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# HYBRIDIZIBLE DISCONTINUOUS GALERKIN METHOD FOR LINEAR ELASTICITY IN CURVED DOMAINS

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**HYBRIDIZIBLE DISCONTINUOUS GALERKIN METHOD FOR LINEAR ELASTICITY IN CURVED DOMAINS**

UN MÉTODO DE GALERKIN DISCONTINUO HIBRIDIZABLE PARA ELASTICIDAD LINEAL EN DOMINIOS CURVOS

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A mi Familia

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# Abstract

This work proposes a hybridize discontinuous Galerkin (HDG) method for the linear elasticity problem in domains $\Omega$ that are not necessarily polyhedral/polygonal. In particular, we approximate the domain by a polyhedral/polygonal computational domain $D_h$ where the HDG solution can be computed. The Dirichlet boundary data is suitable transferred from the boundary $\Gamma := \partial\Omega$ to the computational boundary $\Gamma_h := \partial D_h$. We show that the scheme is well-posed. Moreover, we prove a priori error estimates showing that the method is optimal. In addition, we prove that the numerical trace is superconvergent with order $k+2$ if the distance between $\Gamma$ and $\Gamma_h$ is of order $h^2$. On the other hand, if this distance is of order $h$, then the numerical trace superconvergences with rate $k+3/2$. We validate our theoretical results with numerical experiments in two-dimension.

# Resumen

Este trabajo propone un método de Galerking discontinuo hibridizable (HDG) para el problema de elasticidad lineal en dominios $\Omega$ no necesariamente poliédrico/poligonal. En particular, aproximamos el dominio mediante un dominio computacional poliédrico/poligonal $D_h$ donde la solución HDG puede ser calculada. El dato Dirichlet de la frontera es adecuadamente traspasado desde la frontera $\Gamma := \partial\Omega$ a la frontera computacional $\Gamma_h := \partial D_h$. Mostramos que el esquema esta bien definido. También, proveeemos estimaciones a priori del error mostrando que el método es óptimo. Además, probamos que la traza numérica es superconvergente con orden $k + 2$ si la distancia entre $\Gamma$ y $\Gamma_h$ es de orden $h^2$. Por otra parte, si la distancia es de orden $h$, entonces la traza numérica es superconvergente con tasa $k + 3/2$. Validamos nuestros resultados teóricos con experimentos numéricos en dos dimensiones.

# Chapter 1

# Introduction

This work proposes and analyses a Hybridizable Discontinuous Galerkin (HDG) method for the isotropic Linear Elasticity problem

$$\mathcal{A}\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\epsilon}}(\boldsymbol{u}) = 0 \quad \text{in } \Omega \subset \mathbb{R}^n, \tag{1.1a}$$

$$\nabla \cdot \underline{\boldsymbol{\sigma}} = \boldsymbol{f} \quad \text{in } \Omega, \tag{1.1b}$$

$$\boldsymbol{u} = \boldsymbol{g} \quad \text{on } \Gamma. \tag{1.1c}$$

where $\Omega \in \mathbb{R}^n, (n \in \{2,3\})$ is a bounded domain not necessarily polygonal/polyhedral. Here $\boldsymbol{u}$ is the displacement, $\underline{\boldsymbol{\epsilon}}(\boldsymbol{u}) := \frac{1}{2}(\nabla \boldsymbol{u} + \nabla^T \boldsymbol{u})$ is the strain tensor, $\underline{\boldsymbol{\sigma}}$ is the Cauchy stress tensor, $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ is a source term, $\boldsymbol{g} \in \boldsymbol{H}^{1/2}(\Gamma)$ is a given boundary data, $\Gamma$ is a piecewise $C^2$ and Lipschitz boundary and $\mathcal{A}$ is a bounded symmetric positive definite tensor, i.e., there exist $C_{\mathcal{A}} > 0$ and $C_{el} > 0$ such that

$$\| \mathcal{A}(\underline{\boldsymbol{\xi}}) \|_{[L^2(\Omega)]^{n\times n}} \leq C_{\mathcal{A}} \| \underline{\boldsymbol{\xi}} \|_{[L^2(\Omega)]^{n\times n}} \qquad \text{for all } \underline{\boldsymbol{\xi}} \in [L^2(\Omega)]^{n\times n} \tag{1.2}$$

and

$$(\mathcal{A}(\underline{\boldsymbol{\xi}}), \underline{\boldsymbol{\xi}})_{[L^2(\Omega)]^{n\times n}} \geq C_{el} \| \underline{\boldsymbol{\xi}} \|^2_{[L^2(\Omega)]^{n\times n}} \qquad \text{for all } \underline{\boldsymbol{\xi}} \in [L^2(\Omega)]^{n\times n}. \tag{1.3}$$

In applications, $\mathcal{A}^{-1}$ is the elasticity tensor determined by the Hooke's Law:

$$\mathcal{A}^{-1}(\underline{\boldsymbol{\xi}}) = 2\mu\underline{\boldsymbol{\xi}} + \lambda tr(\underline{\boldsymbol{\xi}})\underline{\boldsymbol{I}} \qquad \text{for all } \underline{\boldsymbol{\xi}} \in [L^2(\Omega)]^{n\times n}$$

and also

$$\mathcal{A}(\underline{\boldsymbol{\xi}}) = \frac{1}{2\mu}\underline{\boldsymbol{\xi}} - \frac{\lambda}{2\mu(n\lambda + 2\mu)} tr(\underline{\boldsymbol{\xi}})\underline{\boldsymbol{I}}. \tag{1.4}$$

Here, $\underline{\boldsymbol{I}}$ denotes the identity tensor, $tr(\underline{\boldsymbol{\xi}}) := \sum_{i=1}^n \underline{\boldsymbol{\xi}}_{ii}$, $\lambda$ and $\mu$ are the Lamé constant such that

$$\mu := \frac{E}{2(1+\nu)} \quad \text{and} \quad \lambda := \frac{E\nu}{(1+\nu)(1-2\nu)},$$

where $E$ is the Young's modulus and $\nu$ is the Poisson ratio. In the current work, where $\mathcal{A}$ is given by (1.4), it is possible to show that $C_{\mathcal{A}} = 1/\mu$ and $\mathcal{A}$ is a symmetric and positive definite tensor if $E > 0$ and $\nu \in ]-1, 1/2[$. Moreover, we can see that $C_{el} = (2\mu + \lambda n|\Omega|^{1/2})^{-1}$.

One of the first HDG methods for the linear elasticity problem has been proposed in [14] for the Formulation (1.1) in polyhedral domains. There, numerical experiments showed optimal convergence rates of the method. However, to the best of our knowledge, its analysis study is still an open problem.

On the other hand, we introduce the rotation $\underline{\boldsymbol{\rho}}(\boldsymbol{u}) = (\nabla \boldsymbol{u} - \nabla^T \boldsymbol{u})/2$ as unknown and rewrite (1.1) as

$$\mathcal{A}\underline{\boldsymbol{\sigma}} - \nabla \boldsymbol{u} + \underline{\boldsymbol{\rho}} = 0 \quad \text{in } \Omega \subset \mathbb{R}^n, \tag{1.5a}$$

$$\nabla \cdot \underline{\boldsymbol{\sigma}} = \boldsymbol{f} \quad \text{in } \Omega, \tag{1.5b}$$

$$\boldsymbol{u} = \boldsymbol{g} \quad \text{on } \Gamma. \tag{1.5c}$$

An HDG method for (1.5), in the case of a polyhedral domain, has been analysed in [5] and it has been shown there that the HDG scheme is optimal.

This work considers an HDG formulation of (1.5) where we emphasize that $\Omega$ is not necessarily polyhedral. For this, we follow the approach in [4, 6] for the Poisson problem, which is based on transferring the boundary condition from $\Gamma$ to the computational boundary $\Gamma_h$. More precisely, [6] used the fact that the gradient of the pressure was part of the unknowns and the boundary condition can be obtained integrating this gradient along a segment. In the case of lineal elasticity the same idea, can be applied because $\nabla \boldsymbol{u}$ is the sum of the unknowns $\mathcal{A}\underline{\boldsymbol{\sigma}}$ and $\underline{\boldsymbol{\rho}}$. In fact, to fix ideas, let $\boldsymbol{x} \in \Gamma_h, \bar{\boldsymbol{x}} \in \Gamma$, $l(\boldsymbol{x}) := |\bar{\boldsymbol{x}} - \boldsymbol{x}|$ and $\boldsymbol{t}(\boldsymbol{x})$ the unit tangent vector of the segment joining $\boldsymbol{x}$ and $\bar{\boldsymbol{x}}$, then integrating $\nabla \boldsymbol{u}$ between $\boldsymbol{x}$ and $\bar{\boldsymbol{x}}$ we get

$$\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{u}(\bar{\boldsymbol{x}}) - \int_0^{l(\boldsymbol{x})} (\mathcal{A}\underline{\boldsymbol{\sigma}} + \underline{\boldsymbol{\rho}})(\boldsymbol{x} + s\boldsymbol{t}(\boldsymbol{x}))\boldsymbol{t}(\boldsymbol{x}) ds,$$

since $\boldsymbol{u}(\bar{\boldsymbol{x}}) = \boldsymbol{g}(\bar{\boldsymbol{x}})$. Defining $\tilde{\boldsymbol{g}}(\boldsymbol{x}) := \boldsymbol{u}(\boldsymbol{x})$, we obtain the following expression for the boundary data $\tilde{\boldsymbol{g}}$ in $\Gamma_h$:

$$\tilde{\boldsymbol{g}}(\boldsymbol{x}) = \boldsymbol{g}(\bar{\boldsymbol{x}}) - \int_0^{l(\boldsymbol{x})} (\mathcal{A}\underline{\boldsymbol{\sigma}} + \underline{\boldsymbol{\rho}})(\boldsymbol{x} + s\boldsymbol{t}(\boldsymbol{x}))\boldsymbol{t}(\boldsymbol{x}) ds. \tag{1.6}$$

Then, we solve the following problem in a computational subdomain $D_h$ of $\Omega$:

$$\mathcal{A}\underline{\boldsymbol{\sigma}} - \nabla \boldsymbol{u} + \underline{\boldsymbol{\rho}} = 0 \quad \text{in } D_h, \tag{1.7a}$$

$$\nabla \cdot \underline{\boldsymbol{\sigma}} = \boldsymbol{f} \quad \text{in } D_h, \tag{1.7b}$$

$$\boldsymbol{u} = \tilde{\boldsymbol{g}} \quad \text{on } \Gamma_h := \partial D_h. \tag{1.7c}$$

As we mentioned above, the idea of transferring the boundary data from $\Gamma$ to $\Gamma_h$ by integrating $\nabla \boldsymbol{u}$ along a segment, was originally introduced and analysed in a one-dimensional diffusion problem [3], where an HDG method was employed. Later, [7] generalized the method to the two-dimensional case and developed the implementation tools. In the same direction, [8] numerically showed that the method performs optimaly in convection-diffusion equations. Also, this technique was use in an exterior diffusion problem in a curved domain [9]. There, the authors coupled the boundary element method to an HDG scheme and experimentally showed that the order of convergence of the resulting method is optimal. Then [4] analysed the method proposed in [7] using the projections-based error analysis of HDG methods [2]. In fact, [4] provided the theoretical framework to analyse this type of techniques of transferring the boundary data. Lately, this approach was applied also to an HDG scheme of the Stokes problem [13] and to an elliptic interface problem where the interface is not polygonal [12].

The rest of this manuscript is organized as follows. In Chapter 2 we construct the computational domain, set the notation associated to the mesh and define the transferring segments. Then, in Chapter 3, we present the HDG scheme and summarize the main results. Chapter 4 is devoted to the proofs of well-posedness and the error estimates. Numerical experiments validating the theoretical results are presented in Chapter 5.

Chapter 2

# Mesh construction and notation

## 2.1 Computational domain

Let us first construct the computational domain $D_h$. We follow the approach in [6]. We begin by choosing a background polyhedral domain $\mathcal{M} \supset \Omega$. Then, given a sequence $\{\mathsf{T}_h\}_{h>0}$ of triangulations of $\mathcal{M}$(made of simplexes), we define $\mathcal{T}_h$ to be the set of all the elements $K \in \mathsf{T}_h$ which are totally included in $\Omega$,then we take the set $D_h := (\cup_{K \in \mathcal{T}_h} \overline{K})^o$. In Figure 2.1 we show a two-dimensional example, the boundary of $D_h$ is denoted by $\Gamma_h$. We assume, by simplicity the triangulation does not have hanging nodes and the elements $K$ are uniformly shape regular, this means, there exists a constant $\beta$ such that $h_K \leq \beta \varrho_K$, where $h_K$ is the diameter of the element $K$ and $\varrho_K$ is the radius of the biggest ball included in $K$. The maximum of the diameters $h_K \in \mathcal{T}_h$ is denoted by $h$.

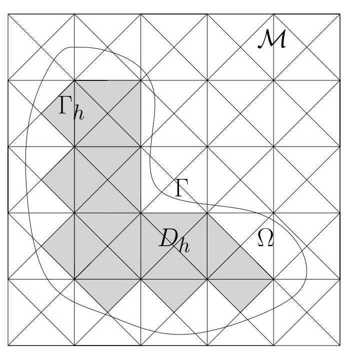



Figure 2.1: Example of a Domain $\Omega$, its boundary $\Gamma$, a background domain $\mathcal{M}$ and the polygonal subdomain $D_h$(gray).

We call $e$ an interior face if there are two elements $K^+$ and $K^-$ in $\mathcal{T}_h$ such that $e = \partial K^+ \cap \partial K^-$. Similarly, $e$ is a boundary face if there is an element $K \in \mathcal{T}_h$ such that $e = \partial K \cap \Gamma_h$. Let $\mathcal{E}_h^0$ be the set of inferior faces of $\mathcal{T}_h, \mathcal{E}_h^\partial$ the set of faces at the boundary and $\mathcal{E}_h := \mathcal{E}_h^0 \cup \mathcal{E}_h^\partial$.

We denote by $\boldsymbol{n}$ the outward unit normal of the elemet $K$. When there is no confusion, we just write $\boldsymbol{n}$ and, when we want to emphasize that $\boldsymbol{n}$ is normal to the face $e$ of $K$, we write $\boldsymbol{n}_e$.

We use the notation $\underline{\boldsymbol{\eta}}$ for tensor-, $\boldsymbol{\eta}$ for vector-, and $\eta$ for scalar-valued functions. Also, given a region $D \subset \mathbb{R}^n$, we define

$$(\underline{\boldsymbol{\eta}}, \underline{\boldsymbol{\varsigma}})_{\mathcal{T}_h} := \sum_{i,j=1}^{n} (\underline{\boldsymbol{\eta}}_{i,j}, \underline{\boldsymbol{\varsigma}}_{i,j})_{\mathcal{T}_h}, \quad (\boldsymbol{\eta}, \boldsymbol{\varsigma})_{\mathcal{T}_h} := \sum_{i=1}^{n} (\boldsymbol{\eta}_i, \boldsymbol{\varsigma}_i)_{\mathcal{T}_h} \text{ and } (\eta, \varsigma)_{\mathcal{T}_h} := \sum_{K \in \mathcal{T}_h} (\eta, \varsigma)_K,$$

where $(\eta, \varsigma)_D$ denotes the integral of $\eta\varsigma$ over $D \subset \mathbb{R}^n$. Similarly, we write

$$\langle \boldsymbol{\eta}, \boldsymbol{\varsigma} \rangle_{\partial \mathcal{T}_h} := \sum_{i=1}^{n} \langle \eta_i, \varsigma_i \rangle_{\partial \mathcal{T}_h} \quad \text{and} \quad \langle \eta, \varsigma \rangle_{\partial \mathcal{T}_h} := \sum_{K \in \mathcal{T}_h} \langle \eta, \varsigma \rangle_{\partial K},$$

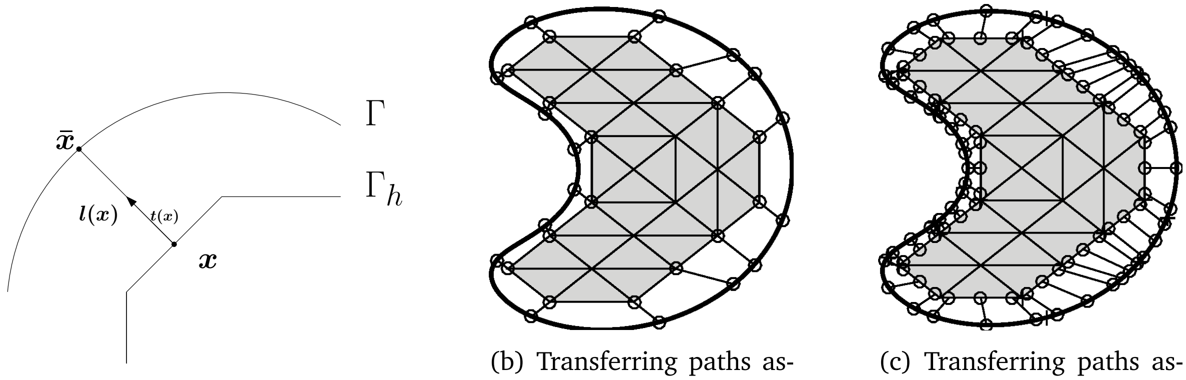
where $\langle \eta, \varsigma \rangle_D$ denotes the integral of $\eta\varsigma$ over $D \subset \mathbb{R}^{n-1}$. We also use the standard notation for Sobolev spaces and the associated norms and seminorms. We define $\| \eta \|_{D,w} := \| \sqrt{w}\eta \|_{L^2(D)}$ and, if $w = 1$, we write $\| \eta \|_D$ . In addition we define the following norm on the skeleton

$$\| \eta \|_h := \left( \sum_{K \in \mathcal{T}_h} h_K \| \eta \|_{\partial K}^2 \right)^{1/2}.$$

Finally, from now on $C$ will denote a positive constant independent of $h$, to simplify notation, $\nabla$ will denote the usual gradient or broken-gradient, depending on the context. Similarly for $\nabla \cdot$.

