $\mathcal{S}(0, M) \subseteq N$ then $N^* \subseteq [\mathcal{S}(0, M)]^*$. (a) of Theorem 4.5.4 implies $[\mathcal{S}(0, M)]^* \subseteq N^*$. Consequently $[\mathcal{S}(0, M)]^* = \mathbb{R}^n_+ - M(\mathbb{R}^n_+)$. Suppose the latter equality holds. Then $N = N^{**} = [\mathcal{S}(0, M)]^{**} \supseteq \mathcal{S}(0, M)$. ■

The next theorem and corollary are sensitivity results for $\mathrm{LCP}(q, M)$ whose data are small perturbations of a given pair (q^0, M^0). It is a extension to the class of $\mathbf{GT}$-matrices of Theorem 11 in [40]. The proof is exactly the same.

Theorem 4.5.11 *Let $M^0 \in \mathbb{R}^{n \times n}$ and $q^0 \in \operatorname{int}[\mathcal{S}(0, M^0)]^*$. Then, there exists $\varepsilon > 0$, such that for all vectors q and matrices M satisfying*

$$||q - q^0|| + ||M - M^0|| < \varepsilon,$$

one has $q \in \operatorname{int}[\mathcal{S}(0, M)]^*$.

Proof. Suppose on the contrary, that there exist sequences $\{q^k, M^k, v^k\}$ satisfying $q^k \to q^0$, $M^k \to M^0$, $0 \neq v^k$, $v^k \in \mathcal{S}(0, M^k)$, $\langle q^k, v^k \rangle \leq 0$. We may assume that $||v^k|| = 1$ and therefore, up to a subsequence, $v^k \to v$, $||v|| = 1$. Whence $0 \neq v \in \mathcal{S}(0, M^0)$ and $\langle q^0, v \rangle \leq 0$, which contradicts the choice of q^0. ■

We obtain the following corollary as a consequence of Theorem 4.5.4 and the above theorem.

Corollary 4.5.12 *Let $M^0 \in \mathbb{R}^{n \times n}$, $q^0 \in \operatorname{int}[\mathcal{S}(0, M^0)]^*$. Then, there exists $\varepsilon > 0$, such that for all vectors q and $\mathbf{GT}$-matrices M satisfying*

$$||q - q^0|| + ||M - M^0|| < \varepsilon,$$

$\mathcal{S}(q, M)$ is a nonempty compact set.

It is worth mentioning that the last two results are consequences of Proposition 2.7.3 and Theorem 2.7.4 respectively, since the mappings $\Phi(x) = Mx$ for $M \in \mathbb{R}^{n\times n}$ are homogeneous of degree $\gamma = 1$, $\Phi(0) = \{0\}$ and $\|M - M^0\|_d = |M - M^0|_d^+$. In the same fashion the next result is a consequence of the above results, Theorems 2.7.5, 3.4.2, and the relationship between the matrix norms $\|\cdot\|$ and $\|\cdot\|_d$.

Corollary 4.5.13 [15, Th. 7.5.1] *Let $M^0 \in \mathbb{R}^{n\times n}$, $q^0 \in \operatorname{int}[\mathcal{S}(0, M^0)]^*$. Then, there exist positive scalars ε, r and L, such that for all vectors q and copositive matrices M satisfying $||q - q^0|| + ||M - M^0|| < \varepsilon$, the following statements hold:*

(a) *$\mathcal{S}(q, M)$ is a nonempty compact set;*

(b) *$\|\mathcal{S}(q, M)\| \leq r$;*

(c) *$\mathcal{S}(q, M) \subseteq \mathcal{S}(q^0, M^0) + L\left(\|q - q^0\| + \|M - M^0\|\right)\mathbb{B}$.*

A particular case of Theorem 4.5.4 is given in the next corollary, which supplements some earlier results appeared in [36].

Corollary 4.5.14 *Let $q \in \mathbb{R}^n$ and $M \in \mathbb{R}^{n\times n}$ be a copositive-star matrix. Then*

(a) *$\operatorname{int}(\mathcal{F}(q, M)) \neq \emptyset \iff q \in \operatorname{int}[\mathrm{S}(0, M)]^* \Longrightarrow \mathcal{S}(q, M) \neq \emptyset$ and compact;*

(b) *$q \in [\mathcal{S}(0, M)]^* \iff \mathcal{S}(q, M) \neq \emptyset \iff \mathcal{F}(q, M) \neq \emptyset$.*

Proof. (a): The first equivalence is a consequence of Proposition 4.5.9 and Theorem 4.5.10, and the remaining implication follows from Theorem 4.5.4.
(b): The first implication "⇒" results from Theorem 4.5.4(a) since every copositive matrix is $\mathbf{GT}(d)$ for all $d > 0$ by Remark 4.3.3 and (b) of Lemma 4.3.5. The second implication "⇒" is straightforward, and the remaining implication to close the cycle follows from (4.21) and by taking into account that $M^{\mathrm{T}}v \leq 0$ for all $v \in \mathcal{S}(0, M)$ whenever M is copositive-star, since in this case the feasibility implies $\langle q, v\rangle \geq 0$, i.e., $q \in [\mathcal{S}(0, M)]^*$. ■

Part of the following corollary was already established in [36]. Its proof follows from the preceding corollary and (d) of Theorem 4.4.1.

Corollary 4.5.15 *Let $q \in \mathbb{R}^n$ and $M \in \mathbb{R}^{n\times n}$ be a copositive-plus matrix. Then,*

$$\operatorname{int}(\mathcal{F}(q,M)) \neq \emptyset \Longleftrightarrow q \in \operatorname{int}[\mathcal{S}(0,M)]^* \Longleftrightarrow \mathcal{S}(q,M) \neq \emptyset \ and\ compact.$$

The equivalence between (a) and (c) of the next theorem was proved in Theorem 6 of [38]. A condition guaranteeing q-pseudomonotonicity is given in (ii) of Remark 4.3.2.