## 2.2 Transferring paths

As we mentioned in the introduction, given a point $\boldsymbol{x} \in \Gamma_h$ we need to specify a point $\bar{\boldsymbol{x}} \in \Gamma$ in order to transfer the boundary data from $\bar{\boldsymbol{x}}$ to $\boldsymbol{x}$ according to (1.6). In principle, $\bar{\boldsymbol{x}}$ could be any point of $\Gamma$ *close enough* to $\boldsymbol{x}$. The segment joining $\boldsymbol{x}$ and $\bar{\boldsymbol{x}}$ will be referred as *transferring path* associated to $\boldsymbol{x}$. We denote by $l(\boldsymbol{x})$ and $\boldsymbol{t}(\boldsymbol{x})$ the length and unit tangent vector, respectively, of the transferring path associated to $\boldsymbol{x}$, see Figure 2.2a. From a practical point of view, this transferring path is required to satisfy three conditions: (1) $\bar{\boldsymbol{x}}$ and $\boldsymbol{x}$ must be as close as possible, (2) two transferring path must not intersect each other before terminating at $\Gamma$ and (3) a transferring path must not intersect the interior of the computational domain $D_h$. The authors in [6], for the two dimensional case, proposed an algorithm to construct a family of transferring paths satisfying the above mentioned condition. The construction in three dimensions can be done using the same ideas. In practice we only need to compute the transferring paths of the quadrature points of all boundary edges (see Figure 2.2c).




(a) Transferring path associated to $\boldsymbol{x}$.

(b) Transferring paths associated to the boundary vertices.

(c) Transferring paths associated to the boundary quadrature points.

Figure 2.2: Examples of the transferring paths

## 2.3 Extrapolation regions

Now, let us introduce the notation associated to the set $D_h^c := \Omega \setminus \overline{D_h}$. For a face $e \in \mathcal{E}_h^\partial$, we denote by $K^e$ the only element of $\mathcal{T}_h$ having $e$ as a face. We define

$$\widetilde{K}_{ext}^e := \{\boldsymbol{x} + s\boldsymbol{t}(\boldsymbol{x}) : 0 \le s \le \boldsymbol{l}(\boldsymbol{x}), \boldsymbol{x} \in e\}.$$

In Figure 2.3 we observe an example of a region $\widetilde{K}_{ext}^e$.

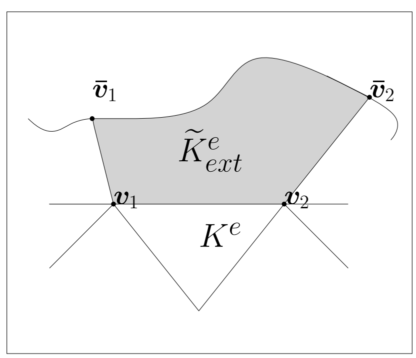



Figure 2.3: Example of $\widetilde{K}_{ext}^e$.

The HDG method will be used to compute an approximation of the solution in $D_h$, which is a polynomial in $K^e$ and can be locally extrapolated from $K^e$ to $\widetilde{K}_{ext}^e$. This procedure provides an approximation of the solution in $D_h^c$ since $\cup_{e\in\mathcal{E}_h^\partial} \widetilde{K}_{ext}^e = D_h^c$. The subscript *ext* in $\widetilde{K}_{ext}^e$ is introduced to indicate that in those regions the discrete solution is being *extrapolated* or *extended*.

Let $p$ a polynomial defined on $K^e$. The extrapolation of $p$ from $K^e$ to $\widetilde{K}_{ext}^e$, denoted by $E_h(p)$, is defined by $E_h(p)(y) := p|_{K^e}(y), \forall y \in \widetilde{K}_{ext}^e$. To simplify notation, from now on we will just write $p(y)$ instead of $E_h(p)(y)$ for $y \in \widetilde{K}_{ext}^e$. The same notation will be used for tensor- and vector-valued polynomial functions defined on $K^e$.

Chapter 3

# The HDG method

## 3.1 Polynomial spaces

We now set the notation associated to the discrete spaces that we will need in the HDG method. Let $P_k(K)$ be the set of polynomials of degree at most $k$ over the element $K$. We set $\mathbf{P}_k(K) := [P_k(K)]^n$, $\underline{\mathbf{P}}_k(K) := [P_k(K)]^{n\times n}$ and $\underline{\boldsymbol{A}}(K) := [\boldsymbol{A}_{i,j}(K)]^{n\times n}$ such that

$$\boldsymbol{A}_{i,j}(K) = \begin{cases} P_k(K) & \text{if } \ i < j \\ 0 & \text{if } \ i = j \\ -P_k(K) & \text{if } \ i > j \end{cases}$$

We notice that $\underline{\boldsymbol{A}}(K) \subset \underline{\boldsymbol{AS}}(K) := \{\underline{\boldsymbol{\eta}} \in \underline{\boldsymbol{L}}^2(K) : \underline{\boldsymbol{\eta}} + \underline{\boldsymbol{\eta}}^T = \underline{\mathbf{0}}\}$ and it is called the space of rotation. In addition we define the polynomial space $\underline{\boldsymbol{B}}(K)$ associate to bubble functions. We proceed as in [5]. In two dimensional case

$$\underline{\boldsymbol{B}}(K) := \nabla \times ((\nabla \times \underline{\boldsymbol{A}}(K))b_k),$$

where $b_k$ is a scalar-valued function. More precisely, for each edge $e$ of the element $K$, let $\eta_e$ be a linear function such that $\eta_e = 0$ on the edge $e$ and $0 \leq \eta_e$ on $K$. Thus,

$$b_k := \prod_{e\subset\partial K} \eta_e.$$

Here, the operator $\nabla\times$ is defined as follows:

$$\nabla \times \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{pmatrix} := \begin{pmatrix} -\partial_y\tau_{11} + \partial_x\tau_{12} \\ -\partial_y\tau_{21} + \partial_x\tau_{22} \end{pmatrix}, \qquad \nabla \times \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} := \begin{pmatrix} -\partial_y\nu_1 & \partial_x\nu_1 \\ -\partial_y\nu_2 & \partial_x\nu_2 \end{pmatrix}.$$

In the three dimensional case we have

$$\underline{\boldsymbol{B}}(K) := \nabla \times ((\nabla \times \underline{\boldsymbol{A}}(K))\underline{\boldsymbol{b}}_k),$$

where the bubble function is defined by

$$\underline{\boldsymbol{b}}_k := \sum_{e\subset\partial K} \Big[ \prod_{e'\subset\partial K\setminus\{e\}} \eta_{e'} \Big] \nabla\eta_e \otimes \nabla\eta_e.$$

Here the operator $\nabla\times$ is defined as:

$$\nabla \times \begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{pmatrix} := \begin{pmatrix} \nabla \times (\tau_{11}, \tau_{12}, \tau_{13}) \\ \nabla \times (\tau_{21}, \tau_{22}, \tau_{23}) \\ \nabla \times (\tau_{31}, \tau_{32}, \tau_{33}) \end{pmatrix}.$$

Then, for an element $K$ and a face $e$, we define the local spaces $\underline{\boldsymbol{V}}(K), \boldsymbol{W}(K), \underline{\boldsymbol{A}}(K)$ and $\boldsymbol{M}(e)$ by

$$\begin{aligned} \underline{\boldsymbol{V}}(K) &:= \underline{\mathbf{P}}_k(K) + \underline{\boldsymbol{B}}(K), \\ \boldsymbol{W}(K) &:= \mathbf{P}_k(K), \\ \boldsymbol{M}(e) &:= \mathbf{P}_k(e). \end{aligned}$$

Finally, we notice that

$$\begin{aligned} \underline{\boldsymbol{V}}(K) &= \underline{\mathbf{P}}_k(K) + \nabla \times ((\nabla \times \underline{\boldsymbol{A}}(K))) b_k) \\ &= \underline{\mathbf{P}}_k(K) \oplus \nabla \times ((\nabla \times \underline{\tilde{\boldsymbol{A}}}(K)) b_k), \end{aligned}$$

where $\underline{\tilde{\boldsymbol{A}}}(K) = \underline{\boldsymbol{A}}(K) \cap \underline{\tilde{\mathbf{P}}}_k(K)$ and $\underline{\tilde{\mathbf{P}}}_k(K)$ is the set of polynomials of degree $k$ exactly.
*Remark.* It is not difficult to realize that any function $\underline{\boldsymbol{v}}$ lying in the space $\underline{\mathcal{B}}_h := \{\underline{\boldsymbol{\eta}} \in \underline{\boldsymbol{L}}^2(D_h) : \underline{\boldsymbol{\eta}}|_K \in \underline{\boldsymbol{B}}(K), K \in \mathcal{T}_h\}$ is such that

(B.1) $\nabla \cdot \underline{\boldsymbol{v}}|_K = 0 \qquad \forall K \in \mathcal{T}_h,$

(B.2) $\underline{\boldsymbol{v}}\boldsymbol{n}|_e = 0 \qquad \forall e \in \mathcal{E}_h.$

## 3.2 The HDG scheme

The method we consider seeks an approximation $(\underline{\boldsymbol{\sigma}}_h, \boldsymbol{u}_h, \underline{\boldsymbol{\rho}}_h, \hat{\boldsymbol{u}}_h)$ of the exact solution $(\underline{\boldsymbol{\sigma}}, \boldsymbol{u}, \underline{\boldsymbol{\rho}}, \boldsymbol{u}|_{\mathcal{E}_h})$ in the finite-dimensional space $\in \underline{\boldsymbol{V}}_h \times \boldsymbol{W}_h \times \underline{\boldsymbol{A}}_h \times \boldsymbol{M}_h \subset \underline{\boldsymbol{L}}^2(D_h) \times \boldsymbol{L}^2(D_h) \times \underline{\boldsymbol{AS}}(D_h) \times \boldsymbol{L}^2(\mathcal{E}_h)$ given by

$$\underline{\boldsymbol{V}}_h = \{\underline{\boldsymbol{v}} \in \underline{\boldsymbol{L}}^2(\mathcal{T}_h) : \underline{\boldsymbol{v}}|_K \in \underline{\boldsymbol{V}}(K), \quad \forall K \in \mathcal{T}_h\}, \tag{3.1a}$$

$$\boldsymbol{W}_h = \{\boldsymbol{w} \in \boldsymbol{L}^2(\mathcal{T}_h) : \boldsymbol{w}|_K \in \boldsymbol{W}(K), \quad \forall K \in \mathcal{T}_h\}, \tag{3.1b}$$

$$\underline{\boldsymbol{A}}_h = \{\underline{\boldsymbol{\eta}} \in \underline{\boldsymbol{L}}^2(\mathcal{T}_h) : \underline{\boldsymbol{\eta}}|_K \in \underline{\boldsymbol{A}}(K), \quad \forall K \in \mathcal{T}_h\}, \tag{3.1c}$$

$$\boldsymbol{M}_h = \{\boldsymbol{\mu} \in \boldsymbol{L}^2(\mathcal{E}_h) : \boldsymbol{\mu}|_e \in P_k(e), \quad \forall e \in \mathcal{E}_h\}. \tag{3.1d}$$

The approximation $(\underline{\boldsymbol{\sigma}}_h, \boldsymbol{u}_h, \underline{\boldsymbol{\rho}}_h, \hat{\boldsymbol{u}}_h)$ is the solution of the following linear system of equations:

$$(\mathcal{A}\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{v}})_{\mathcal{T}_h} + (\boldsymbol{u}_h, \nabla \cdot \underline{\boldsymbol{v}})_{\mathcal{T}_h} + (\underline{\boldsymbol{\rho}}_h, \underline{\boldsymbol{v}})_{\mathcal{T}_h} - \langle \hat{\boldsymbol{u}}_h, \underline{\boldsymbol{v}}\boldsymbol{n} \rangle_{\partial\mathcal{T}_h} = 0, \tag{3.2a}$$

$$(\underline{\boldsymbol{\sigma}}_h, \nabla \boldsymbol{w})_{\mathcal{T}_h} - \langle \hat{\underline{\boldsymbol{\sigma}}}_h \boldsymbol{n}, \boldsymbol{w} \rangle_{\partial\mathcal{T}_h} = -(\boldsymbol{f}, \boldsymbol{w})_{\mathcal{T}_h}, \tag{3.2b}$$

$$(\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{\eta}})_{\mathcal{T}_h} = 0, \tag{3.2c}$$

$$\langle \hat{\underline{\boldsymbol{\sigma}}}_h \boldsymbol{n}, \boldsymbol{\mu} \rangle_{\partial\mathcal{T}_h/\Gamma_h} = 0, \tag{3.2d}$$

$$\langle \hat{\boldsymbol{u}}_h, \boldsymbol{\mu} \rangle_{\Gamma_h} = \langle \tilde{\boldsymbol{g}}_h, \boldsymbol{\mu} \rangle_{\Gamma_h}, \tag{3.2e}$$

for all $(\underline{\boldsymbol{v}}, \boldsymbol{w}, \underline{\boldsymbol{\eta}}, \boldsymbol{\mu}) \in \underline{\boldsymbol{V}}_h \times \boldsymbol{W}_h \times \underline{\boldsymbol{A}}_h \times \boldsymbol{M}_h$, where

$$\hat{\underline{\boldsymbol{\sigma}}}_h \boldsymbol{n} = \underline{\boldsymbol{\sigma}}_h \boldsymbol{n} - \alpha(\boldsymbol{u}_h - \hat{\boldsymbol{u}}_h) \qquad \text{on} \qquad \partial\mathcal{T}_h, \tag{3.2f}$$

$$\tilde{\boldsymbol{g}}_h(\boldsymbol{x}) := \boldsymbol{g}(\bar{\boldsymbol{x}}) - \int_0^{l(\boldsymbol{x})} (\mathcal{A}\underline{\boldsymbol{\sigma}}_h + \underline{\boldsymbol{\rho}}_h)(\boldsymbol{x} + s\boldsymbol{t}(\boldsymbol{x}))\boldsymbol{t}(\boldsymbol{x}) ds \tag{3.2g}$$

and $\alpha$ is a positive scalar-valued stabilization function define on $\partial\mathcal{T}_h$. For a face $e$, we set $\alpha_e := \alpha|_e$. We observe here that (3.2g) is discrete version of (1.6).

This HDG scheme has been original introduced by [5] in the case of a polyhedral domain. In our case, since the domain $\Omega$ is not necessarily a polyhedron, the boundary data is transferred to the computational boundary $\Gamma_h$ according to (3.2g). Hence, the Dirichlet data $\tilde{\boldsymbol{g}}_h$ on $\Gamma_h$ depends on the unknowns $\underline{\boldsymbol{\sigma}}_h$ and $\underline{\boldsymbol{\rho}}_h$.

Chapter 4

# Analysis of the HDG method

## 4.1 Preliminaries

As we will see through this chapter, the analysis of the method uses several technicalities and most of the estimates involve a large number of terms. In order to keep the proofs as clean as possible, we assume the vector $\boldsymbol{t}(\boldsymbol{x})$ of the transferring paths associated to $\boldsymbol{x} \in e, e \in \mathcal{E}_h^{\partial}$, to be normal to $e$, i.e., $\boldsymbol{t}(\boldsymbol{x}) = \boldsymbol{n}_e$. In the general case where $\boldsymbol{t}(\boldsymbol{x})$ is not necessary equal to $\boldsymbol{n}_e$, as it happens in the construction of transferring paths mentioned in Section 2.2, terms of the type $\max_{\boldsymbol{x}\in e} \boldsymbol{t}(\boldsymbol{x})\cdot\boldsymbol{n}_e$ and $\max_{\boldsymbol{x}\in e} \frac{1}{\boldsymbol{t}(\boldsymbol{x})\cdot\boldsymbol{n}_e}$ would appear in the estimates. We emphasize that this assumption is only made to simplify the analysis and we consider that it is not crucial to explain the theory. Moreover, in the numerical experiments we consider the construction of transferring paths of Section 2.2 and we will see that results are optimal. Following the discussion in Section 2.3, for each $e \in \mathcal{E}_h^{\partial}$, let us define

$$K_{ext}^e := \{\boldsymbol{x} + s\boldsymbol{n}_e : 0 \le s \le \boldsymbol{l}(\boldsymbol{x}), \boldsymbol{x} \in e\}.$$

In addition, we define auxiliary constants that will be used in the analysis of the HDG method. Let $K^e$ the element with face $e$. We denote by $h_e^{\perp}$ the biggest distance of a point in $K^e$ to the plane determined by the face $e$. Similarly, we denote by $H_e^{\perp}$ the biggest distance of a point of $K_{ext}^e$ to the plane determined by the face $e$, and set the ratio

$$r_e := H_e^{\perp}/h_e^{\perp}. \tag{4.1}$$

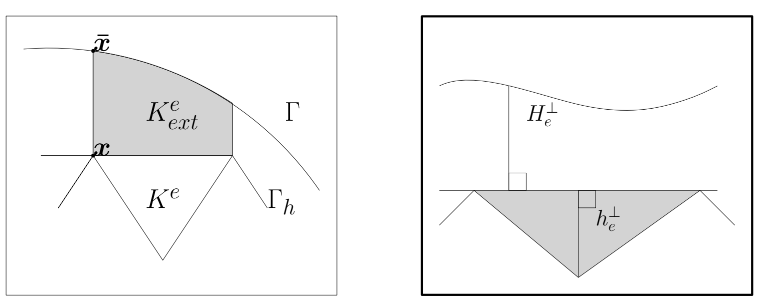



Figure 4.1: Examples of $K_{ext}^e, H_e^{\perp}$ and $h_e^{\perp}$.