Theorem 4.5.16 *Let $q \in \mathbb{R}^n$ and $M \in \mathbb{R}^{n\times n}$ be a q-pseudomonotone matrix. The following assertions are equivalent:*

(a) $\mathcal{F}(q,M) \neq \emptyset$;

(b) $x \geq 0,\ M^{\mathrm{T}}x \leq 0$, $\langle q,x\rangle \leq 0 \Longrightarrow \langle q,x\rangle = 0$;

(c) $\mathcal{S}(q,M) \neq \emptyset$ (*it is already closed and convex*).

Proof. It follows from Corollary 3.3.6(ii) for $\Phi(x) = Mx$, which is linear (homogeneous of degree $\gamma = 1$), and the discussion after (2.7). ■

Part of the next theorem was first observed in [16].

Theorem 4.5.17 *Let $q \in \mathbb{R}^n$ and $M \in \mathbb{R}^{n\times n}$ be a q-pseudomonotone matrix. The following assertions are equivalent:*

(a) $\mathcal{F}(q,M) \neq \emptyset$ *and* $[x \geq 0,\ M^{\mathrm{T}}x \leq 0,\ \langle q,x\rangle = 0 \Longrightarrow x = 0]$;

(b) $\mathcal{S}(q,M)$ *is nonempty and compact;*

(c) $x \geq 0$, $M^{\mathrm{T}}x \leq 0$, $\langle q,x\rangle \leq 0 \Longrightarrow x = 0$;

(d) $\mathcal{F}_s(q,M)) \neq \emptyset$;

(e) $\mathrm{int}(\mathcal{F}(q, M)) \neq \emptyset$.

Proof. Taking into account (4.21), easy computations show that (a) and (c) are equivalent each other. The equivalence between (b)-(c)-(d) follows from Corollary 3.3.5 for $\Phi(x) = Mx$, which is polyhedral. Clearly (d) is equivalent to (e). ■

There is a counterpart to Theorem 4.5.11 under the pseudomonotonicity assumption. This result is new.

Theorem 4.5.18 *Let $M^0 \in \mathbb{R}^{n\times n}$ and $q^0 \in \mathbb{R}^n$ such that*

$$v \geq 0,\ (M^0)^{\mathrm{T}} v \leq 0,\ \langle q^0, v\rangle \leq 0 \Longrightarrow v = 0. \tag{4.23}$$

Then, there exists $\varepsilon > 0$, such that for all vectors q and matrices M satisfying

$$||q - q^0|| + ||M - M^0|| < \varepsilon,$$

one has $v \geq 0,\ M^{\mathrm{T}} v \leq 0,\ \langle q, v\rangle \leq 0 \Longrightarrow v = 0$.

Proof. If on the contrary there exist sequences $\{q^k, M^k, v^k\}$, satisfying $q^k \to q^0$, $M^k \to M^0$, $0 \neq v^k \geq 0$, $(M^k)^{\mathrm{T}} v^k \leq 0$, $\langle q^k, v^k\rangle \leq 0$. We may assume, as before, that $||v^k|| = 1$ and therefore, up to a subsequence, $v^k \to v$, $||v|| = 1$, and hence $\langle q^0, v\rangle \leq 0$ and $(M^0)^{\mathrm{T}} v \leq 0$. This contradicts the choice of q^0. ■

The following proof was provided by one of the referees of [30]: condition (4.23) is equivalent to (by a theorem of alternative)

$$\exists\, u \geq 0,\ \exists\, t \geq 0 :\ M^0 u + t q^0 \geq \mathbb{1}.$$

Clearly, for (M, q) close enough to (M^0, q^0), it holds $Mu + tq \geq \frac{1}{2}\mathbb{1}$, from which the theorem follows.

We obtain the following corollary as a consequence of Theorems 4.5.17 and 4.5.18.

Corollary 4.5.19 *Let $M^0 \in \mathbb{R}^{n\times n}$ and $q^0 \in \mathbb{R}^n$ such that*

$$v \geq 0,\ (M^0)^{\mathrm{T}} v \leq 0,\ \langle q^0, v\rangle \leq 0 \Longrightarrow v = 0.$$

Then, there exists $\varepsilon > 0$, such that for all vectors q and matrices M which are q-pseudomonotone satisfying

$$||q - q^0|| + ||M - M^0|| < \varepsilon,$$

$\mathcal{S}(q, M)$ is a nonempty and compact set.

4.6 Further remarks

We presented various equivalent existence results valid for general matrices based on the asymptotic description of solutions to approximate variational inequality problems. One of these existence results ((a) of Theorem 4.1.6), regarding the limit of θ_k, is due to Gowda and Pang ([40]), but the condition (c), which is used in our proof methodology, regards its exact expression: $\theta_k = -\langle Mx^k + q, v\rangle$. This fact is strongly exploited to obtain more applicable existence results (Theorems 4.5.2 and 4.5.4). It allowed us to extend the main results of [40] (including some sensitivity ones) to the larger class of $\mathbf{GT}$-matrices (a subfamily of the $\mathbf{G}$-matrices), which includes situations in which there are KKT-points (of the quadratic programming problem associated to $\mathrm{LCP}(q, M)$) that are not solutions to $\mathrm{LCP}(q, M)$. To see this simply consider the matrix M_1 (belonging to $\mathbf{GT}(\mathbb{1})$) of Example 4.3.4 and $q = (1, 1)^{\mathrm{T}} \in [\mathcal{S}(0, M_1)]^*$. Then it is not difficult to check that $(0, \frac{1}{2})^{\mathrm{T}}$ is a KKT-point but is not in $\mathcal{S}(q, M_1)$. Thus, our existence results neither contain nor are contained in that of [18].

The following existence theorem is the main result in [40]. It follows from (d) of Theorem 4.1.6 (take for instance $u^{k_m} = x^{k_m} - \gamma_m y$ for suitable $\gamma_m > 0$).

Theorem 4.6.1 [40, Th. 3] *Let $q \in \mathbb{R}^n$ and $M \in \mathbb{R}^{n\times n}$. Suppose that there exists a vector $d > 0$ such that for any three index sets α, β and γ which partition I, the* *following implication holds:*

$$\left.\begin{aligned} v_\alpha > 0,\ \tau \geq 0, \\ M_{\alpha\alpha}v_\alpha + \tau d_\alpha = 0, \\ M_{\beta\alpha}v_\alpha + \tau d_\beta = 0, \\ M_{\gamma\alpha}v_\alpha + \tau d_\gamma > 0. \end{aligned}\right\} \Longrightarrow \begin{cases} \forall\ \bar\beta \subseteq \beta,\ \exists\ 0 \neq (y_\alpha, y_{\bar\beta}) \geq 0 \text{ such that} \\ y_\alpha^{\mathrm{T}} M_{\alpha\alpha} + y_{\bar\beta}^{\mathrm{T}} M_{\bar\beta\alpha} = 0, \\ y_\alpha^{\mathrm{T}} M_{\alpha\bar\beta} + y_{\bar\beta}^{\mathrm{T}} M_{\bar\beta\bar\beta} \geq 0, \\ y_\alpha^{\mathrm{T}} q_\alpha + y_{\bar\beta}^{\mathrm{T}} q_{\bar\beta} \geq 0. \end{cases} \tag{4.24}$$

Then $\mathcal{S}(q, M)$ is nonempty.

In spite of the generality of its formulation, the previous theorem is applicable only to $\mathbf{G}$-matrices. More precisely, we show that condition (4.24) implies that M is a $\mathbf{G}(d)$-matrix. Indeed, let $v \in \mathcal{S}(d, M)$, $v \neq 0$. One can write $\alpha \doteq \operatorname{supp}\{v\}$, $\beta \doteq \{i \in I \setminus \alpha : (Mv)_i + d_i = 0\}$, $\gamma = I \setminus (\alpha \cup \beta)$. Hence, if (4.24) holds (with $\tau = 1$), then for $\bar\beta = \beta$ there exists $0 \neq (y_\alpha, y_\beta) \geq 0$ such that $y_\alpha^{\mathrm{T}} M_{\alpha\alpha} + y_\beta^{\mathrm{T}} M_{\beta\alpha} = 0,\ y_\alpha^{\mathrm{T}} M_{\alpha\beta} + y_\beta^{\mathrm{T}} M_{\beta\beta} \geq 0$, or equivalently, $(M_{\alpha\cup\beta,\alpha})^{\mathrm{T}} y_{\alpha\cup\beta} \geq 0$. The left-hand side of (4.24) gives $M_{\alpha\cup\beta,\alpha}v_\alpha + d_{\alpha\cup\beta} = 0$. Thus $0 = \langle M_{\alpha\cup\beta,\alpha}v_\alpha + d_{\alpha\cup\beta}, y_{\alpha\cup\beta}\rangle = \langle v_\alpha, (M_{\alpha\cup\beta,\alpha})^{\mathrm{T}} y_{\alpha\cup\beta}\rangle + \langle d_{\alpha\cup\beta}, y_{\alpha\cup\beta}\rangle > 0$, which is a contradiction. Consequently $\mathcal{S}(d, M) = \{0\}$. Therefore, the only $\tau \geq 0$ for which implication (4.24) holds is $\tau = 0$.

As a consequence, the above theorem is equivalent (in the sense that one can be derived from the other) to the following result

Theorem 4.6.2 [40, Th. 4] *Let M be a $\mathbf{G}$-matrix. Suppose that for any index sets α, β and γ as stated in the above theorem, the following implication holds:*

$$\left.\begin{aligned} v_\alpha > 0, \\ M_{\alpha\alpha}v_\alpha = 0, \\ M_{\beta\alpha}v_\alpha = 0, \\ M_{\gamma\alpha}v_\alpha > 0. \end{aligned}\right\} \Longrightarrow \begin{cases} \forall\ \bar\beta \subseteq \beta,\ \exists\ 0 \neq (y_\alpha, y_{\bar\beta}) \geq 0 \text{ such that} \\ y_\alpha^{\mathrm{T}} M_{\alpha\alpha} + y_{\bar\beta}^{\mathrm{T}} M_{\bar\beta\alpha} = 0, \\ y_\alpha^{\mathrm{T}} M_{\alpha\bar\beta} + y_{\bar\beta}^{\mathrm{T}} M_{\bar\beta\bar\beta} \geq 0, \\ y_\alpha^{\mathrm{T}} q_\alpha + y_{\bar\beta}^{\mathrm{T}} q_{\bar\beta} \geq 0. \end{cases} \tag{4.25}$$

Then $\mathcal{S}(q, M)$ is nonempty.

For the sake of completeness we establish various equivalent conditions to the non-emptiness of $\mathcal{S}(q,M)$ when M is q-pseudomonotone. In order to do that, we first state such conditions:

(C_0) if the sequence $x_k \geq 0$, $||x_k|| \to +\infty$ is such that $\frac{x_k}{||x_k||} \to v \in V$ and for all $y \geq 0$, there exists k_y such that $\langle Mx^k + q, y - x^k \rangle \geq 0 \;\text{ when }\; k \geq k_y$ then, for k sufficiently large, there exists $u \geq 0$ such that $||u|| < ||x^k||$ and $\langle Mx^k + q, u - x^k \rangle \leq 0$.