In Figure 4.1 we display examples of $K^e_{ext}$ and the constant $H_e^{\perp}$ and $h_e^{\perp}$. Moreover, we consider the following norms:

$$\| \eta \|_{\Gamma_h,l^{-1}} := \left\{ \sum_{e\in\mathcal{E}_h^{\partial}} \| \eta \|^2_{e,l^{-1}} \right\}^{1/2}, \ \| \eta \|_{K^e_{ext},(h^{\perp})^2} := \left\{ \sum_{e\in\mathcal{E}_h^{\partial}} (h_e^{\perp})^2 \| \eta \|^2_{K^e_{ext}} \right\}^{1/2},$$

where $\| \eta \|_{e,l^{-1}} = \| l^{-1/2}\eta \|_e$ . Finally, we define the constants:

$$C^e_{ext} := \frac{1}{\sqrt{r_e}} \sup_{\underline{\boldsymbol{\eta}}\in\underline{\boldsymbol{V}}(K^e)\boldsymbol{n}_e\setminus\{0\}} \frac{\| \underline{\boldsymbol{\eta}} \|_{K^e_{ext}}}{\| \underline{\boldsymbol{\eta}} \|_{K^e}} \qquad C^e_{inv} := h_e^{\perp} \sup_{\underline{\boldsymbol{\eta}}\in\underline{\boldsymbol{V}}(K^e)\boldsymbol{n}_e\setminus\{0\}} \frac{\| \partial_{\boldsymbol{n}_e}\underline{\boldsymbol{\eta}} \|_{K^e}}{\| \underline{\boldsymbol{\eta}} \|_{K^e}}. \tag{4.2}$$

The constants $C^e_{ext}$ and $C^e_{inv}$ are independent of $h$, but depend on the polynomial degree $k$ as shown in Lemma A.2 of [4].

### 4.1.1 Auxiliary estimates

In this section we will state estimates that will be used in the proof of the main results.

**Definition 4.1.** For any face $e \in \mathcal{E}_h^{\partial}$, any point $\boldsymbol{x}$ lying on $e$ and any smooth enough function tensor $\underline{\boldsymbol{v}}$ given in $K^e_{ext}$, we define the auxiliary function

$$\boldsymbol{\Lambda}^{\underline{\boldsymbol{v}}}(\boldsymbol{x}) := \frac{1}{l(\boldsymbol{x})} \int_0^{l(\boldsymbol{x})} [\underline{\boldsymbol{v}}(\boldsymbol{x}+s\boldsymbol{n}_e) - \underline{\boldsymbol{v}}(\boldsymbol{x})]\boldsymbol{n}_e ds.$$

**Lemma 4.1.** *For each $e \in \mathcal{E}_h^{\partial}$, we have that*

$$\| \boldsymbol{\Lambda}^{\underline{\boldsymbol{v}}} \|_{e,l} \leq \frac{1}{\sqrt{3}} {r_e}^{3/2} C^e_{ext} C^e_{inv} \| \underline{\boldsymbol{v}} \|_{K^e}, \qquad \text{for all } \underline{\boldsymbol{v}} \in \underline{\mathbf{P}}_k(K^e) \text{ and } l \text{ denotes } |l(x)|.$$

*Proof.* It follows from Lemma 5.2 of [4] applied to each of row of $\underline{\boldsymbol{v}}$. □

**Lemma 4.2.** *(Lemma 1.46 of [10])(Discrete Trace Inequality) Let $K \in \mathcal{T}_h$ and $e$ a face of $K$. Then, for $p \in P_k(K)$ we have*

$$\| p \|_{L^2(e)} \leq C^e_{tr} h_e^{-1/2} \| p \|_{L^2(K)},$$

*where $C^e_{tr} > 0$ is independent of $h$.*

**Lemma 4.3.** *(Lemma A.1 in [4]) For any polynomial $p$ of degree $l$ in $K^e \cup K^e_{ext}$, we have*

$$\| p \|_{K^e_{ext}} \leq C^e_{ext} {r_e}^{1/2} \| p \|_{K^e} .$$

**Lemma 4.4.** *(Lemma 2.8 in [11]) Given $\underline{\boldsymbol{\eta}} \in \underline{\boldsymbol{A}}_h^0 := \{\underline{\boldsymbol{\eta}} \in \underline{\boldsymbol{A}}_h : (\underline{\boldsymbol{\eta}}, \underline{\boldsymbol{v}})_K = 0$ for all $\underline{\boldsymbol{v}} \in \underline{\mathbf{P}}_0(K)$ and for all $K \in \mathcal{T}_h\}$, there exists $\underline{\boldsymbol{v}} \in \underline{\mathcal{B}}_h$ such that*

$$(\underline{\boldsymbol{\eta}}, \underline{\boldsymbol{\gamma}})_{\mathcal{T}_h} = (\underline{\boldsymbol{v}}, \underline{\boldsymbol{\gamma}})_{\mathcal{T}_h}, \quad \text{for all } \boldsymbol{\gamma} \in \underline{\boldsymbol{A}}_h, \tag{4.3a}$$

$$\| \underline{\boldsymbol{v}} \|_{D_h} \leq C^0_{\eta} \| \underline{\boldsymbol{\eta}} \|_{D_h}, \tag{4.3b}$$

*where $C^0_{\eta} > 0$ where is independent on $h$.*

We also need to define the auxiliary space $\underline{\mathcal{G}}_h := \{\underline{\boldsymbol{v}} \in \underline{\boldsymbol{H}}(div; D_h) : \underline{\boldsymbol{v}}|_K \in \underline{\mathbf{P}}_1(K) \forall K \in \mathcal{T}_h\}$.

**Lemma 4.5.** *(Lemma 3.9 of [4] ) Given* $\underline{\boldsymbol{\eta}} \in \underline{\boldsymbol{A}}_h^c := \underline{\boldsymbol{A}}_h \cap \underline{\mathbf{P}}_0(\mathcal{T}_h)$*, there exists* $\underline{\boldsymbol{v}} \in \underline{\mathcal{G}}_h$ *such that*

$$\nabla \cdot \underline{\boldsymbol{v}} = 0, \tag{4.4a}$$

$$(\underline{\boldsymbol{v}}, \underline{\boldsymbol{\gamma}})_{\mathcal{T}_h} = (\underline{\boldsymbol{\eta}}, \underline{\boldsymbol{\gamma}})_{\mathcal{T}_h} \quad \text{for all } \underline{\boldsymbol{\gamma}} \in \underline{\boldsymbol{A}}_h^c, \tag{4.4b}$$

$$\| \underline{\boldsymbol{v}} \|_{D_h} \leq C_\eta^c \| \underline{\boldsymbol{\eta}} \|_{D_h}, \tag{4.4c}$$

*where* $C_\eta^c > 0$ *where is independent of* $h$*.*

### 4.1.2 Auxiliary constant

We define constant involving the ratio $r_e$ that will help us to simplify the notation because most of the estimates presented in this work depend on these quantities. Roughly speaking, they indicate how close $\Gamma_h$ and $\Gamma$ must be in order to ensure the stability of the method:

$$R := \max_{e \in \mathcal{E}_h^\partial} r_e, \qquad R_T := \max\{1, (1 + C_{\mathcal{A}})\} \max_{e \in \mathcal{E}_h^\partial} C_{tr}^e r_e,$$

$$R_c := \max\{1, C_{\mathcal{A}}\} \max_{e \in \mathcal{E}_h^\partial} r_e^2 C_{ext}^e C_{inv}^e, \qquad R_A := \max_{e \in \mathcal{E}_h^\partial} r_e(1 + C_{\mathcal{A}}), \qquad R_\alpha := \max_{e \in \mathcal{E}_h^\partial} r_e \alpha_e,$$

## 4.2 Existence and uniqueness

We proceed now to show existence and uniqueness of the HDG Scheme (3.2), under the following assumptions.
Assumptions C
For every face $e \in \mathcal{E}_h$, we assume

(C.1) $h_e^\perp \leq h_e$,

(C.2) $\max\{1, C_\eta\} C_{tr}^e C_{ext}^e r_e \leq \sqrt{2}/4$,

(C.3) $\max\{(1 + C_{\mathcal{A}}), C_\eta C_{\mathcal{A}}\} C_{el}^{-1/2} C_{tr}^e r_e^{1/2} \leq 1/10$,

(C.4) $\max\{1, C_\eta\} C_{el}^{-1/2} C_{\mathcal{A}} r_e^{3/2} C_{ext}^e C_{inv}^e \leq 1/10$,

(C.5) $r_e \alpha_e h_e^\perp \leq 1/5$,

(C.6) $r_e \leq C$,

where $C_\eta$ is a positive constant independent of the discretization parameters that will be introduced in Lemma 4.3.

**Lemma 4.6.** *Let* $\boldsymbol{f} = \mathbf{0}$ *and* $\boldsymbol{g} = \mathbf{0}$*. Then, the approximation in* (3.2) *satisfies*

$$C_{el} \| \underline{\boldsymbol{\sigma}}_h \|_{D_h}^2 + \| \boldsymbol{u}_h - \hat{\boldsymbol{u}}_h \|_{\partial \mathcal{T}_h, \alpha}^2 \leq \mathbb{T},$$

*where* $\mathbb{T} := \langle \tilde{\boldsymbol{g}}_h, \hat{\underline{\boldsymbol{\sigma}}}_h \boldsymbol{n} \rangle_{\Gamma_h}$*.*

*Proof.* We take $\underline{\boldsymbol{v}} = \underline{\boldsymbol{\sigma}}_h, \boldsymbol{w} = \boldsymbol{u}_h, \underline{\boldsymbol{\eta}} = \underline{\boldsymbol{\rho}}_h, \boldsymbol{\mu} = \hat{\boldsymbol{u}}_h, \boldsymbol{\mu} = \hat{\underline{\boldsymbol{\sigma}}}_h \boldsymbol{n}$, in Equations (3.2a)-(3.2e), respectively. Then

$$(\mathcal{A}\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{\sigma}}_h)_{\mathcal{T}_h} + (\boldsymbol{u}_h, \nabla \cdot \underline{\boldsymbol{\sigma}}_h)_{\mathcal{T}_h} + (\underline{\boldsymbol{\rho}}_h, \underline{\boldsymbol{\sigma}}_h)_{\mathcal{T}_h} - \langle \hat{\boldsymbol{u}}_h, \underline{\boldsymbol{\sigma}}_h \boldsymbol{n} \rangle_{\partial \mathcal{T}_h} = 0, \tag{4.5a}$$

$$(\underline{\boldsymbol{\sigma}}_h, \nabla \boldsymbol{u}_h)_{\mathcal{T}_h} - \langle \hat{\underline{\boldsymbol{\sigma}}}_h \boldsymbol{n}, \boldsymbol{u}_h \rangle_{\partial \mathcal{T}_h} = 0, \tag{4.5b}$$

$$(\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{\rho}}_h)_{\mathcal{T}_h} = 0, \tag{4.5c}$$

$$\langle \hat{\underline{\boldsymbol{\sigma}}}_h \boldsymbol{n}, \hat{\boldsymbol{u}}_h \rangle_{\partial \mathcal{T}_h / \Gamma_h} = 0, \tag{4.5d}$$

$$\langle \hat{\boldsymbol{u}}_h, \hat{\underline{\boldsymbol{\sigma}}}_h \boldsymbol{n} \rangle_{\Gamma_h} = \langle \tilde{\boldsymbol{g}}_h, \hat{\underline{\boldsymbol{\sigma}}}_h \boldsymbol{n} \rangle_{\Gamma_h}. \tag{4.5e}$$

Integrating by parts Equation (4.5b), we obtain

$$(\underline{\boldsymbol{\sigma}}_h, \nabla \boldsymbol{u}_h)_{\mathcal{T}_h} - \langle \hat{\underline{\boldsymbol{\sigma}}}_h \boldsymbol{n}, \boldsymbol{u}_h\rangle_{\partial\mathcal{T}_h} = -(\nabla \cdot \underline{\boldsymbol{\sigma}}_h, \boldsymbol{u}_h)_{\mathcal{T}_h} + \langle \underline{\boldsymbol{\sigma}}_h \boldsymbol{n} - \hat{\underline{\boldsymbol{\sigma}}}_h \boldsymbol{n}, \boldsymbol{u}_h\rangle_{\partial\mathcal{T}_h} = 0. \tag{4.6}$$

Adding Equations (4.5a) and (4.6) and using Equation (4.5c), we get

$$(\mathcal{A}\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{\sigma}}_h)_{\mathcal{T}_h} + \langle \underline{\boldsymbol{\sigma}}_h \boldsymbol{n} - \hat{\underline{\boldsymbol{\sigma}}}_h \boldsymbol{n}, \boldsymbol{u}_h\rangle_{\partial\mathcal{T}_h} - \langle \hat{\boldsymbol{u}}_h, \underline{\boldsymbol{\sigma}}_h \boldsymbol{n}\rangle_{\partial\mathcal{T}_h} = 0.$$

Next, note that $\langle \hat{\boldsymbol{u}}_h, \underline{\boldsymbol{\sigma}}_h \boldsymbol{n}\rangle_{\partial\mathcal{T}_h} = \langle \hat{\boldsymbol{u}}_h, \underline{\boldsymbol{\sigma}}_h \boldsymbol{n} - \hat{\underline{\boldsymbol{\sigma}}}_h \boldsymbol{n}\rangle_{\partial\mathcal{T}_h} + \langle \hat{\boldsymbol{u}}_h, \hat{\underline{\boldsymbol{\sigma}}}_h \boldsymbol{n}\rangle_{\partial\mathcal{T}_h}$ and $\langle \hat{\boldsymbol{u}}_h, \hat{\underline{\boldsymbol{\sigma}}}_h \boldsymbol{n}\rangle_{\partial\mathcal{T}_h} = \langle \hat{\boldsymbol{u}}_h, \hat{\underline{\boldsymbol{\sigma}}}_h \boldsymbol{n}\rangle_{\Gamma_h}$ by Equation (4.5e). Using Equation (3.2f) we have

$$\begin{aligned}(\mathcal{A}\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{\sigma}}_h)_{\mathcal{T}_h} + \langle \underline{\boldsymbol{\sigma}}_h \boldsymbol{n} - \hat{\underline{\boldsymbol{\sigma}}}_h \boldsymbol{n}, \boldsymbol{u}_h\rangle_{\partial\mathcal{T}_h} &= \langle \hat{\boldsymbol{u}}_h, \hat{\underline{\boldsymbol{\sigma}}}_h \boldsymbol{n}\rangle_{\partial\mathcal{T}_h},\\ (\mathcal{A}\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{\sigma}}_h)_{\mathcal{T}_h} + \langle \alpha_e(\boldsymbol{u}_h - \hat{\boldsymbol{u}}_h), \boldsymbol{u}_h - \hat{\boldsymbol{u}}_h\rangle_{\partial\mathcal{T}_h} &= \langle \hat{\boldsymbol{u}}_h, \hat{\underline{\boldsymbol{\sigma}}}_h \boldsymbol{n}\rangle_{\Gamma_h}.\end{aligned}$$

Using the fact that $C_{el} \parallel \underline{\boldsymbol{\sigma}}_h \parallel^2_{\mathcal{T}_h} \leq (\mathcal{A}\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{\sigma}}_h)_{\mathcal{T}_h}$, we have

$$C_{el} \parallel \underline{\boldsymbol{\sigma}}_h \parallel^2_{D_h} + \parallel \boldsymbol{u}_h - \hat{\boldsymbol{u}}_h \parallel^2_{\partial\mathcal{T}_h,\alpha} \leq \langle \hat{\boldsymbol{u}}_h, \hat{\underline{\boldsymbol{\sigma}}}_h \boldsymbol{n}\rangle_{\Gamma_h} = \langle \tilde{\boldsymbol{g}}_h, \hat{\underline{\boldsymbol{\sigma}}}_h \boldsymbol{n}\rangle_{\Gamma_h}$$

and the result follows from the definition of $\mathbb{T}$. □

*Remark.* In the case of a polyhedral domain $\Omega$, the previous results holds with $\mathbb{T} = 0$ since $\tilde{\boldsymbol{g}}_h = \boldsymbol{g} = \boldsymbol{0}$, and well-posedness of the method follows by standard arguments. In our case, $\mathbb{T}$ is not zero and we proceed now to bound it.

**Lemma 4.7.** *We have $\mathbb{T} = \sum_{i=1}^6 \mathbb{T}_i$, where*

$$\begin{aligned}&\mathbb{T}_1 = \langle l^{-1/2}\tilde{\boldsymbol{g}}_h, l^{1/2}(\underline{\boldsymbol{\sigma}}_h - \mathcal{A}\underline{\boldsymbol{\sigma}}_h)\boldsymbol{n}\rangle_{\Gamma_h}, \quad \mathbb{T}_2 = \langle l^{-1}\tilde{\boldsymbol{g}}_h, \tilde{\boldsymbol{g}}_h\rangle_{\Gamma_h}, \quad \mathbb{T}_3 = \langle l^{-1/2}\tilde{\boldsymbol{g}}_h, l^{1/2}\boldsymbol{\Lambda}^{\mathcal{A}(\underline{\boldsymbol{\sigma}}_h)}\rangle_{\Gamma_h},\\ &\mathbb{T}_4 = \langle l^{-1/2}\tilde{\boldsymbol{g}}_h, l^{1/2}\boldsymbol{\Lambda}^{\underline{\boldsymbol{\rho}}_h}\rangle_{\Gamma_h}, \quad \mathbb{T}_5 = \langle l^{-1/2}\tilde{\boldsymbol{g}}_h, l^{1/2}\underline{\boldsymbol{\rho}}_h\boldsymbol{n}\rangle_{\Gamma_h}, \quad \mathbb{T}_6 = \langle l^{-1/2}\tilde{\boldsymbol{g}}_h, l^{1/2}\alpha(\boldsymbol{u}_h - \hat{\boldsymbol{u}}_h)\rangle_{\Gamma_h}.\end{aligned}$$

*Proof.* First of all, we use Definition 4.1 and rewrite $\tilde{\boldsymbol{g}}_h(\boldsymbol{x})$ as follows

$$\begin{aligned}\tilde{\boldsymbol{g}}_h(\boldsymbol{x}) &= -\int_0^{l(\boldsymbol{x})} (\mathcal{A}\underline{\boldsymbol{\sigma}}_h + \underline{\boldsymbol{\rho}}_h)(\boldsymbol{x} + s\boldsymbol{n}_e)\boldsymbol{n}_e ds\\ &= -\int_0^{l(\boldsymbol{x})} [\mathcal{A}\underline{\boldsymbol{\sigma}}_h(\boldsymbol{x} + s\boldsymbol{n}_e) - \mathcal{A}\underline{\boldsymbol{\sigma}}_h(\boldsymbol{x})]\boldsymbol{n}_e ds - l(\boldsymbol{x})\mathcal{A}\underline{\boldsymbol{\sigma}}_h\boldsymbol{n}_e - \int_0^{l(\boldsymbol{x})} [\underline{\boldsymbol{\rho}}_h(\boldsymbol{x} + s\boldsymbol{n}_e) - \underline{\boldsymbol{\rho}}_h(\boldsymbol{x})]\boldsymbol{n}_e ds - l(\boldsymbol{x})\underline{\boldsymbol{\rho}}_h\boldsymbol{n}_e\\ &= -l(\boldsymbol{x})\left(\boldsymbol{\Lambda}^{\mathcal{A}(\underline{\boldsymbol{\sigma}}_h)}(\boldsymbol{x}) + \mathcal{A}\underline{\boldsymbol{\sigma}}_h\boldsymbol{n}_e + \boldsymbol{\Lambda}^{\underline{\boldsymbol{\rho}}_h}(\boldsymbol{x}) + \underline{\boldsymbol{\rho}}_h\boldsymbol{n}_e\right)\end{aligned}$$

and obtain

$$\mathcal{A}\underline{\boldsymbol{\sigma}}_h\boldsymbol{n}_e = -\frac{1}{l(\boldsymbol{x})}\tilde{\boldsymbol{g}}_h(\boldsymbol{x}) - \boldsymbol{\Lambda}^{\mathcal{A}(\underline{\boldsymbol{\sigma}}_h)}(\boldsymbol{x}) - \boldsymbol{\Lambda}^{\underline{\boldsymbol{\rho}}_h}(\boldsymbol{x}) - \underline{\boldsymbol{\rho}}_h\boldsymbol{n}_e.$$

By replacing the last identity in definition (3.2f), we obtain

$$\begin{aligned}\hat{\underline{\boldsymbol{\sigma}}}_h\boldsymbol{n}_e &= \underline{\boldsymbol{\sigma}}_h\boldsymbol{n}_e - \alpha_e(\boldsymbol{u}_h - \hat{\boldsymbol{u}}_h)\\ &= (\underline{\boldsymbol{\sigma}}_h - \mathcal{A}\underline{\boldsymbol{\sigma}}_h)\boldsymbol{n}_e + \mathcal{A}\underline{\boldsymbol{\sigma}}_h\boldsymbol{n}_e - \alpha_e(\boldsymbol{u}_h - \hat{\boldsymbol{u}}_h)\\ &= (\underline{\boldsymbol{\sigma}}_h - \mathcal{A}\underline{\boldsymbol{\sigma}}_h)\boldsymbol{n}_e - \frac{1}{l(\boldsymbol{x})}\tilde{\boldsymbol{g}}(\boldsymbol{x}) - \boldsymbol{\Lambda}^{\mathcal{A}(\underline{\boldsymbol{\sigma}}_h)}(\boldsymbol{x}) - \boldsymbol{\Lambda}^{\underline{\boldsymbol{\rho}}_h}(\boldsymbol{x}) - \underline{\boldsymbol{\rho}}_h\boldsymbol{n}_e - \alpha_e(\boldsymbol{u}_h - \hat{\boldsymbol{u}}_h).\end{aligned}$$

Finally, the result is obtained replacing the last expression in the definition of $\mathbb{T}$. □

**Corollary 4.1.** *Let us suppose that Assumption (C.1) holds. Then,*

$$
\begin{aligned}
|\mathbb{T}| \;\; \leq \;\; & -\frac{1}{2} \parallel \tilde{\boldsymbol{g}}_h \parallel^2_{\Gamma_h, l^{-1}} \\
& + \frac{10}{4}(1+C_{\mathcal{A}})^2 \max_{e\in\mathcal{E}_h^\partial} (C^e_{tr})^2 r_e \parallel \underline{\boldsymbol{\sigma}}_h \parallel^2_{D_h} + \frac{10}{12} C^2_{\mathcal{A}} \max_{e\in\mathcal{E}_h^\partial} r^3_e (C^e_{ext})^2 (C^e_{inv})^2 \parallel \underline{\boldsymbol{\sigma}}_h \parallel^2_{D_h} \\
& + \frac{10}{12} \max_{e\in\mathcal{E}_h^\partial} r^3_e (C^e_{ext})^2 (C^e_{inv})^2 \parallel \underline{\boldsymbol{\rho}}_h \parallel^2_{D_h} + \max_{e\in\mathcal{E}_h^\partial} \frac{10}{4} (C^e_{tr})^2 r_e \parallel \underline{\boldsymbol{\rho}}_h \parallel^2_{D_h} \\
& + \frac{10}{4} \max_{e\in\mathcal{E}_h^\partial} r_e h^\perp_e \alpha_e \parallel \boldsymbol{u}_h - \hat{\boldsymbol{u}}_h \parallel^2_{\Gamma_h, \alpha} . 
\end{aligned} \tag{4.7}
$$

*Moreover, if Assumption (C.3)-(C.5) also hold, then*

$$
|\mathbb{T}| \leq -\frac{1}{2} \parallel \tilde{\boldsymbol{g}}_h \parallel^2_{\Gamma_h, l^{-1}} + \frac{C_{el}}{30} \parallel \underline{\boldsymbol{\sigma}}_h \parallel^2_{D_h} + \frac{C_{el}}{30} \frac{1}{C^2_{\mathcal{A}} (C_\eta)^2} \parallel \underline{\boldsymbol{\rho}}_h \parallel^2_{D_h} + \frac{1}{2} \parallel \boldsymbol{u}_h - \hat{\boldsymbol{u}}_h \parallel^2_{\Gamma_h, \alpha} . \tag{4.8}
$$

*Proof.* Let $\varepsilon > 0$. We proceed to bound the terms $\mathbb{T}_i, i = 1, ..., 6.$ of Lemma 4.1. For $\mathbb{T}_1$, we use Cauchy-Schwarz inequality, the fact that $l(\boldsymbol{x}) \leq H^\perp_e$, Lemma 4.2, Assumptions (C.1), and Young's inequality:

$$
\begin{aligned}
\mathbb{T}_1 &= \sum_{e\in\mathcal{E}_h^\partial} \langle l^{-1/2} \tilde{\boldsymbol{g}}_h, l^{1/2} (\underline{\boldsymbol{\sigma}}_h - \mathcal{A}\underline{\boldsymbol{\sigma}}_h) \boldsymbol{n}_e \rangle_e \leq \sum_{e\in\mathcal{E}_h^\partial} \parallel \tilde{\boldsymbol{g}}_h \parallel_{e, l^{-1}} (H^\perp_e)^{1/2} \parallel (\underline{\boldsymbol{\sigma}}_h - \mathcal{A}\underline{\boldsymbol{\sigma}}_h) \boldsymbol{n}_e \parallel_e \\
&\leq \sum_{e\in\mathcal{E}_h^\partial} \parallel \tilde{\boldsymbol{g}}_h \parallel_{e, l^{-1}} C^e_{tr} (H^\perp_e)^{1/2} h_e^{-1/2} \parallel \underline{\boldsymbol{\sigma}}_h - \mathcal{A}\underline{\boldsymbol{\sigma}}_h \parallel_{K^e} \leq \sum_{e\in\mathcal{E}_h^\partial} \parallel \tilde{\boldsymbol{g}}_h \parallel_{e, l^{-1}} C^e_{tr} r_e^{1/2} (1+C_{\mathcal{A}}) \parallel \underline{\boldsymbol{\sigma}}_h \parallel_{K^e}, \\
&\leq \sum_{e\in\mathcal{E}_h^\partial} \left( \varepsilon \parallel \tilde{\boldsymbol{g}}_h \parallel^2_{e, l^{-1}} + \frac{1}{4\varepsilon} (C^e_{tr})^2 r_e (1+C_{\mathcal{A}})^2 \parallel \underline{\boldsymbol{\sigma}}_h \parallel^2_{K^e} \right).
\end{aligned}
$$

It is clear that $\mathbb{T}_2 = - \parallel \tilde{\boldsymbol{g}}_h \parallel^2_{\Gamma_h, l^{-1}}$. For $\mathbb{T}_3$, we use Cauchy-Schwarz inequality, Lemma 4.1 and Young's inequality

$$
\begin{aligned}
\mathbb{T}_3 &= \sum_{e\in\mathcal{E}_h^\partial} \langle l^{-1/2} \tilde{\boldsymbol{g}}_h, l^{1/2} \boldsymbol{\Lambda}^{\mathcal{A}(\underline{\boldsymbol{\sigma}}_h)} \rangle_e \leq \sum_{e\in\mathcal{E}_h^\partial} \parallel \tilde{\boldsymbol{g}}_h \parallel_{e, l^{-1}} \parallel \boldsymbol{\Lambda}^{\mathcal{A}(\underline{\boldsymbol{\sigma}}_h)} \parallel_{e,l} \leq \sum_{e\in\mathcal{E}_h^\partial} \parallel \tilde{\boldsymbol{g}}_h \parallel_{e, l^{-1}} \frac{1}{\sqrt{3}} r_e^{3/2} C_{\mathcal{A}} C^e_{ext} C^e_{inv} \parallel \underline{\boldsymbol{\sigma}}_h \parallel_{K^e} \\
&\leq \sum_{e\in\mathcal{E}_h^\partial} \left( \varepsilon \parallel \tilde{\boldsymbol{g}}_h \parallel^2_{e, l^{-1}} + \frac{1}{12\varepsilon} C^2_{\mathcal{A}} r^3_e (C^e_{ext})^2 (C^e_{inv})^2 \parallel \underline{\boldsymbol{\sigma}}_h \parallel^2_{K^e} \right).
\end{aligned}
$$

For $\mathbb{T}_4$, we use the same arguments as in the bound of $\mathbb{T}_3$ and obtain

$$
\mathbb{T}_4 \leq \sum_{e\in\mathcal{E}_h^\partial} \left( \varepsilon \parallel \tilde{\boldsymbol{g}}_h \parallel^2_{e, l^{-1}} + \frac{1}{12\varepsilon} r^3_e (C^e_{ext})^2 (C^e_{inv})^2 \parallel \underline{\boldsymbol{\rho}}_h \parallel^2_{K^e} \right).
$$

Analogously to the bound of $\mathbb{T}_1$ we get

$$
\mathbb{T}^e_5 \leq \sum_{e\in\mathcal{E}_h^\partial} \left( \varepsilon \parallel \tilde{\boldsymbol{g}}_h \parallel^2_{e, l^{-1}} + \frac{1}{4\varepsilon} (C^e_{tr})^2 r_e \parallel \underline{\boldsymbol{\rho}}_h \parallel^2_{K^e} \right).
$$

Finally, for $\mathbb{T}_6$ we use Cauchy-Schwarz inequality, the fact $l(\boldsymbol{x}) \leq H^\perp_e$ and the Young's inequality

$$
\begin{aligned}
\mathbb{T}_6 &= \sum_{e\in\mathcal{E}_h^\partial} \langle l^{-1/2} \tilde{\boldsymbol{g}}_h, l^{1/2} \alpha_e (\boldsymbol{u}_h - \hat{\boldsymbol{u}}_h) \rangle_e \leq \sum_{e\in\mathcal{E}_h^\partial} \parallel \tilde{\boldsymbol{g}}_h \parallel_{e, l^{-1}} (H^\perp_e)^{1/2} \alpha_e^{1/2} \parallel \boldsymbol{u}_h - \hat{\boldsymbol{u}}_h \parallel_{e, \alpha_e} \\
&\leq \sum_{e\in\mathcal{E}_h^\partial} \parallel \tilde{\boldsymbol{g}}_h \parallel_{e, l^{-1}} (r_e h^\perp_e \alpha_e)^{1/2} \parallel \boldsymbol{u}_h - \hat{\boldsymbol{u}}_h \parallel_{e, \alpha_e} \leq \sum_{e\in\mathcal{E}_h^\partial} \left( \varepsilon \parallel \tilde{\boldsymbol{g}}_h \parallel^2_{e, l^{-1}} + \frac{1}{4\varepsilon} r_e h^\perp_e \alpha_e \parallel \boldsymbol{u}_h - \hat{\boldsymbol{u}}_h \parallel^2_{e, \alpha_e} \right).
\end{aligned}
$$

We obtain (4.7) gathering all the above bounds and considering $\varepsilon = 1/10$. Moreover, considering (C.3)-(C.5), (4.7) implies (4.8). $\square$

**Lemma 4.8.** *Let* $\boldsymbol{f} = \mathbf{0}$ *and* $\boldsymbol{g} = \mathbf{0}$. *Then*

$$\| \tilde{\boldsymbol{g}}_h \|_{e,l^{-1}}^2 \leq 2C_{\mathcal{A}}^2 (C_{ext}^e)^2 r_e \| \underline{\boldsymbol{\sigma}}_h \|_{K^e}^2 + 2(C_{ext}^e)^2 r_e \| \underline{\boldsymbol{\rho}}_h \|_{K^e}^2,$$

*for all* $e$ *in* $\mathcal{E}_h$.