(C_1) [48] there exists a compact set $\emptyset \neq D \subseteq \mathbb{R}^n_+$ such that $\forall\, x \in \mathbb{R}^n_+ \backslash D$, $\exists\, y \in D$: $\langle Mx + q, y - x \rangle \leq 0$.

(C_2) [19] there exist $u \geq 0$ and $r > ||u||$ such that $\langle Mx + q, u - x \rangle \leq 0 \;\forall\, x \geq 0$, $||x|| = r$.

(C_3) [19] there exists $r > 0$ such that $\forall\, x \geq 0$, $||x|| = r$ there exists $u \geq 0$, $||u|| < r$: $\langle Mx + q, u - x \rangle \leq 0$.

Clearly $(C_1) \Longrightarrow (C_0)$ and $(C_2) \Longrightarrow (C_3)$.

In [29], when equilibrium problems are specialized to linear complementarity problems, the following theorem is proved. Compare with Theorem 2.2 in [19].

Theorem 4.6.3 [29] *Let $q \in \mathbb{R}^n$ and $M \in \mathbb{R}^{n\times n}$ be q-pseudomonotone. Then, the following assertions are equivalent:*

(a) (C_0) *is satisfied;*

(b) $\mathcal{S}(q,M) \neq \emptyset$ (*it is already closed and convex*)*;*

(c) (C_1) *is satisfied;*

(d) (C_2) *is satisfied;*

(e) (C_3) *is satisfied.*

In the case when M is positive subdefinite, the next result is obtained.

Corollary 4.6.4 [56, Th. 3.5] *Let $q \in \mathbb{R}^n$ and $M \in \mathbb{R}^{n\times n}$ be positive subdefinite with rank greater or equal than two. Then*

$$\mathcal{F}(q, M) \neq \emptyset \Longleftrightarrow \mathcal{S}(q, M) \neq \emptyset.$$

Proof. In Theorem 2.2. and Lemma 3.2 of [56] is proved that for such a matrix M at least one of the following conditions holds: M is copositive-star or $M \leq 0$. In the first case the result follows from Corollary 4.5.14; while when $M \leq 0$, the feasibility implies $q \geq 0$, thus $0 \in \mathcal{S}(q, M)$. ■

If the rank of M is equal to one, the previous corollary may be false as shown in [56].

Chapter 5

On Q-matrices

A matrix M is said to be a **Q**-*matrix*, or $M \in \mathbf{Q}$, if $\mathcal{S}(q, M) \neq \emptyset$ for all $q \in \mathbb{R}^n$. Strict copositive matrices, nonnegative matrices with positive diagonal entries and copositive-plus matrices with a positive column vector are examples of **Q**-matrices (see [60]). This class has proved to be very important from algorithmic view's point. Indeed, many iterative methods for solving the $\mathrm{LCP}(q, M)$ can be described by means of a *matrix splitting* (see [15, Ch. 5]). To do this, we split the matrix M as the sum of two matrices B and C, i.e. $M = B + C$. For such a splitting (B, C) of M, the $\mathrm{LCP}(q, M)$ can be transformed into a fixed-point problem; indeed, for an arbitrary vector z, we may consider the $\mathrm{LCP}(q^z, B)$ where $q^z \doteq q + Cz$ and the multifunction $z \mapsto \Omega(z) \doteq \mathcal{S}(q^z, B)$ which associates with this vector z the solution set of the $\mathrm{LCP}(q^z, B)$. Clearly z solves the $\mathrm{LCP}(q, M)$ if and only if z is a fixed-point of $\Omega(\cdot)$. In terms of this fixed-point formulation, the following iterative method for solving the $\mathrm{LCP}(q, M)$ may be introduced.

Algorithm.(The Basic Splitting Method)

Step 0: Initialization. `Let` z^0 `be an arbitrary positive vector, set` $k = 0$`;`

Step 1: General iteration. `Given` z^k`, solve the` $\mathrm{LCP}(q^k, B)$ `where`

$q^k = q + Cz^k$ and let z^{k+1} be an arbitrary solution;

Step 2: Test for termination. If z^{k+1} satisfies a prescribed stopping rule, terminate. Otherwise, return to Step 1 with k replaced by $k+1$.

In order for this algorithm to be well-defined each subproblem $\mathrm{LCP}(q^k, B)$ must have at least one solution, for this reason we must assume that B is a **Q**-matrix. It is worth mentioning that the computational complexity of checking whether a given matrix B is a **Q**-matrix is not known (to our best knowledge). In [60, Ex. 3.87] is provided a finite algorithm for checking the **Q**-property and when it is applied on a matrix of order n, this algorithm requires the solution of at most n^{2^n} systems of linear inequalities, hence this method though finite, is utterly impractical even for $n = 4$. Moreover, no polynomially bounded algorithms are known so far, and it is not known whether this problem is $\mathcal{NP}$-complete. For this reason, characterizations of **Q**-matrices within some classes of matrices are of great interest.

5.1 Classes of matrices and previous results

In the following we recall some important classes of matrices arising in the study of the linear complementarity problem and which we shall use in this chapter.