*Proof.* To prove this Lemma we use Equation (3.2f) and the fact $|(\boldsymbol{x} + s\boldsymbol{n}_e)\boldsymbol{n}_e| \leq |\bar{\boldsymbol{x}} - \boldsymbol{x}| = l(\boldsymbol{x})$, for all $e \in \mathcal{E}_h^\partial$, by (3.2g) and Cauchy-Schwarz inequality

$$\| \tilde{\boldsymbol{g}}_h \|_{e,l^{-1}}^2 = \left| \int_e \frac{1}{l(x)} \left[ \int_0^{l(x)} (\mathcal{A}\underline{\boldsymbol{\sigma}}_h + \underline{\boldsymbol{\rho}}_h)(\boldsymbol{x} + s\boldsymbol{n}_e)\boldsymbol{n}_e ds \right]^2 dx \right| \leq \| \mathcal{A}\underline{\boldsymbol{\sigma}}_h + \underline{\boldsymbol{\rho}}_h \|_{K_{ext}^e}^2 .$$

Then, using the Lemma 4.3, we obtain

$$\| \tilde{\boldsymbol{g}}_h \|_{e,l^{-1}}^2 \leq \| \mathcal{A}\underline{\boldsymbol{\sigma}}_h + \underline{\boldsymbol{\rho}}_h \|_{K_{ext}^e}^2 \leq 2C_{\mathcal{A}}^2 \| \underline{\boldsymbol{\sigma}}_h \|_{K_{ext}^e}^2 + 2 \| \underline{\boldsymbol{\rho}}_h \|_{K_{ext}^e}^2 \leq 2C_{\mathcal{A}}^2 (C_{ext}^e)^2 r_e \| \underline{\boldsymbol{\sigma}}_h \|_{K^e}^2 + 2(C_{ext}^e)^2 r_e \| \underline{\boldsymbol{\rho}}_h \|_{K^e}^2 .$$

$\square$

**Lemma 4.9.** *Let* $\boldsymbol{f} = \mathbf{0}$ *and* $\boldsymbol{g} = \mathbf{0}$. *Suppose that* $k \geq 1$ *and Assumption (C.1) holds, then there exits* $C_\eta > 0$ *independent of* $h$ *such that*

$$\| \underline{\boldsymbol{\rho}}_h \|_{D_h}^2 \leq 2(C_\eta)^2 C_{\mathcal{A}}^2 \| \underline{\boldsymbol{\sigma}}_h \|_{D_h}^2 + 2 \sum_{e \in \mathcal{E}_h^\partial} (C_\eta)^2 (C_{tr}^e)^2 r_e \| \tilde{\boldsymbol{g}}_h \|_{e,l^{-1}}^2 .$$

*Proof.* We follow the ideas in [5]. We consider the next orthogonal decomposition:

$$\underline{\boldsymbol{\rho}}_h = \underline{\boldsymbol{\rho}}_h^0 + \underline{\boldsymbol{\rho}}_h^c, \quad \underline{\boldsymbol{\rho}}_h^c|_{K^e} := \frac{1}{|K^e|} \int_{K^e} \underline{\boldsymbol{\rho}}_h, \quad \underline{\boldsymbol{\rho}}_h^0 = \underline{\boldsymbol{\rho}}_h - \underline{\boldsymbol{\rho}}_h^c,$$

where $\underline{\boldsymbol{\rho}}_h^0 \in \underline{\boldsymbol{A}}_h^0$ and $\underline{\boldsymbol{\rho}}_h^c \in \underline{\boldsymbol{A}}_h^c$ (we recall that $\underline{\boldsymbol{A}}_h^0$ and $\underline{\boldsymbol{A}}_h^c$ have been introduced in Lemmas 4.4 and 4.5). We proceed in two steps to bound the $\underline{\boldsymbol{\rho}}_h^0$ and $\underline{\boldsymbol{\rho}}_h^c$.
**Step 1** Let $\underline{\boldsymbol{\eta}} := \underline{\boldsymbol{\rho}}_h^0$ in Lemma 4.4, then there exist $\underline{\boldsymbol{v}} \in \underline{\mathcal{B}}_h \subset \underline{\boldsymbol{V}}_h$ satisfying (4.3a) and (4.3b). Then we rewrite the Equation (3.2a) as

$$(\mathcal{A}\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{v}})_{\mathcal{T}_h} + (\boldsymbol{u}_h, \nabla \cdot \underline{\boldsymbol{v}})_{\mathcal{T}_h} + (\underline{\boldsymbol{\rho}}_h^0, \underline{\boldsymbol{v}})_{\mathcal{T}_h} + (\underline{\boldsymbol{\rho}}_h^c, \underline{\boldsymbol{v}})_{\mathcal{T}_h} - \langle \hat{\boldsymbol{u}}_h, \underline{\boldsymbol{v}}\boldsymbol{n} \rangle_{\partial\mathcal{T}_h} = 0. \tag{4.9}$$

By property (B.1) and (B.2), we have $(\boldsymbol{u}_h, \nabla \cdot \underline{\boldsymbol{v}})_{\mathcal{T}_h} = 0$ and $\langle \hat{\boldsymbol{u}}_h, \underline{\boldsymbol{v}}\boldsymbol{n} \rangle_{\partial\mathcal{T}_h} = 0$. Now considering $\underline{\boldsymbol{\gamma}} := \underline{\boldsymbol{\rho}}_h^c$ in Lemma 4.4, we have that $(\underline{\boldsymbol{\rho}}_h^c, \underline{\boldsymbol{v}})_{\mathcal{T}_h} = (\underline{\boldsymbol{\rho}}_h^0, \underline{\boldsymbol{\rho}}_h^c)_{\mathcal{T}_h} = 0$, since the decomposition of $\underline{\boldsymbol{\rho}}_h$ is orthogonal in $\underline{\boldsymbol{L}}^2$. Moreover, by taking $\underline{\boldsymbol{\gamma}} = \underline{\boldsymbol{\eta}} = \underline{\boldsymbol{\rho}}_h^0$ in (4.3a) we have that $(\underline{\boldsymbol{\rho}}_h^0, \underline{\boldsymbol{v}})_{\mathcal{T}_h} = \| \underline{\boldsymbol{\rho}}_h^0 \|_{D_h}^2$.

Thus, replacing the above terms in Equation (4.9) and using the Equation (4.3b) we get

$$\| \underline{\boldsymbol{\rho}}_h^0 \|_{D_h}^2 = (\underline{\boldsymbol{v}}, \underline{\boldsymbol{\rho}}_h^0)_{\mathcal{T}_h} = -(\mathcal{A}\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{v}})_{\mathcal{T}_h} \leq C_{\mathcal{A}} \| \underline{\boldsymbol{\sigma}}_h \|_{D_h} \| \underline{\boldsymbol{v}} \|_{\mathcal{T}_h} \leq C_\eta^0 C_{\mathcal{A}} \| \underline{\boldsymbol{\sigma}}_h \|_{D_h} \| \underline{\boldsymbol{\rho}}_h^0 \|_{D_h} .$$

Then, we obtain

$$\| \underline{\boldsymbol{\rho}}_h^0 \|_{D_h} \leq C_\eta^0 C_{\mathcal{A}} \| \underline{\boldsymbol{\sigma}}_h \|_{D_h} .$$

**Step 2** Let $\boldsymbol{\eta} := \underline{\boldsymbol{\rho}}_h^c$ in Lemma 4.5, then there exist $\underline{\boldsymbol{v}} \in \underline{\mathcal{G}}_h$ satisfying (4.4a)-(4.4c). Then $(\boldsymbol{u}_h, \nabla \cdot \underline{\boldsymbol{v}})_{\mathcal{T}_h} = 0$ and $\langle \hat{\boldsymbol{u}}_h, \underline{\boldsymbol{v}}\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} = \langle \tilde{\boldsymbol{g}}_h, \underline{\boldsymbol{v}}\boldsymbol{n}\rangle_{\Gamma_h}$, thanks to Equation (3.2e) and the fact that $\underline{\boldsymbol{v}} \in \underline{\boldsymbol{H}}(div; D_h)$ (we recall that we are assuming $k \geq 1$). Thus, with the decomposition of $\underline{\boldsymbol{\rho}}_h$, Equation (3.2a) yields

$$(\mathcal{A}\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{v}})_{\mathcal{T}_h} + (\underline{\boldsymbol{\rho}}_h^0, \underline{\boldsymbol{v}})_{\mathcal{T}_h} + (\underline{\boldsymbol{\rho}}_h^c, \underline{\boldsymbol{v}})_{\mathcal{T}_h} - \langle \tilde{\boldsymbol{g}}_h, \underline{\boldsymbol{v}}\boldsymbol{n}\rangle_{\Gamma_h} = 0. \tag{4.10}$$

Moreover, taking $\underline{\boldsymbol{\gamma}} := \underline{\boldsymbol{\rho}}_h^c$ in (4.4b) we have $(\underline{\boldsymbol{\rho}}_h^c, \underline{\boldsymbol{v}})_{\mathcal{T}_h} = \| \underline{\boldsymbol{\rho}}_h^c \|_{D_h}$ and then from Equation (4.10) we obtain

$$\| \underline{\boldsymbol{\rho}}_h^c \|_{D_h}^2 = -(\mathcal{A}\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{v}})_{\mathcal{T}_h} - (\underline{\boldsymbol{\rho}}_h^0, \underline{\boldsymbol{v}})_{\mathcal{T}_h} - \langle \tilde{\boldsymbol{g}}_h, \underline{\boldsymbol{v}}\boldsymbol{n}\rangle_{\Gamma_h}.$$

Using Cauchy-Schwarz inequality, Lemma 4.2, the bound (4.2), Equation (4.4c) of Lemma 4.5, the fact that $l(\boldsymbol{x}) \leq H_e^\perp$ and Assumption (C.1) we get

$$\begin{aligned} \| \underline{\boldsymbol{\rho}}_h^c \|_{D_h}^2 &\leq C_{\mathcal{A}} \| \underline{\boldsymbol{\sigma}}_h \|_{D_h} \| \underline{\boldsymbol{v}} \|_{D_h} + \| \underline{\boldsymbol{\rho}}_h^0 \|_{D_h} \| \underline{\boldsymbol{v}} \|_{D_h} + \sum_{e \in \mathcal{E}_h^\partial} |l(x)|^{1/2} \| \tilde{\boldsymbol{g}}_h \|_{e,l^{-1}} \| \underline{\boldsymbol{v}}\boldsymbol{n} \|_e \\ &\leq \Big\{ C_{\mathcal{A}} \| \underline{\boldsymbol{\sigma}}_h \|_{D_h} + C_\eta^0 C_{\mathcal{A}} \| \underline{\boldsymbol{\sigma}}_h \|_{D_h} + \sum_{e \in \mathcal{E}_h^\partial} C_{tr}^e h_e^{-1/2} (H_e^\perp)^{1/2} \| \tilde{\boldsymbol{g}}_h \|_{e,l^{-1}} \Big\} C_\eta^c \| \underline{\boldsymbol{\rho}}_h^c \|_{D_h} . \end{aligned}$$

Then,

$$\| \underline{\boldsymbol{\rho}}_h^c \|_{D_h} \leq (C_{\mathcal{A}} C_\eta^c + C_{\mathcal{A}} C_\eta^0 C_\eta^c) \| \underline{\boldsymbol{\sigma}}_h \|_{D_h} + \sum_{e \in \mathcal{E}_h^\partial} C_\eta^c C_{tr}^e r_e^{1/2} \| \tilde{\boldsymbol{g}}_h \|_{e,l^{-1}} .$$

Finally, using Steps 1 and 2, we get

$$\begin{aligned} \| \underline{\boldsymbol{\rho}}_h \|_{D_h} &\leq \| \underline{\boldsymbol{\rho}}_h^0 \|_{D_h} + \| \underline{\boldsymbol{\rho}}_h^c \|_{D_h} \\ &\leq C_{\mathcal{A}} C_\eta^0 \| \underline{\boldsymbol{\sigma}}_h \|_{D_h} + (C_{\mathcal{A}} C_\eta^c + C_{\mathcal{A}} C_\eta^0 C_\eta^c) \| \underline{\boldsymbol{\sigma}}_h \|_{D_h} + \sum_{e \in \mathcal{E}_h^\partial} C_\eta^c C_{tr}^e r_e^{1/2} \| \tilde{\boldsymbol{g}}_h \|_{e,l^{-1}} \\ &\leq (C_\eta^0 + C_\eta^0 C_\eta^c + C_\eta^c) C_{\mathcal{A}} \| \underline{\boldsymbol{\sigma}}_h \|_{D_h} + \sum_{e \in \mathcal{E}_h^\partial} C_\eta C_{tr}^e r_e^{1/2} \| \tilde{\boldsymbol{g}}_h \|_{e,l^{-1}} . \end{aligned}$$

We finish the proof setting $C_\eta = (C_\eta^0 + C_\eta^0 C_\eta^c + C_\eta^c)$ and considering the fact that $C_\eta^c \leq C_\eta$. ☐

**Corollary 4.2.** *Let $\boldsymbol{f} = \boldsymbol{0}$ and $\boldsymbol{g} = \boldsymbol{0}$. If $k \geq 1$ and Assumption (C.2) holds, then*

$$\| \underline{\boldsymbol{\rho}}_h \|_{D_h}^2 \leq 3 (C_\eta)^2 (C_{\mathcal{A}})^2 \| \underline{\boldsymbol{\sigma}}_h \|_{D_h}^2$$

*Proof.* We replace the estimate given by Lemma 4.8 in the terms of the right hand side in Lemma 4.9, obtain

$$\| \underline{\boldsymbol{\rho}}_h \|_{D_h}^2 \leq \Big\{ 2(C_\eta)^2 C_{\mathcal{A}}^2 + 4 (C_\eta)^2 C_{\mathcal{A}}^2 \max_{e \in \mathcal{E}_h^\partial} (C_{tr}^e)^2 (C_{ext}^e)^2 r_e^2 \Big\} \| \underline{\boldsymbol{\sigma}}_h \|_{D_h}^2 + 4 \max_{e \in \Gamma_h} (C_\eta)^2 (C_{tr}^e)^2 (C_{ext}^e)^2 r_e^2 \| \underline{\boldsymbol{\rho}}_h \|_{D_h}^2 .$$

The result follows from rearranging terms in last expression and considering Assumption (C.2). ☐

**Theorem 4.10.** *If Assumptions C are satisfied and $k \geq 1$, then the scheme* (3.2) *has a unique solution.*

*Proof.* We replace (4.8) in the right hand side of the estimate in Lemma 4.1 and arrange terms to obtain

$$\frac{29}{30} C_{el} \| \underline{\boldsymbol{\sigma}}_h \|_{D_h}^2 + \frac{1}{2} \| \boldsymbol{u}_h - \hat{\boldsymbol{u}}_h \|_{\partial\mathcal{T}_h,\alpha}^2 + \frac{1}{2} \| \tilde{\boldsymbol{g}}_h \|_{\Gamma_h,l^{-1}}^2 \leq \frac{C_{el}}{30} \frac{1}{C_{\mathcal{A}} (C_\eta)^2} \| \underline{\boldsymbol{\rho}}_h \|_{D_h}^2 .$$

Then, using the inequality in Corollary 4.2, we obtain

$$\frac{26}{30}C_{el} \parallel \underline{\boldsymbol{\sigma}}_h \parallel^2_{D_h} + \frac{1}{2} \parallel \boldsymbol{u}_h - \hat{\boldsymbol{u}}_h \parallel^2_{\partial\mathcal{T}_h,\alpha} + \frac{1}{2} \parallel \tilde{\boldsymbol{g}}_h \parallel^2_{\Gamma_h,l^{-1}} \leq 0.$$

Thus, we have $\underline{\boldsymbol{\sigma}}_h = \underline{\mathbf{0}}$ in $D_h$, $\tilde{\boldsymbol{g}}_h = \mathbf{0}$ in $\Gamma_h$ and $\hat{\boldsymbol{u}}_h = \boldsymbol{u}_h$ in $\Gamma_h$. In addition, by Lemma 4.9 we conclude that $\underline{\boldsymbol{\rho}}_h = \underline{\mathbf{0}}$. Finally, from (3.2a) we now have $(\boldsymbol{u}_h, \nabla \cdot \underline{\boldsymbol{v}})_{\mathcal{T}_h} - \langle \boldsymbol{u}_h, \underline{\boldsymbol{v}}\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} = 0$ for all $\underline{\boldsymbol{v}} \in \underline{\boldsymbol{V}}_h$, which implies, after integration by parts $\nabla \boldsymbol{u}_h$ is constant. However by (3.2e) $\boldsymbol{u}_h = \mathbf{0}$ in $\Gamma_h$ and then $\boldsymbol{u}_h = \mathbf{0}$ in $D_h$. □

## 4.3 Error analysis

In this section we provide detailed proofs for our a $priori$ error estimates. We employ the projection-based analysis of HDG method introduced for the diffusion problem [2], combined with the analysis in [4, 5]. Through this section we will use the following result

**Lemma 4.11.** *(Lemma 5.2 in [5])For each $e \in \mathcal{E}_h^\partial$, we have that*

$$\parallel \mathbf{\Lambda}^{\delta_{\underline{\boldsymbol{v}}}} \parallel_{e,l} \leq \frac{1}{\sqrt{3}} \parallel \partial_{\boldsymbol{n}}(\delta_{\underline{\boldsymbol{v}}}\boldsymbol{n}) \parallel_{K^e_{ext},(h^\perp)^2},$$

*for all $\underline{\boldsymbol{v}}$ smooth enough tensor function.*

Similarly to Assumptions C in Section 4.2, we have to suppose the following conditions in order to obtain optimal error estimates:

Assumptions S

For every face $e \in \mathcal{E}_h$, we assume

(S.1) $\max\{1, C_\eta\}C^e_{tr}C^e_{ext}r_e \leq \sqrt{2}/8$,

(S.2) $(1 + C_{\mathcal{A}})C_{el}^{-1/2}C^e_{tr}r_e^{1/2} \leq \sqrt{5}/30$,

(S.3) $\max\{1, C_{\mathcal{A}}\}C_{el}^{-1/2}r_e^{3/2}C^e_{ext}C^e_{inv} \leq \sqrt{15}/30$,

(S.4) $r_e\alpha_e h_e^\perp \leq 1/9$.