We say that $M \in \mathbb{R}^{n\times n}$ is

- *copositive* if $\langle Mx, x\rangle \geq 0 \;\; \forall\, x \geq 0$;
- a *star*-matrix if $[x \in \mathcal{S}(0, M) \Longrightarrow M^{\mathrm{T}} x \leq 0]$;
- *copositive-star* if M is copositive and star-matrix;
- *semimonotone*, or $M \in \mathbf{E}_0$, if $\mathcal{S}(p, M) = \{0\}$ for all $p > 0$, or equivalently, $[0 \neq x \geq 0 \Longrightarrow \exists\, i: \; x_i > 0, \; (Mx)_i \geq 0]$;

- (given $p > 0$) an $\mathbf{R}(p)$-matrix, if $\mathcal{S}(\tau p, M) = \{0\}$ for all $\tau \geq 0$;
- *positive subdefinite* (PSBD) if $[\langle Mx, x\rangle < 0 \implies M^{\mathrm{T}}x \leq 0 \text{ or } M^{\mathrm{T}}x \geq 0]$;
- (given $p > 0$) a $\mathbf{G}(p)$-matrix, if $\mathcal{S}(p, M) = \{0\}$;
- an $\mathbf{R}_0$-matrix ($\mathbf{E}^*(0)$ in [33]), if $\mathcal{S}(0, M) = \{0\}$;
- an $\mathbf{E}_1$-matrix if, given $0 \neq v \in \mathcal{S}(0, M)$ there exist nonnegative diagonal matrices D_1 and D_2 such $D_2 v \neq 0$ and $(D_1 M + M^{\mathrm{T}} D_2)v = 0$;
- an $\mathbf{L}$-matrix, if $M \in \mathbf{E}_0 \cap \mathbf{E}_1$;
- a $\mathbf{P}_0$-matrix, if its principal minors are nonnegative;
- an $\mathbf{S}$-matrix, if there is $x \geq 0$ such that $Mx > 0$, or equivalently, there is $x > 0$ such that $Mx > 0$;
- a $\mathbf{Q}_b$-matrix, if $\mathcal{S}(q, M)$ is nonempty and compact for all $q \in \mathbb{R}^n$;
- a $\mathbf{Q}_0$-matrix, if $[\mathcal{F}(q, M) \neq \emptyset \Longrightarrow \mathcal{S}(q, M) \neq \emptyset]$;
- a $\mathbf{Z}$-matrix, if $M_{ij} \leq 0$ for $i \neq j$;
- a *nonnegative* $M \geq 0$ (resp. *nonpositive* $M \leq 0$) matrix, if its entries are nonnegative (resp. nonpositive).

Finally, we say that a matrix $M \in \mathbb{R}^{n\times n}$ has the *property* (T) [5] if for every nonempty $\alpha \subseteq \{1, \ldots, n\}$ the existence of a solution z_α to the system

$$z_\alpha > 0,\ M_{\alpha\alpha} z_\alpha \leq 0,\ M_{\bar{\alpha}\alpha} z_\alpha \geq 0, \tag{5.1}$$

implies there exists a nonzero vector $y_{\alpha_0} \geq 0$ such that

$$y_{\alpha_0}^{\mathrm{T}} M_{\alpha_0 \alpha} = 0 \text{ and } y_{\alpha_0}^{\mathrm{T}} M_{\alpha_0 \bar{\alpha}} \leq 0, \tag{5.2}$$

where $\alpha_0 = \{i \in \alpha :\ M_{i\alpha} z_\alpha = 0\}$.

This property was employed to characterize constructively $\mathbf{Q}_0 \cap \mathbf{P}_0$, while a characterization of $\mathbf{Q}_0$ by solving linear complementarity problems as linear programs was provided in [61] following the line of Mangasarian [54] (see also [2]).

Given $d > 0$, as we already know $M \in \mathbf{R}(d)$ is equivalent to the inconsistency of the following system:

$$\begin{aligned} (Mv)_i + td_i = 0 \quad & i \in \operatorname{supp}\{v\}, \\ (Mv)_i + td_i \geq 0 \quad & i \notin \operatorname{supp}\{v\}, \ . \\ 0 \neq v \geq 0, \quad & t \geq 0, \end{aligned}$$

Such a system has its origin in [47], where only the case $d = \mathbb{1}$ (the vector of ones) is considered, and further developed in [49]. The class $\mathbf{R}(d)$ is extremely important since for matrices in that class, $\mathrm{LCP}(q, M)$ cannot have a secondary ray; hence Lemke's complementarity pivoting scheme would actually compute a solution in a finite number of iterations for any $q \in \mathbb{R}^n$ ([33]).

It is not difficult to check that ([15, Prop. 3.15])

$$M \in \mathbf{S} \Longleftrightarrow \mathcal{F}(q, M) \neq \emptyset \quad \forall\, q \in \mathbb{R}^n. \tag{5.3}$$

Moreover, as a consequence of the Farkas' Lemma,

$$\mathcal{F}(q, M) \neq \emptyset \Longleftrightarrow [x \geq 0,\ M^{\mathrm{T}} x \leq 0 \Longrightarrow \langle q, x \rangle \geq 0]. \tag{5.4}$$

In Chapter 4 it is proved that

$$[\mathcal{S}(q, M)]^\infty \subseteq \mathcal{S}(0, M). \tag{5.5}$$

Therefore

$$\mathbf{Q}_b = \mathbf{Q} \cap \mathbf{R}_0, \tag{5.6}$$

since $A^\infty = \{0\}$ if and only if A is bounded. On the other hand, we also have

$$\mathbf{Q} = \mathbf{Q}_0 \cap \mathbf{S}. \tag{5.7}$$

Two main results were established in connection with the characterization of $\mathbf{Q}$-matrices. The first one is due to Aganagič and Cottle [4] valid for $\mathbf{P}_0$-matrices, and read as follows:

Theorem 5.1.1 (Aganagič-Cottle) *Let $M \in \mathbf{P}_0$. Then*

$$M \in \mathbf{R}(\mathbb{1}) \Longleftrightarrow M \in \mathbf{R}_0 \Longleftrightarrow M \in \mathbf{Q} \Longleftrightarrow M \in \mathbf{Q}_b \Longrightarrow M \in \mathbf{S}.$$

By choosing the matrix

$$M_1 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

we have that $\mathbf{P}_0 \cap \mathbf{S} \not\subseteq \mathbf{Q}$. The second result was proved by Pang in [62] for the class $\mathbf{L} = \mathbf{E}_0 \cap \mathbf{E}_1$ which is distinct from $\mathbf{P}_0$:

Theorem 5.1.2 (Pang) *Let $M \in \mathbf{L}$. Then*

$$M \in \mathbf{R}(\mathbb{1}) \Longleftrightarrow M \in \mathbf{R}_0 \Longleftrightarrow M \in \mathbf{Q} \Longleftrightarrow M \in \mathbf{S} \Longleftrightarrow M \in \mathbf{Q}_b.$$

Since $\mathbf{P}_0 \subseteq \mathbf{E}_0$ and $\mathbf{L} \subseteq \mathbf{E_0}$, Pang asked the question whether Theorem 5.1.1 is valid for $\mathbf{E}_0$ instead of $\mathbf{P}_0$. This question was completely solved in the affirmative in the symmetric case [39], and in the negative for nonsymmetric matrices [45]. To be more precise, in that paper it is exhibited a matrix M satisfying

$$M \in (\mathbf{E}_0 \cap \mathbf{Q}) \setminus \mathbf{R}_0.$$

We investigate a possible extension of Theorem 5.1.1, without the equivalence with $\mathbf{Q}$, to a class larger than $\mathbf{P}_0$ (Corollary 5.3.3), and also a generalization of Theorem 5.1.2 (Theorem 5.3.4). We start by providing a wide class of matrices, namely $\mathbf{F}_1$, for which $\mathbf{Q} \cap \mathbf{F}_1 = \mathbf{Q}_b \cap \mathbf{F}_1$ (Theorem 5.3.1), where eventually $\mathbf{Q} \cap \mathbf{F}_1 \neq \mathbf{R}_0$. Positive subdefinite matrices with rank equal one are particularly analyzed as well. The latter class of matrices has been recently studied in [17].

Our main results apply to matrices which are not in $\mathbf{L}$, such as

$$M_2 = \begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix} \in (\mathbf{R}_0 \cap \mathbf{G}(\mathbb{1})) \setminus \mathbf{E}_0,$$

or to matrices which are not in $\mathbf{E}_1$ [40] as

$$M_3 = \begin{pmatrix} 0 & -1 & -2 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix} \in (\mathbf{E}_0 \setminus \mathbf{P_0}) \setminus \mathbf{E_1}.$$

Since $\mathcal{S}(0, M_3) = \{(v_1, 0, 0)^{\mathrm{T}} : \ v_1 \geq 0\}$, $M_3 \notin \mathbf{R}_0$. In this spirit a new class of matrices will be introduced in the next section.

5.2 A new class of matrices

Our main theorems require the following new class of matrices, where the notation $\mathbf{F}_1$ is used since extends $\mathbf{E}_1$.

Definition 5.2.1 *A matrix $M \in \mathbb{R}^{n\times n}$ is said to be an* $\mathbf{F}_1$-matrix, *if*

$$0 \neq v \in \mathcal{S}(0, M) \Longrightarrow \left\{ \begin{array}{l} \text{there exists a nonnegative diagonal matrix } \Lambda, \\ \Lambda v \neq 0 \text{ and } M^{\mathrm{T}}\Lambda v \leq 0. \end{array} \right.$$

Equivalently, $M \in \mathbf{F}_1$ if and only if for any nonempty set $\alpha \subseteq \{1, \ldots, n\}$, the following implication holds:

$$\left. \begin{array}{c} x_\alpha > 0, \\ M_{\alpha\alpha} x_\alpha = 0, \\ M_{\bar{\alpha}\alpha} x_\alpha \geq 0. \end{array} \right\} \Longrightarrow \left\{ \begin{array}{l} \exists\, 0 \neq w_\alpha \geq 0 : \\ w_\alpha^{\mathrm{T}} M_{\alpha\alpha} = 0, \\ w_\alpha^{\mathrm{T}} M_{\alpha\bar{\alpha}} \leq 0. \end{array} \right. \tag{5.8}$$

Now we list some classes of matrices contained in the new class $\mathbf{F_1}$.

Proposition 5.2.2 $M \in \mathbf{F}_1$ *if any of the conditions below is satisfied:*

(a) *M satisfies property* (T)*;*

(b) $M \in \mathbf{P}_0 \cap \mathbf{Q}_0$*;*

(c) $M \in \mathbf{Z} \cap \mathbf{P}_0$*;*

(d) *M is a star-matrix (in particular if M is positive semidefinite or $M \leq 0$);*

(e) $M \in \mathbf{E}_1$ *(in particular if $M \in \mathbf{R}_0$);*

(f) *M is a PSBD-matrix with rank greater or equal than two.*

Proof. If M has property (T) then obviously $M \in \mathbf{F}_1$ since for $z \in \mathcal{S}(0, M)$, $z \neq 0$, we get $\alpha_0 = \alpha$ in (5.2) with $\alpha = \mathrm{supp}\{i : z_i \neq 0\}$), and then the right-hand side of (5.8) is satisfied for $w_\alpha = y_\alpha$. In case $M \in \mathbf{P}_0 \cap \mathbf{Q}_0$, by Proposition 3 in [5], M has property (T) and thus $M \in \mathbf{F}_1$. If $M \in \mathbf{Z} \cap \mathbf{P}_0$ then, by Theorem 3.11.6 in [15], $M \in \mathbf{Q}_0 \cap \mathbf{P}_0$. Hence $M \in \mathbf{F}_1$. Under either (d) or (e), the result follows easily. In case M is PSBD we apply Theorem 2.2 and Lemma 3.2 of [56] to conclude that M is either copositive-star or $M \leq 0$. Thus by (d), we get the desired result. ■