### 4.3.1 Projection operators

On each element $K$, for $(\underline{\boldsymbol{\sigma}}, \boldsymbol{u}) \in \boldsymbol{H}^1(K) \times \underline{\boldsymbol{H}}^1(K)$, we consider the projection $(\underline{\mathbf{\Pi}}^{\boldsymbol{D}}\underline{\boldsymbol{\sigma}}, \mathbf{\Pi}_{\boldsymbol{W}}\boldsymbol{u}) \in \underline{\mathbf{P}}_k(K) \times \mathbf{P}_k(K)$ such that

$$(\underline{\mathbf{\Pi}}^{\boldsymbol{D}}\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{v}})_K = (\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{v}})_K \qquad \text{for all } \underline{\boldsymbol{v}} \in \underline{\mathbf{P}}_{k-1}(K), \tag{4.11a}$$

$$(\mathbf{\Pi}_{\boldsymbol{W}}\boldsymbol{u}, \boldsymbol{w})_K = (\boldsymbol{u}, \boldsymbol{w})_K \qquad \text{for all } \boldsymbol{w} \in \mathbf{P}_{k-1}(K), \tag{4.11b}$$

$$\langle \underline{\mathbf{\Pi}}^{\boldsymbol{D}}\underline{\boldsymbol{\sigma}}\boldsymbol{n} - \alpha_e(\mathbf{\Pi}_{\boldsymbol{W}}\boldsymbol{u}), \boldsymbol{\mu}\rangle_e = \langle \underline{\boldsymbol{\sigma}}\boldsymbol{n} - \alpha_e(\boldsymbol{\mathcal{P}}_{\boldsymbol{M}}\boldsymbol{u}), \boldsymbol{\mu}\rangle_e \qquad \text{for all } \boldsymbol{\mu} \in \boldsymbol{M}(e), \tag{4.11c}$$

for all faces $e$ of the element $K$. Here $\boldsymbol{\mathcal{P}}_{\boldsymbol{M}}$ denotes the $L^2$ projection onto $\boldsymbol{M}(e)$.

**Lemma 4.12.** *On each element $K$, the projection $(\underline{\mathbf{\Pi}}^{\boldsymbol{D}}\underline{\boldsymbol{\sigma}}, \mathbf{\Pi}_{\boldsymbol{W}}\boldsymbol{u})$ of $(\underline{\boldsymbol{\sigma}}, \boldsymbol{u}) \in \boldsymbol{H}^1(K) \times \underline{\boldsymbol{H}}^1(K)$ is well-defined. Moreover, if $(\underline{\boldsymbol{\sigma}}, \boldsymbol{u}) \in \boldsymbol{H}^{k+1}(K) \times \underline{\boldsymbol{H}}^{k+1}(K)$. Then, there exist $C > 0$, independent of $h$ such that,*

$$\parallel \underline{\mathbf{\Pi}}^{\boldsymbol{D}}\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}} \parallel_K \leq Ch_K^{k+1}(|\boldsymbol{u}|_{\boldsymbol{H}^{k+1}(K)} + |\underline{\boldsymbol{\sigma}}|_{\underline{\boldsymbol{H}}^{k+1}(K)}), \tag{4.12a}$$

$$\parallel \mathbf{\Pi}_{\boldsymbol{W}}\boldsymbol{u} - \boldsymbol{u} \parallel_K \leq Ch_K^{k+1}(|\boldsymbol{u}|_{\boldsymbol{H}^{k+1}(K)} + |\nabla \cdot \underline{\boldsymbol{\sigma}}|_{\underline{\boldsymbol{H}}^k(K)}). \tag{4.12b}$$

*Proof.* It follows from Theorem 2.1 in [2]. □

On the other hand, on each element $K$, we denote by $\underline{\boldsymbol{\Pi_A\rho}}$ the $\underline{\boldsymbol{L}}^2(K)$-projection of $\underline{\boldsymbol{\rho}} \in \underline{\boldsymbol{L}}^2(K)$ into $\underline{\boldsymbol{A}}(K)$. If $\underline{\boldsymbol{\rho}} \in \underline{\boldsymbol{H}}^{k+1}(K)$, we have that (Lemma 1.58 in [10])

$$\| \underline{\boldsymbol{\Pi_A\rho}} - \underline{\boldsymbol{\rho}} \|_K \le C h_K^{k+1} |\underline{\boldsymbol{\rho}}|_{\boldsymbol{H}^{k+1}(K)}. \tag{4.13}$$

We define the projection of the errors

$$\underline{\boldsymbol{e_\sigma}} := \underline{\boldsymbol{\Pi}}^{\boldsymbol{D}}\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}_h,\ \ \boldsymbol{e_u} := \boldsymbol{\Pi_W u} - \boldsymbol{u}_h,\ \ \underline{\boldsymbol{e_\rho}} := \underline{\boldsymbol{\Pi_A\rho}} - \underline{\boldsymbol{\rho}}_h,\ \ \boldsymbol{e_{\hat{u}}} := \boldsymbol{\mathcal{P}_M u} - \boldsymbol{u}_h,\ \ \underline{\boldsymbol{e_{\hat{\sigma}}}}\boldsymbol{n} := \boldsymbol{\mathcal{P}_M}(\underline{\boldsymbol{\sigma}}\boldsymbol{n}) - \underline{\hat{\boldsymbol{\sigma}}}_h\boldsymbol{n}$$

and the interpolation error,

$$\underline{\boldsymbol{\delta_\sigma}} := \underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\Pi}}^{\boldsymbol{D}}\underline{\boldsymbol{\sigma}},\quad \boldsymbol{\delta_u} := \boldsymbol{u} - \boldsymbol{\Pi_W u},\quad \underline{\boldsymbol{\delta_\rho}} := \underline{\boldsymbol{\rho}} - \underline{\boldsymbol{\Pi_A\rho}}.$$

### 4.3.2 Energy argument

We first present the equations of the projection of the error.

**Lemma 4.13.** *The projection of the errors satisfy*

$$(\mathcal{A}\underline{\boldsymbol{e_\sigma}}, \underline{\boldsymbol{v}})_{\mathcal{T}_h} + (\boldsymbol{e_u}, \nabla\cdot\underline{\boldsymbol{v}})_{\mathcal{T}_h} + (\underline{\boldsymbol{e_\rho}}, \underline{\boldsymbol{v}})_{\mathcal{T}_h} - \langle \boldsymbol{e_{\hat{u}}}, \underline{\boldsymbol{v}}\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} = -(\mathcal{A}\underline{\boldsymbol{\delta_\sigma}}, \underline{\boldsymbol{v}})_{\mathcal{T}_h} - (\underline{\boldsymbol{\delta_\rho}}, \underline{\boldsymbol{v}})_{\partial\mathcal{T}_h}, \tag{4.14a}$$

$$(\underline{\boldsymbol{e_\sigma}}, \nabla\boldsymbol{w})_{\mathcal{T}_h} - \langle \underline{\boldsymbol{e_{\hat{\sigma}}}}\boldsymbol{n}, \boldsymbol{w}\rangle_{\partial\mathcal{T}_h} = 0, \tag{4.14b}$$

$$(\underline{\boldsymbol{e_\sigma}}, \underline{\boldsymbol{\eta}})_{\mathcal{T}_h} = -(\underline{\boldsymbol{\delta_\sigma}}, \underline{\boldsymbol{\eta}})_{\mathcal{T}_h}, \tag{4.14c}$$

$$\langle \underline{\boldsymbol{e_{\hat{\sigma}}}}\boldsymbol{n}, \boldsymbol{\mu}\rangle_{\partial\mathcal{T}_h/\Gamma_h} = 0, \tag{4.14d}$$

$$\langle \boldsymbol{e_{\hat{u}}}, \boldsymbol{\mu}\rangle_{\Gamma_h} = \langle \tilde{\boldsymbol{g}} - \tilde{\boldsymbol{g}}_h, \boldsymbol{\mu}\rangle_{\Gamma_h}. \tag{4.14e}$$

*for all* $(\underline{\boldsymbol{v}}, \boldsymbol{w}, \underline{\boldsymbol{\eta}}, \boldsymbol{\mu}) \in \underline{\boldsymbol{V}}_h \times \boldsymbol{W}_h \times \underline{\boldsymbol{A}}_h \times \boldsymbol{M}_h$.

*Proof.* Let $(\underline{\boldsymbol{v}}, \boldsymbol{w}, \underline{\boldsymbol{\eta}}, \boldsymbol{\mu}) \in \underline{\boldsymbol{V}}_h \times \boldsymbol{W}_h \times \underline{\boldsymbol{A}}_h \times \boldsymbol{M}_h$. We note that the exact solution $(\underline{\boldsymbol{\sigma}}, \boldsymbol{u}, \underline{\boldsymbol{\rho}}, \boldsymbol{u}|_{\mathcal{E}_h})$ also satisfies (3.2). Then, if we do a simple algebraic manipulations, we obtain that

$$\begin{aligned}
(\mathcal{A}\underline{\boldsymbol{\Pi}}^{\boldsymbol{D}}\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{v}})_{\mathcal{T}_h} + (\boldsymbol{\Pi_W u}, \nabla\cdot\underline{\boldsymbol{v}})_{\mathcal{T}_h} + (\underline{\boldsymbol{\Pi_A\rho}}, \underline{\boldsymbol{v}})_{\mathcal{T}_h} - \langle \boldsymbol{\mathcal{P}_M u}, \underline{\boldsymbol{v}}\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} &= -(\mathcal{A}\underline{\boldsymbol{\delta_\sigma}}, \underline{\boldsymbol{v}})_{\mathcal{T}_h} - (\boldsymbol{\delta_u}, \nabla\cdot\underline{\boldsymbol{v}})_{\mathcal{T}_h} \\
&\quad - (\underline{\boldsymbol{\delta_\rho}}, \underline{\boldsymbol{v}})_{\mathcal{T}_h} + \langle \boldsymbol{u} - \boldsymbol{\mathcal{P}_M u}, \underline{\boldsymbol{v}}\boldsymbol{n}\rangle_{\partial\mathcal{T}_h}, \\
(\underline{\boldsymbol{\Pi}}^{\boldsymbol{D}}\underline{\boldsymbol{\sigma}}, \nabla\boldsymbol{w})_{\mathcal{T}_h} - \langle \boldsymbol{\mathcal{P}_M}(\underline{\boldsymbol{\sigma}}\boldsymbol{n}), \boldsymbol{w}\rangle_{\partial\mathcal{T}_h} &= -(\boldsymbol{f}, \boldsymbol{w})_{\mathcal{T}_h} - (\underline{\boldsymbol{\delta_\sigma}}, \nabla\boldsymbol{w})_{\mathcal{T}_h} \\
&\quad + \langle \underline{\boldsymbol{\sigma}}\boldsymbol{n} - \boldsymbol{\mathcal{P}_M}(\underline{\boldsymbol{\sigma}}\boldsymbol{n}), \boldsymbol{w}\rangle_{\partial\mathcal{T}_h}, \\
(\underline{\boldsymbol{\Pi}}^{\boldsymbol{D}}\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\eta}})_{\mathcal{T}_h} &= -(\underline{\boldsymbol{\delta_\sigma}}, \underline{\boldsymbol{\eta}})_{\mathcal{T}_h}.
\end{aligned}$$

In addition, by the definition of the projection $\boldsymbol{\mathcal{P}_M}$, we have that

$$\begin{aligned}
\langle \boldsymbol{\mathcal{P}_M}(\underline{\boldsymbol{\sigma}}\boldsymbol{n}, \boldsymbol{\mu}\rangle_{\partial\mathcal{T}_h/\Gamma_h} &= 0, \\
\langle \boldsymbol{\mathcal{P}_M}(\boldsymbol{u}), \boldsymbol{\mu}\rangle_{\Gamma_h} &= \langle \tilde{\boldsymbol{g}}, \boldsymbol{\mu}\rangle_{\Gamma_h}.
\end{aligned}$$

Let $K \in \mathcal{T}_h$. By (4.12a) and (4.12b), we have $(\underline{\boldsymbol{\delta_\sigma}}, \nabla\boldsymbol{w})_{\mathcal{T}_h} = 0$ and $(\boldsymbol{\delta_u}, \nabla\cdot\underline{\boldsymbol{v}})_{\mathcal{T}_h} = 0$ for any $(\boldsymbol{w}, \underline{\boldsymbol{v}}) \in \underline{\mathbf{P}}_k(K) \times \underline{\mathbf{P}}_k(K)$. Here we have used by (B.1) the fact that

$$\nabla\cdot\underline{\boldsymbol{V}}(K) = \nabla\cdot(\underline{\boldsymbol{V}}(K) + \underline{\boldsymbol{B}}(K)) = \nabla\cdot(\underline{\boldsymbol{V}}(K)).$$

Now, by (B.2) on each face $e \in \partial K$, we notice that

$$\underline{\mathbf{P}}_k(K)|_e \subset \underline{\mathbf{P}}_k(e), \qquad \underline{\boldsymbol{V}}(K)\boldsymbol{n}|_e = \underline{\mathbf{P}}_k(K)\boldsymbol{n}|_e + \underline{\boldsymbol{B}}(K)|\boldsymbol{n}|_e = \underline{\mathbf{P}}_k(K)\boldsymbol{n}|_e \subset \boldsymbol{M}(e).$$

This implies that

$$\langle \underline{\boldsymbol{\sigma}}\boldsymbol{n} - \boldsymbol{\mathcal{P}}_{\boldsymbol{M}}(\underline{\boldsymbol{\sigma}}\boldsymbol{n}), \boldsymbol{w}\rangle_{\partial\mathcal{T}_h} = 0, \quad \langle \boldsymbol{u} - \boldsymbol{\mathcal{P}}_{\boldsymbol{M}}(\boldsymbol{u}), \underline{\boldsymbol{v}}\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} = 0$$

Using all the above identities, we have :

$$\begin{aligned}
(\mathcal{A}\underline{\boldsymbol{\Pi}}^{\boldsymbol{D}}\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{v}})_{\mathcal{T}_h} + (\boldsymbol{\Pi}_{\boldsymbol{W}}\boldsymbol{u}, \nabla\cdot\underline{\boldsymbol{v}})_{\mathcal{T}_h} + (\underline{\boldsymbol{\Pi}}_{\boldsymbol{A}}\underline{\boldsymbol{\rho}}, \underline{\boldsymbol{v}})_{\mathcal{T}_h} - \langle \boldsymbol{\mathcal{P}}_{\boldsymbol{M}}\boldsymbol{u}, \underline{\boldsymbol{v}}\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} &= -(\mathcal{A}\underline{\boldsymbol{\delta}}_{\boldsymbol{\sigma}}, \underline{\boldsymbol{v}})_{\mathcal{T}_h} - (\underline{\boldsymbol{\delta}}_{\boldsymbol{\rho}}, \underline{\boldsymbol{v}})_{\mathcal{T}_h},\\
(\underline{\boldsymbol{\Pi}}^{\boldsymbol{D}}\underline{\boldsymbol{\sigma}}, \nabla\boldsymbol{w})_{\mathcal{T}_h} - \langle \boldsymbol{\mathcal{P}}_{\boldsymbol{M}}(\underline{\boldsymbol{\sigma}}\boldsymbol{n}, \boldsymbol{w}\rangle_{\partial\mathcal{T}_h} &= -(\boldsymbol{f}, \boldsymbol{w})_{\mathcal{T}_h},\\
(\underline{\boldsymbol{\Pi}}^{\boldsymbol{D}}\underline{\boldsymbol{\sigma}}\boldsymbol{n}, \underline{\boldsymbol{\eta}})_{\mathcal{T}_h} &= -(\underline{\boldsymbol{\delta}}_{\boldsymbol{\sigma}}, \underline{\boldsymbol{\eta}})_{\mathcal{T}_h},\\
\langle \boldsymbol{\mathcal{P}}_{\boldsymbol{M}}(\underline{\boldsymbol{\sigma}}\boldsymbol{n}), \boldsymbol{\mu}\rangle_{\partial\mathcal{T}_h/\Gamma_h} &= 0,\\
\langle \boldsymbol{\mathcal{P}}_{\boldsymbol{M}}\boldsymbol{u}, \boldsymbol{\mu}\rangle_{\Gamma_h} &= \langle \tilde{\boldsymbol{g}}, \boldsymbol{\mu}\rangle_{\Gamma_h}.
\end{aligned}$$

for all $(\underline{\boldsymbol{v}}, \boldsymbol{w}, \underline{\boldsymbol{\eta}}, \boldsymbol{\mu}) \in \underline{\boldsymbol{V}}_h \times \boldsymbol{W}_h \times \underline{\boldsymbol{A}}_h \times \boldsymbol{M}_h$. Finally we subtracts (3.2) of above system and we obtain the result. □

**Lemma 4.14.** *We have*

$$(\mathcal{A}\underline{\boldsymbol{e}}_{\boldsymbol{\sigma}}, \underline{\boldsymbol{e}}_{\boldsymbol{\sigma}})_{\mathcal{T}_h} + \langle \alpha(\boldsymbol{e}_{\boldsymbol{u}} - \boldsymbol{e}_{\hat{\boldsymbol{u}}}), (\boldsymbol{e}_{\boldsymbol{u}} - \boldsymbol{e}_{\hat{\boldsymbol{u}}})\rangle_{\partial\mathcal{T}_h} = (\underline{\boldsymbol{\delta}}_{\boldsymbol{\sigma}}, \underline{\boldsymbol{e}}_{\boldsymbol{\rho}})_{\mathcal{T}_h} - (\mathcal{A}\underline{\boldsymbol{\delta}}_{\boldsymbol{\sigma}}, \underline{\boldsymbol{e}}_{\boldsymbol{\sigma}})_{\mathcal{T}_h} - (\underline{\boldsymbol{\delta}}_{\boldsymbol{\rho}}, \underline{\boldsymbol{e}}_{\boldsymbol{\sigma}})_{\mathcal{T}_h} + \mathbb{T},$$

*where* $\mathbb{T} = \langle \tilde{\boldsymbol{g}} - \tilde{\boldsymbol{g}}_h, \underline{\boldsymbol{e}}_{\hat{\boldsymbol{\sigma}}}\boldsymbol{n}\rangle_{\Gamma_h}$ .

*Proof.* We take $\underline{\boldsymbol{v}} = \underline{\boldsymbol{e}}_{\boldsymbol{\sigma}}$ and $\boldsymbol{w} = \boldsymbol{e}_{\boldsymbol{u}}$, in Equations (4.14a), (4.14b), respectively. Then, summing both equations and using $(\boldsymbol{e}_{\boldsymbol{u}}, \nabla\cdot\underline{\boldsymbol{e}}_{\boldsymbol{\sigma}})_{\mathcal{T}_h} = -(\underline{\boldsymbol{e}}_{\boldsymbol{\sigma}}, \nabla\boldsymbol{e}_{\boldsymbol{u}})_{\mathcal{T}_h} + \langle \boldsymbol{e}_{\boldsymbol{u}}, \underline{\boldsymbol{e}}_{\boldsymbol{\sigma}}\boldsymbol{n}\rangle_{\partial\mathcal{T}_h}$, we obtain

$$(\mathcal{A}\underline{\boldsymbol{e}}_{\boldsymbol{\sigma}}, \underline{\boldsymbol{e}}_{\boldsymbol{\sigma}})_{\mathcal{T}_h} + (\underline{\boldsymbol{e}}_{\boldsymbol{\rho}}, \underline{\boldsymbol{e}}_{\boldsymbol{\sigma}})_{\mathcal{T}_h} - \langle \boldsymbol{e}_{\hat{\boldsymbol{u}}}, \underline{\boldsymbol{e}}_{\boldsymbol{\sigma}}\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} + \langle \underline{\boldsymbol{e}}_{\boldsymbol{\sigma}}\boldsymbol{n}, \boldsymbol{e}_{\boldsymbol{u}}\rangle_{\partial\mathcal{T}_h} - \langle \underline{\boldsymbol{e}}_{\hat{\boldsymbol{\sigma}}}\boldsymbol{n}, \boldsymbol{e}_{\boldsymbol{u}}\rangle_{\partial\mathcal{T}_h} = -(\mathcal{A}\underline{\boldsymbol{\delta}}_{\boldsymbol{\sigma}}, \underline{\boldsymbol{e}}_{\boldsymbol{\sigma}})_{\mathcal{T}_h} - (\underline{\boldsymbol{\delta}}_{\boldsymbol{\rho}}, \underline{\boldsymbol{e}}_{\boldsymbol{\sigma}})_{\mathcal{T}_h}. \quad (4.15)$$

Next, note that by taking $\underline{\boldsymbol{\eta}} = \underline{\boldsymbol{e}}_{\boldsymbol{\rho}}$ in (4.14c), we have $(\underline{\boldsymbol{e}}_{\boldsymbol{\sigma}}, \underline{\boldsymbol{e}}_{\boldsymbol{\rho}})_{\mathcal{T}_h} = -(\underline{\boldsymbol{\delta}}_{\boldsymbol{\sigma}}, \underline{\boldsymbol{e}}_{\boldsymbol{\rho}})_{\mathcal{T}_h}$. Also, taking $\boldsymbol{\mu} = \boldsymbol{e}_{\hat{\boldsymbol{u}}}$ and $\boldsymbol{\mu} = \underline{\boldsymbol{e}}_{\hat{\boldsymbol{\sigma}}}\boldsymbol{n}$ in (4.14d) and (4.14e), respectively, we have

$$\langle \boldsymbol{e}_{\hat{\boldsymbol{u}}}, \underline{\boldsymbol{e}}_{\hat{\boldsymbol{\sigma}}}\boldsymbol{n}\rangle_{\partial\mathcal{T}_h/\Gamma_h} = 0, \qquad \langle \boldsymbol{e}_{\hat{\boldsymbol{u}}}, \underline{\boldsymbol{e}}_{\hat{\boldsymbol{\sigma}}}\boldsymbol{n}\rangle_{\Gamma_h} = \langle \tilde{\boldsymbol{g}} - \tilde{\boldsymbol{g}}_h, \underline{\boldsymbol{e}}_{\hat{\boldsymbol{\sigma}}}\boldsymbol{n}\rangle_{\Gamma_h}.$$

Then,

$$\begin{aligned}
-\langle \boldsymbol{e}_{\hat{\boldsymbol{u}}}, \underline{\boldsymbol{e}}_{\boldsymbol{\sigma}}\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} &= -\langle \boldsymbol{e}_{\hat{\boldsymbol{u}}}, \underline{\boldsymbol{e}}_{\boldsymbol{\sigma}}\boldsymbol{n}\rangle_{\partial\mathcal{T}_h/\Gamma_h} - \langle \boldsymbol{e}_{\hat{\boldsymbol{u}}}, \underline{\boldsymbol{e}}_{\boldsymbol{\sigma}}\boldsymbol{n}\rangle_{\Gamma_h}\\
&= -\langle \boldsymbol{e}_{\hat{\boldsymbol{u}}}, (\underline{\boldsymbol{e}}_{\boldsymbol{\sigma}} - \underline{\boldsymbol{e}}_{\hat{\boldsymbol{\sigma}}})\boldsymbol{n}\rangle_{\partial\mathcal{T}_h/\Gamma_h} - \langle \boldsymbol{e}_{\hat{\boldsymbol{u}}}, \underline{\boldsymbol{e}}_{\boldsymbol{\sigma}}\boldsymbol{n} + \underline{\boldsymbol{e}}_{\hat{\boldsymbol{\sigma}}}\boldsymbol{n} - \underline{\boldsymbol{e}}_{\hat{\boldsymbol{\sigma}}}\boldsymbol{n}\rangle_{\Gamma_h}\\
&= -\langle \boldsymbol{e}_{\hat{\boldsymbol{u}}}, (\underline{\boldsymbol{e}}_{\boldsymbol{\sigma}} - \underline{\boldsymbol{e}}_{\hat{\boldsymbol{\sigma}}})\boldsymbol{n}\rangle_{\partial\mathcal{T}_h/\Gamma_h} - \langle \boldsymbol{e}_{\hat{\boldsymbol{u}}}, (\underline{\boldsymbol{e}}_{\boldsymbol{\sigma}} - \underline{\boldsymbol{e}}_{\hat{\boldsymbol{\sigma}}})\boldsymbol{n}\rangle_{\Gamma_h} - \langle \boldsymbol{e}_{\hat{\boldsymbol{u}}}, \underline{\boldsymbol{e}}_{\hat{\boldsymbol{\sigma}}}\boldsymbol{n}\rangle_{\Gamma_h}\\
&= -\langle \boldsymbol{e}_{\hat{\boldsymbol{u}}}, (\underline{\boldsymbol{e}}_{\boldsymbol{\sigma}} - \underline{\boldsymbol{e}}_{\hat{\boldsymbol{\sigma}}})\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} - \langle \boldsymbol{e}_{\hat{\boldsymbol{u}}}, \underline{\boldsymbol{e}}_{\hat{\boldsymbol{\sigma}}}\boldsymbol{n}\rangle_{\Gamma_h}.
\end{aligned}$$

Thus, replacing the above identities in (4.15) we obtain

$$(\mathcal{A}\underline{\boldsymbol{e}}_{\boldsymbol{\sigma}}, \underline{\boldsymbol{e}}_{\boldsymbol{\sigma}})_{\mathcal{T}_h} + \langle (\boldsymbol{e}_{\boldsymbol{u}} - \boldsymbol{e}_{\hat{\boldsymbol{u}}}), (\underline{\boldsymbol{e}}_{\boldsymbol{\sigma}} - \underline{\boldsymbol{e}}_{\hat{\boldsymbol{\sigma}}})\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} = -(\mathcal{A}\underline{\boldsymbol{\delta}}_{\boldsymbol{\sigma}}, \underline{\boldsymbol{e}}_{\boldsymbol{\sigma}})_{\mathcal{T}_h} + (\underline{\boldsymbol{\delta}}_{\boldsymbol{\sigma}}, \underline{\boldsymbol{e}}_{\boldsymbol{\rho}})_{\mathcal{T}_h} - (\underline{\boldsymbol{\delta}}_{\boldsymbol{\rho}}, \underline{\boldsymbol{e}}_{\boldsymbol{\sigma}})_{\mathcal{T}_h} + \langle \tilde{\boldsymbol{g}} - \tilde{\boldsymbol{g}}_h, \underline{\boldsymbol{e}}_{\hat{\boldsymbol{\sigma}}}\boldsymbol{n}\rangle_{\Gamma_h}.$$

To end the proof , we need to show that

$$\langle \boldsymbol{e}_{\boldsymbol{u}} - \boldsymbol{e}_{\hat{\boldsymbol{u}}}, \underline{\boldsymbol{e}}_{\boldsymbol{\sigma}}\boldsymbol{n} - \underline{\boldsymbol{e}}_{\hat{\boldsymbol{\sigma}}}\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} = \langle \boldsymbol{e}_{\boldsymbol{u}} - \boldsymbol{e}_{\hat{\boldsymbol{u}}}, \alpha_e(\boldsymbol{e}_{\boldsymbol{u}} - \boldsymbol{e}_{\hat{\boldsymbol{u}}})\rangle_{\partial\mathcal{T}_h}.$$

In fact, on each $K$, by the definition of the numerical trace (3.2f) and properties of projections, we have

$$\begin{aligned}\langle \boldsymbol{e_u} - \boldsymbol{e_{\hat{u}}}, \underline{\boldsymbol{e_\sigma}}\boldsymbol{n} - \underline{\boldsymbol{e_{\hat{\sigma}}}}\boldsymbol{n}\rangle_{\partial K} &= \langle \boldsymbol{e_u} - \boldsymbol{e_{\hat{u}}}, \underline{\boldsymbol{\Pi}}^{\boldsymbol{D}}\underline{\boldsymbol{\sigma}}\boldsymbol{n} - \underline{\boldsymbol{\sigma}}_h\boldsymbol{n} - \boldsymbol{\mathcal{P}_M}(\underline{\boldsymbol{\sigma}}\boldsymbol{n}) + \underline{\hat{\boldsymbol{\sigma}}}_h\boldsymbol{n}\rangle_{\partial K} \\ &= \langle \boldsymbol{e_u} - \boldsymbol{e_{\hat{u}}}, \underline{\boldsymbol{\Pi}}^{\boldsymbol{D}}\underline{\boldsymbol{\sigma}}\boldsymbol{n} - \boldsymbol{\mathcal{P}_M}(\underline{\boldsymbol{\sigma}}\boldsymbol{n}) + \underline{\hat{\boldsymbol{\sigma}}}_h\boldsymbol{n} - \underline{\boldsymbol{\sigma}}_h\boldsymbol{n}\rangle_{\partial K} \\ &= \langle \boldsymbol{e_u} - \boldsymbol{e_{\hat{u}}}, \alpha_e(\boldsymbol{\Pi_W u} - \boldsymbol{\mathcal{P}_M u} + \hat{\boldsymbol{u}}_h - \boldsymbol{u}_h)\rangle_{\partial K} \\ &= \langle \boldsymbol{e_u} - \boldsymbol{e_{\hat{u}}}, \alpha_e(\boldsymbol{e_u} - \boldsymbol{e_{\hat{u}}})\rangle_{\partial K}. \end{aligned} \tag{4.16}$$

We complete the proof by taking the sum of the above equation over all $K \in \mathcal{T}_h$. □

Following the structure in Section 4.2, we need to rewrite the term $\mathbb{T}$ to facilitate the bound in the estimate of $\underline{\boldsymbol{e_\sigma}}$.

**Lemma 4.15.** *We have* $\mathbb{T} = \sum_{i=1}^{6} \mathbb{T}_i$ *, with*

$$\mathbb{T}_1 = -\langle \tilde{\boldsymbol{g}} - \tilde{\boldsymbol{g}}_h, \frac{1}{l}(\tilde{\boldsymbol{g}} - \tilde{\boldsymbol{g}}_h)\rangle_{\Gamma_h}, \qquad \mathbb{T}_2 = -\langle \tilde{\boldsymbol{g}} - \tilde{\boldsymbol{g}}_h, \mathbb{T}^\rho\rangle_{\Gamma_h},$$

$$\mathbb{T}_3 = -\langle \tilde{\boldsymbol{g}} - \tilde{\boldsymbol{g}}_h, \boldsymbol{\Lambda}^{\mathcal{A}(\underline{\boldsymbol{\delta_\sigma}})} + \mathcal{A}(\underline{\boldsymbol{\delta_\sigma}})\boldsymbol{n}\rangle_{\Gamma_h}, \qquad \mathbb{T}_4 = -\langle \tilde{\boldsymbol{g}} - \tilde{\boldsymbol{g}}_h, \boldsymbol{\Lambda}^{\mathcal{A}\underline{\boldsymbol{e_\sigma}}}\rangle_{\Gamma_h},$$

$$\mathbb{T}_5 = -\langle \tilde{\boldsymbol{g}} - \tilde{\boldsymbol{g}}_h, \alpha(\boldsymbol{e_u} - \boldsymbol{e_{\hat{u}}})\rangle_{\Gamma_h}, \qquad \mathbb{T}_6 = \langle \tilde{\boldsymbol{g}} - \tilde{\boldsymbol{g}}_h, \underline{\boldsymbol{e_\sigma}}\boldsymbol{n} - \mathcal{A}(\underline{\boldsymbol{e_\sigma}})\boldsymbol{n}\rangle_{\Gamma_h}.$$

*where* $\mathbb{T}^\rho := \boldsymbol{\Lambda}^{\underline{\boldsymbol{\delta_\rho}}}(\boldsymbol{x}) + \underline{\boldsymbol{\delta_\rho}}\boldsymbol{n}_e + \boldsymbol{\Lambda}^{\underline{\boldsymbol{e_\rho}}}(\boldsymbol{x}) + \underline{\boldsymbol{e_\rho}}\boldsymbol{n}_e$.

*Proof.* First of all, we rewrite $\tilde{\boldsymbol{g}} - \tilde{\boldsymbol{g}}_h$ as follows

$$\tilde{\boldsymbol{g}}(\boldsymbol{x}) - \tilde{\boldsymbol{g}}_h(\boldsymbol{x}) = -\int_0^{l(\boldsymbol{x})} \mathcal{A}(\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}_h)(\boldsymbol{x} + s\boldsymbol{n}_e)\boldsymbol{n}_e ds - \int_0^{l(\boldsymbol{x})} (\underline{\boldsymbol{\rho}} - \underline{\boldsymbol{\rho}}_h)(\boldsymbol{x} + s\boldsymbol{n}_e)\boldsymbol{n}_e ds.$$

We define $T^\sigma := \int_0^{l(\boldsymbol{x})} \mathcal{A}(\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}_h)(\boldsymbol{x} + s\boldsymbol{n}_e)\boldsymbol{n}_e ds$ and $T^\rho := \int_0^{l(\boldsymbol{x})} (\underline{\boldsymbol{\rho}} - \underline{\boldsymbol{\rho}}_h)(\boldsymbol{x} + s\boldsymbol{n}_e)\boldsymbol{n}_e ds$. We use Definition 4.1 and rewrite $T^\sigma$ as follows

$$\begin{aligned} T^\sigma(\boldsymbol{x}) &= \int_0^{l(\boldsymbol{x})} \mathcal{A}(\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\Pi}}^{\boldsymbol{D}}(\underline{\boldsymbol{\sigma}}))(\boldsymbol{x} + s\boldsymbol{n}_e)\boldsymbol{n}_e ds + \int_0^{l(\boldsymbol{x})} \mathcal{A}(\underline{\boldsymbol{\Pi}}^{\boldsymbol{D}}(\underline{\boldsymbol{\sigma}}) - \underline{\boldsymbol{\sigma}}_h)(\boldsymbol{x} + s\boldsymbol{n}_e)\boldsymbol{n}_e ds \\ &= \int_0^{l(\boldsymbol{x})} \mathcal{A}(\underline{\boldsymbol{\delta_\sigma}})(\boldsymbol{x} + s\boldsymbol{n}_e)\boldsymbol{n}_e ds + \int_0^{l(\boldsymbol{x})} \mathcal{A}(\underline{\boldsymbol{e_\sigma}})(\boldsymbol{x} + s\boldsymbol{n}_e)\boldsymbol{n}_e ds \\ &= \int_0^{l(\boldsymbol{x})} [\mathcal{A}(\underline{\boldsymbol{\delta_\sigma}})(\boldsymbol{x} + s\boldsymbol{n}_e)\boldsymbol{n}_e - \mathcal{A}(\underline{\boldsymbol{\delta_\sigma}})(\boldsymbol{x})\boldsymbol{n}_e] ds + l(\boldsymbol{x})\mathcal{A}(\underline{\boldsymbol{\delta_\sigma}})(\boldsymbol{x})\boldsymbol{n}_e \\ &\quad + \int_0^{l(\boldsymbol{x})} [\mathcal{A}(\underline{\boldsymbol{e_\sigma}})(\boldsymbol{x} + s\boldsymbol{n}_e)\boldsymbol{n}_e - \mathcal{A}(\underline{\boldsymbol{e_\sigma}})(\boldsymbol{x})\boldsymbol{n}_e] ds + l(\boldsymbol{x})\mathcal{A}(\underline{\boldsymbol{e_\sigma}})(\boldsymbol{x})\boldsymbol{n}_e \\ &= l(\boldsymbol{x}) \left( \boldsymbol{\Lambda}^{\mathcal{A}(\underline{\boldsymbol{\delta_\sigma}})}(\boldsymbol{x}) + \mathcal{A}(\underline{\boldsymbol{\delta_\sigma}})(\boldsymbol{x})\boldsymbol{n}_e + \boldsymbol{\Lambda}^{\mathcal{A}\underline{\boldsymbol{e_\sigma}}}(\boldsymbol{x}) + \mathcal{A}(\underline{\boldsymbol{e_\sigma}})(\boldsymbol{x})\boldsymbol{n}_e \right). \end{aligned}$$