PSBD-matrices with rank equal one may be not in $\mathbf{F}_1$ as the matrix

$$M_4 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \in (\mathbf{S} \cap \mathbf{E}_0) \setminus (\mathbf{F}_1 \cup \mathbf{Q})$$

shows. Here $\mathcal{S}(0, M_4) = \{(0, v_2)^{\mathrm{T}} : \ v_2 \geq 0\}$. On the other hand, we observe that $\mathbf{F}_1$ properly contains the union of the classes mentioned in (b)–(f). In fact, we simply take

$$M_5 = \begin{pmatrix} 0 & 1 & -2 \\ 0 & -1 & 2 \\ -1 & 1 & -2 \end{pmatrix}.$$

Here $\mathcal{S}(0, M_5) = \{(0, 2t, t)^{\mathrm{T}} : t \geq 0\}$, and M_5 verifies Definition 5.2.1 by taking

$$\Lambda = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Moreover, the matrix

$$M_6 = \begin{pmatrix} 1 & -2 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in (\mathbf{G}(\mathbb{1}) \cap \mathbf{F}_1) \setminus \mathbf{E}_0$$

but does not satisfy property (T). Since $\mathcal{S}(0, M_6) = \{(0, 0, v_3)^{\mathrm{T}} : v_3 \geq 0\}$, $M_6 \notin \mathbf{R}_0$.

5.3 Main results

Now, we are ready to establish the first main theorem in this Chapter. Its first part unifies Lemma 2 of [4] and Lemma 3 of [62] since $\mathbf{P}_0 \cap \mathbf{Q} = \mathbf{P}_0 \cap \mathbf{Q}_0 \cap \mathbf{S} \subseteq \mathbf{F}_1 \cap \mathbf{S}$ and $\mathbf{E}_1 \cap \mathbf{Q} \subseteq \mathbf{F}_1 \cap \mathbf{S}$ by Proposition 5.2.2; whereas the second part extends Theorem 4.11 of [20].

Theorem 5.3.1 *Let $M \in \mathbf{F}_1 \cap \mathbf{S}$, then $M \in \mathbf{R}_0$. As a consequence, if $M \in \mathbf{F}_1$ then*

$$M \in \mathbf{Q}_b \Longleftrightarrow M \in \mathbf{Q}.$$

Proof. Let $M \in \mathbf{F}_1 \cap \mathbf{S}$. If $M \notin \mathbf{R}_0$, take $0 \neq v \in \mathcal{S}(0, M)$. By assumption there is a nonnegative diagonal matrix Λ such that $\Lambda v \neq 0$ and $M^{\mathrm{T}} \Lambda v \leq 0$. Since $\langle \Lambda v, v \rangle > 0$, we obtain $\langle Mx - v, \Lambda v \rangle < 0$ for all $x \geq 0$. This shows that $\mathcal{F}(-v, M) = \emptyset$ ($M \notin \mathbf{S}$), a contradiction. This establishes the first implication. Obviously $\mathbf{Q}_b \subseteq \mathbf{Q}$; for the other inclusion we use (5.5) and the first part of the theorem. ■

The equivalence in Theorem 5.3.1 may be false if $M \notin \mathbf{F}_1$ as the matrix [45]

$$M_7 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & -1 & 1 \end{pmatrix} \in (\mathbf{E}_0 \cap \mathbf{Q}) \setminus (\mathbf{R}_0 \cup \mathbf{F}_1)$$

shows; while the matrix

$$M_8 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \in \mathbf{R}_0 \setminus (\mathbf{S} \cup \mathbf{G}(d)) \;\; \forall\, d > 0,$$

illustrates that $\mathbf{R}_0 \neq \mathbf{F}_1 \cap \mathbf{S}$.

The characterization of regular matrices (Theorem 4.5.6) was the starting point of the results of this section. We present an alternative proof.

Theorem 5.3.2 *Let $d > 0$. Then*

$$M \in \mathbf{G}(d) \cap \mathbf{Q}_b \Longleftrightarrow M \in \mathbf{R}(d).$$

Proof. From (5.6) we can write $\mathbf{G}(d) \cap \mathbf{Q}_b = \mathbf{G}(d) \cap \mathbf{Q} \cap \mathbf{R}_0 = \mathbf{R}(d) \cap \mathbf{Q}$ since $\mathbf{G}(d) \cap \mathbf{R}_0 = \mathbf{R}(d)$. Thus, one implication is straightforward. The other is a consequence of Theorems 9 and 11 of [40] (see also Theorem 3.1 of [49] and (5.5)). ■

From the previous theorem we obtain the following result which generalizes Lemma 1 of [62] and thus also Lemma 1 of [4],

Corollary 5.3.3 *Let $M \in \mathbf{G}(d)$ for some $d > 0$. Then*

$$M \in \mathbf{S} \Longleftarrow M \in \mathbf{Q} \Longleftarrow M \in \mathbf{Q}_b \Longleftrightarrow M \in \mathbf{R}_0 \Longleftrightarrow M \in \mathbf{R}(d).$$

The above matrix M_7 and $M_9 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ show that $\mathbf{E_0} \cap \mathbf{Q} \not\subseteq \mathbf{Q_b}$ and $\mathbf{E}_0 \cap \mathbf{S} \not\subseteq \mathbf{Q}$ respectively. Here, $\mathcal{S}(0, M_9) = \{(0, v_2)^{\mathrm{T}} : \ v_2 \geq 0\}$.

Our second main theorem generalizes Theorem 1.1 of [62].

Theorem 5.3.4 *Let $d > 0$ and $M \in \mathbf{G}(d) \cap \mathbf{F}_1$. The following assertions are equivalent:*

(a) $M \in \mathbf{R}(d)$;

(b) $M \in \mathbf{R}_0$;

(c) $M \in \mathbf{Q}_b$;

(d) $M \in \mathbf{Q}$;

(e) $M \in \mathbf{S}$.

Proof. The equivalences (a)-(d) follow from Theorem 5.3.1 and Corollary 5.3.3. Obviously (d) implies (e). For the implication $(e) \Rightarrow (b)$ use Theorem 5.3.1. ■

We now give matrices for which we can apply the previous theorem and not Theorem 5.1.2 (since they are not in $\mathbf{L}$): the matrices $M_3 \in (\mathbf{G}(d) \cap \mathbf{F}_1)$ for all $d > 0$ and $M_6 \in \mathbf{G}(\mathbb{1}) \cap \mathbf{F}_1$ are not $\mathbf{Q}$-matrices since they are not in $\mathbf{R}_0$; the matrix $M_2 \in (\mathbf{R}_0 \cap \mathbf{G}(\mathbb{1})) \setminus \mathbf{E}_0$ is a $\mathbf{Q}$-matrix.
Finally, the matrices M_7 and

$$M_{10} = \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix} \in (\mathbf{R}_0 \cap \mathbf{Q}) \setminus \mathbf{G}(d) \ \ \forall\, d > 0$$

show that Theorem 5.3.4 may be false if $M \notin \mathbf{G}(d) \cap \mathbf{F}_1$.

Concerning PSBD-matrices, the following result is obtained.

Theorem 5.3.5 *Let M be a PSBD-matrix:*
If $\operatorname{rank}(M) = 1$. *The following assertions are equivalent.*

(a) $M \in \mathbf{E}_0 \cap \mathbf{R}_0$;

(b) $M \in \mathbf{R}(d)$ *for some* $d > 0$;

(c) $M \in \mathbf{Q}_b$;

(d) $M \in \mathbf{R}_0 \cap \mathbf{S}$.

If $\operatorname{rank}(M) \geq 2$*, then* M *is either copositive-star (hence* $M \in \mathbf{L}$ *and Theorem 5.1.2 could be applied) or* $M \notin \mathbf{S}$*.*

Proof. The implication $(a) \Rightarrow (b)$ is straightforward; that (b) implies (c) and (c) implies (d) follow from Theorem 5.3.2 and (5.6) respectively since $\mathbf{Q} \subseteq \mathbf{S}$. It remains to check that $\mathbf{R}_0 \cap \mathbf{S} \subseteq \mathbf{E}_0$. Thus, we need to prove that $\mathcal{S}(d', M) = \{0\}$ for all $d' > 0$. Take $0 \neq v \in \mathcal{S}(d', M)$, then $\langle Mv, v\rangle < 0$. By hypothesis, $M^{\mathrm{T}}v \leq 0$. We apply (5.3) and (5.4) to obtain $\langle q, v\rangle \geq 0$ for all $q \in \mathbb{R}^n$. Whence $v = 0$, a contradiction.
In the case when $\operatorname{rank}(M) \geq 2$, by Theorem 2.2 and Lemma 3.2 of [56], M is either copositive-star (and hence $M \in \mathbf{L}$ by Proposition 7 of [35]), or $M \leq 0$, and then $M \notin \mathbf{S}$. ■

The PSBD-matrices with rank equal one $M_4 \in (\mathbf{S} \cap \mathbf{E}_0) \setminus (\mathbf{R}_0 \cup \mathbf{F}_1 \cup \mathbf{Q})$ and

$$M_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in (\mathbf{E}_0 \cap \mathbf{E}_1) \setminus \mathbf{R}_0,$$

show that assertions (d) and (a) in the previous theorem cannot be substituted by assertions $M \in \mathbf{S}$ and $M \in \mathbf{E}_0 \cap \mathbf{E}_1 = \mathbf{L}$) respectively. The matrix M_4 also shows that in general $M \in \mathbf{S}$ does not imply $M \in \mathbf{Q}$. Here $\mathcal{S}(0, M_4) = \mathcal{S}(0, M_{11}) = \{(0, v_2)^{\mathrm{T}} : v_2 \geq 0\}$ and $\mathcal{S}(q, M_4) = \emptyset$ if $q = (-1, -2)^{\mathrm{T}}$. Theorem 5.3.5 can be applied for the PSBD-matrix

$$M_{12} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in \mathbf{S} \cap \mathbf{R}_0 \cap \mathbf{E}_0.$$

As is recalled above PSBD-matrices with rank greater than one are either copositive-star or nonpositive and we can use Theorem 5.1.2 for them. It remains to deal with PSBD-matrices with rank equal one, such matrices are easily studied with the help of definitions (see Proposition 2.1 of [17]). The next corollary is a consequence of the previous theorem. To the best of our knowledge, such a corollary has not been pointed out before.

Corollary 5.3.6 *Let $M \in \mathbf{R}_0$ be a PSBD-matrix with $\mathrm{rank}(M) = 1$. Then*

$$M \in \mathbf{E}_0 \Longleftrightarrow M \in \mathbf{G}(d) \text{ for some } d > 0 \Longleftrightarrow M \in \mathbf{Q} \Longleftrightarrow M \in \mathbf{S}.$$

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