Similarly for $T^\rho$, we obtain

$$T^\rho(\boldsymbol{x}) = l(\boldsymbol{x}) \left( \boldsymbol{\Lambda}^{\underline{\boldsymbol{\delta_\rho}}}(\boldsymbol{x}) + \underline{\boldsymbol{\delta_\rho}}\boldsymbol{n}_e + \boldsymbol{\Lambda}^{\underline{\boldsymbol{e_\rho}}}(\boldsymbol{x}) + \underline{\boldsymbol{e_\rho}}\boldsymbol{n}_e \right).$$

Thus, replacing the above terms in expression $\tilde{\boldsymbol{g}}(\boldsymbol{x}) - \tilde{\boldsymbol{g}}_h(\boldsymbol{x})$, we have

$$\tilde{\boldsymbol{g}}(\boldsymbol{x}) - \tilde{\boldsymbol{g}}_h(\boldsymbol{x}) = -l(\boldsymbol{x})[\boldsymbol{\Lambda}^{\mathcal{A}(\underline{\boldsymbol{\delta_\sigma}})}(\boldsymbol{x}) + \mathcal{A}(\underline{\boldsymbol{\delta_\sigma}})(\boldsymbol{x})\boldsymbol{n}_e + \boldsymbol{\Lambda}^{\mathcal{A}\underline{\boldsymbol{e_\sigma}}}(\boldsymbol{x}) + \mathcal{A}(\underline{\boldsymbol{e_\sigma}})(\boldsymbol{x})\boldsymbol{n}_e] - l(\boldsymbol{x})[\boldsymbol{\Lambda}^{\underline{\boldsymbol{\delta_\rho}}}(\boldsymbol{x}) + \underline{\boldsymbol{\delta_\rho}}\boldsymbol{n}_e + \boldsymbol{\Lambda}^{\underline{\boldsymbol{e_\rho}}}(\boldsymbol{x}) + \underline{\boldsymbol{e_\rho}}\boldsymbol{n}_e].$$

Since $\mathbb{T}^\rho := \boldsymbol{\Lambda}^{\underline{\boldsymbol{\delta_\rho}}}(\boldsymbol{x}) + \underline{\boldsymbol{\delta_\rho}}\boldsymbol{n}_e + \boldsymbol{\Lambda}^{\underline{\boldsymbol{e_\rho}}}(\boldsymbol{x}) + \underline{\boldsymbol{e_\rho}}\boldsymbol{n}_e$, we obtain

$$\mathcal{A}(\underline{\boldsymbol{e_\sigma}})(\boldsymbol{x})\boldsymbol{n}_e = \frac{-1}{l(\boldsymbol{x})}(\tilde{\boldsymbol{g}}(\boldsymbol{x}) - \tilde{\boldsymbol{g}}_h(\boldsymbol{x})) - \mathbb{T}^\rho - \boldsymbol{\Lambda}^{\mathcal{A}(\underline{\boldsymbol{\delta_\sigma}})}(\boldsymbol{x}) - \mathcal{A}(\underline{\boldsymbol{\delta_\sigma}})(\boldsymbol{x})\boldsymbol{n}_e - \boldsymbol{\Lambda}^{\mathcal{A}\underline{\boldsymbol{e_\sigma}}}(\boldsymbol{x}).$$

Thanks to Equation (3.2f) we have that $\underline{e_{\hat{\sigma}}}n_e = \underline{e_{\sigma}}n_e - \alpha_e(e_u - e_{\hat{u}})$ for all $e \in \mathcal{E}_h$. Then, using the last identity we can rewrite this expression as

$$\begin{aligned}\underline{e_{\hat{\sigma}}}n_e &= \mathcal{A}\underline{e_{\sigma}}(x)n_e - \mathcal{A}(\underline{e_{\sigma}}(x))n_e + \underline{e_{\sigma}}n_e - \alpha_e(e_u - e_{\hat{u}}) \\ &= \mathcal{A}\underline{e_{\sigma}}(x)n_e - \alpha_e(e_u - e_{\hat{u}}) + (\underline{e_{\sigma}}n_e - \mathcal{A}(\underline{e_{\sigma}})n_e)(x) \\ &= \frac{-1}{l(x)}(\tilde{g}(x) - \tilde{g}_h(x)) - \mathbb{T}^{\rho} - \Lambda^{\mathcal{A}(\underline{\delta_{\sigma}})}(x) - \mathcal{A}(\underline{\delta_{\sigma}})(x)n_e - \Lambda^{\mathcal{A}\underline{e_{\sigma}}}(x) - \alpha_e(e_u - e_{\hat{u}}) + (\underline{e_{\sigma}}n_e - \mathcal{A}(\underline{e_{\sigma}})n_e)(x).\end{aligned} \tag{4.17}$$

Finally, the result is obtained replacing the last expression in the definition of $\mathbb{T}$. □

**Corollary 4.3.** *Let us suppose the Assumption (C.1) holds. Then,*

$$\begin{aligned}|\mathbb{T}| \leq\ & -\frac{1}{2} \| \tilde{g} - \tilde{g}_h \|^2_{\Gamma_h, l^{-1}} + \frac{18}{4} \max_{e\in\mathcal{E}_h^{\partial}} \alpha_e r_e h_e^{\perp} \| e_u - e_{\hat{u}} \|^2_{\Gamma_h,\alpha} \\ & + \frac{18}{12} C_{\mathcal{A}}^2 \max_{e\in\mathcal{E}_h^{\partial}} r_e^3 (C_{ext}^e)^2 (C_{inv}^e)^2 \| \underline{e_{\sigma}} \|^2_{D_h} + \frac{18}{4}(1 + C_{\mathcal{A}})^2 \max_{e\in\mathcal{E}_h^{\partial}} (C_{tr}^e)^2 r_e \| \underline{e_{\sigma}} \|^2_{D_h} \\ & + \frac{18}{12} \max_{e\in\mathcal{E}_h^{\partial}} r_e^3 (C_{ext}^e)^2 (C_{inv}^e)^2 \| \underline{e_{\rho}} \|^2_{D_h} + \frac{18}{4} \max_{e\in\mathcal{E}_h^{\partial}} (C_{tr}^e)^2 r_e \| \underline{e_{\rho}} \|^2_{D_h} \\ & + \frac{18}{12} \max_{e\in\mathcal{E}_h^{\partial}} r_e^2 \| \partial_n(\underline{\delta_{\rho}}n) \|^2_{D_h^c,(h^{\perp})^2} + \frac{18}{4} \max_{e\in\mathcal{E}_h^{\partial}} (C_{tr}^e)^2 r_e \| \underline{\delta_{\rho}} \|^2_{D_h} \\ & + \frac{18}{4} C_{\mathcal{A}}^2 \max_{e\in\mathcal{E}_h^{\partial}} (C_{tr}^e)^2 r_e \| \underline{\delta_{\sigma}} \|^2_{D_h} + \frac{18}{12} \max_{e\in\mathcal{E}_h^{\partial}} r_e^2 \| \partial_n(\mathcal{A}\underline{\delta_{\sigma}}n) \|^2_{D_h^c,(h^{\perp})^2} .\end{aligned} \tag{4.18}$$

*Moreover, if Assumption (S.2)-(S.4) also holds, then*

$$\begin{aligned}|\mathbb{T}| \leq\ & -\frac{1}{2} \| \tilde{g} - \tilde{g}_h \|^2_{\Gamma_h,l^{-1}} + \frac{1}{2} \| e_u - e_{\hat{u}} \|^2_{\Gamma_h,\alpha} + \frac{C_{el}}{20} \| \underline{e_{\sigma}} \|^2_{D_h} + \frac{C_{el}}{20(C_{\eta})^2 C_{\mathcal{A}}^2} \| \underline{e_{\rho}} \|^2_{D_h} \\ & + C\Big\{ \| \partial_n(\underline{\delta_{\rho}}n) \|^2_{D_h^c,(h^{\perp})^2} + \| \underline{\delta_{\rho}} \|^2_{D_h} + \| \underline{\delta_{\sigma}} \|^2_{D_h} + \| \partial_n(\mathcal{A}\underline{\delta_{\sigma}}n) \|^2_{D_h^c,(h^{\perp})^2} \Big\},\end{aligned} \tag{4.19}$$

*where $C$ is a constant independent of $h$.*

*Proof.* Let $\varepsilon > 0$. We proceed to bound the terms $\mathbb{T}_i, i = 1, \cdots, 6.$ of Lemma (4.15). It is clear that $\mathbb{T}_1 = - \| \tilde{g} - \tilde{g}_h \|^2_{\Gamma_h,l^{-1}}$. For $\mathbb{T}_2$ and $\mathbb{T}_2$, we use Cauchy-Schwarz inequality, Lemma 4.11, the fact that $l(x) \leq H_e^{\perp}$, Lemma 4.2, Assumption (C.1) and Young's inequality:

$$\begin{aligned}\mathbb{T}_2 \leq & \sum_{e\in\mathcal{E}_h^{\partial}} \| \tilde{g} - \tilde{g}_h \|_{e,l^{-1}} \Big\{ \| \Lambda^{\underline{\delta_{\rho}}}(x) \|_{e,l} + \| \underline{\delta_{\rho}}n_e \|_{e,l} + \| \Lambda^{\underline{e_{\rho}}}(x) \|_{e,l} + \| \underline{e_{\rho}}n_e \|_{e,l} \Big\} \\ \leq & \sum_{e\in\mathcal{E}_h^{\partial}} \| \tilde{g} - \tilde{g}_h \|_{e,l^{-1}} \Big\{ \frac{1}{\sqrt{3}} r_e \| \partial_n(\underline{\delta_{\rho}}n) \|_{K_{ext}^e,(h^{\perp})^2} + C_{tr}^e r_e^{1/2} \| \underline{\delta_{\rho}} \|_{K^e} \\ & + \frac{1}{\sqrt{3}} r_e^{3/2} C_{ext}^e C_{inv}^e \| \underline{e_{\rho}} \|_{K^e} + C_{tr}^e r_e^{1/2} \| \underline{e_{\rho}} \|_{K^e} \Big\} \\ \leq & 4\varepsilon \| \tilde{g} - \tilde{g}_h \|^2_{\Gamma_h,l^{-1}} + \frac{1}{12\varepsilon} \max_{e\in\mathcal{E}_h^{\partial}} r_e^2 \| \partial_n(\underline{\delta_{\rho}}n) \|^2_{D_h^c,(h^{\perp})^2} + \frac{1}{4\varepsilon} \max_{e\in\mathcal{E}_h^{\partial}} (C_{tr}^e)^2 r_e \| \underline{\delta_{\rho}} \|^2_{D_h} \\ & + \frac{1}{12\varepsilon} \max_{e\in\mathcal{E}_h^{\partial}} r_e^3 (C_{ext}^e)^2 (C_{inv}^e)^2 \| \underline{e_{\rho}} \|^2_{D_h} + \max_{e\in\mathcal{E}_h^{\partial}} (C_{tr}^e)^2 r_e \| \underline{e_{\rho}} \|^2_{D_h} .\end{aligned}$$

$$\mathbb{T}_3 \leq \sum_{e\in\mathcal{E}_h^\partial} \| \tilde{\boldsymbol{g}} - \tilde{\boldsymbol{g}}_h \|_{e,l^{-1}} \left\{ \| \boldsymbol{\Lambda}^{\mathcal{A}(\underline{\boldsymbol{\delta_\sigma}})} \|_{e,l} + \| \mathcal{A}\underline{\boldsymbol{\delta_\sigma}}\boldsymbol{n}_e \|_{e,l} \right\}$$

$$\leq \sum_{e\in\mathcal{E}_h^\partial} \| \tilde{\boldsymbol{g}} - \tilde{\boldsymbol{g}}_h \|_{e,l^{-1}} \left\{ \frac{1}{\sqrt{3}} r_e \| \partial_{\boldsymbol{n}}(\mathcal{A}\underline{\boldsymbol{\delta_\sigma}}\boldsymbol{n}) \|_{K^e_{ext},(h^\perp)^2} + C_{\mathcal{A}} C^e_{tr} r_e^{1/2} \| \underline{\boldsymbol{\delta_\sigma}} \|_{K^e} \right\}$$

$$\leq 2\varepsilon \| \tilde{\boldsymbol{g}} - \tilde{\boldsymbol{g}}_h \|^2_{\Gamma_h,l^{-1}} + \frac{1}{12\varepsilon} \max_{e\in\mathcal{E}_h^\partial} r_e^2 \| \partial_{\boldsymbol{n}}(\mathcal{A}\underline{\boldsymbol{\delta_\sigma}}\boldsymbol{n}) \|^2_{D_h^c,(h^\perp)^2} + \frac{1}{4\varepsilon} C^2_{\mathcal{A}} \max_{e\in\mathcal{E}_h^\partial} (C^e_{tr})^2 r_e \| \underline{\boldsymbol{\delta_\sigma}} \|^2_{D_h} .$$

For $\mathbb{T}_4$, Cauchy-Schwarz inequality, Lemma 4.2 and Young's inequality to imply

$$\mathbb{T}_4 \leq \sum_{e\in\mathcal{E}_h^\partial} \| \tilde{\boldsymbol{g}} - \tilde{\boldsymbol{g}}_h \|_{e,l^{-1}} \| \mathcal{A}\underline{\boldsymbol{e_\sigma}} \|_{e,l} \leq \sum_{e\in\mathcal{E}_h^\partial} \| \tilde{\boldsymbol{g}} - \tilde{\boldsymbol{g}}_h \|_{e,l^{-1}} \frac{1}{\sqrt{3}} r_e^{3/2} C^e_{ext} C^e_{inv} C_{\mathcal{A}} \| \underline{\boldsymbol{e_\sigma}} \|_{K^e}$$

$$\leq \varepsilon \| \tilde{\boldsymbol{g}} - \tilde{\boldsymbol{g}}_h \|^2_{\Gamma_h,l^{-1}} + \frac{1}{12\varepsilon} C^2_{\mathcal{A}} \max_{e\in\mathcal{E}_h^\partial} r_e^3 (C^e_{ext})^2 (C^e_{inv})^2 \| \underline{\boldsymbol{e_\sigma}} \|^2_{D_h} .$$

For $\mathbb{T}_5$, we use Cauchy-Schwarz inequality and Young's inequality to get

$$\mathbb{T}_5 \leq \varepsilon \| \tilde{\boldsymbol{g}} - \tilde{\boldsymbol{g}}_h \|^2_{\Gamma_h,l^{-1}} + \frac{1}{4\varepsilon} \| \boldsymbol{e_u} - \boldsymbol{e_{\hat{u}}} \|^2_{\Gamma_h,\alpha} .$$

For $\mathbb{T}_6$, we use Cauchy-Schwarz inequality, the fact that $l(\boldsymbol{x}) \leq H_e^\perp$, Lemma 4.2, Assumption (C.1) and Young's inequality to obtain

$$\mathbb{T}_6 \leq \sum_{e\in\mathcal{E}_h^\partial} \| \tilde{\boldsymbol{g}} - \tilde{\boldsymbol{g}}_h \|_{e,l^{-1}} \| \underline{\boldsymbol{e_\sigma}}\boldsymbol{n}_e - \mathcal{A}\underline{\boldsymbol{e_\sigma}}\boldsymbol{n}_e \|_{e,l} \leq \sum_{e\in\mathcal{E}_h^\partial} \| \tilde{\boldsymbol{g}} - \tilde{\boldsymbol{g}}_h \|_{e,l^{-1}} (1 + C_{\mathcal{A}}) C^e_{tr} r_e^{1/2} \| \underline{\boldsymbol{e_\sigma}} \|_{K^e}$$

$$\leq \varepsilon \| \tilde{\boldsymbol{g}} - \tilde{\boldsymbol{g}}_h \|^2_{\Gamma_h,l^{-1}} + \frac{1}{4\varepsilon} (1 + C_{\mathcal{A}})^2 \max_{e\in\mathcal{E}_h^\partial} (C^e_{tr})^2 r_e \| \underline{\boldsymbol{e_\sigma}} \|^2_{D_h} .$$

We obtain (4.18) gathering all the above bounds and considering $\varepsilon = 1/18$. Moreover, considering (S.2)-(S.4), (4.18) implies (4.19). ☐

**Lemma 4.16.** *We have*

$$\| \tilde{\boldsymbol{g}} - \tilde{\boldsymbol{g}}_h \|^2_{e,l^{-1}} \leq 4C^2_{\mathcal{A}} (C^e_{ext})^2 r_e (\| \underline{\boldsymbol{\delta_\sigma}} \|^2_{K^e} + \| \underline{\boldsymbol{e_\sigma}} \|^2_{K^e}) + 4(C^e_{ext})^2 r_e (\| \underline{\boldsymbol{\delta_\rho}} \|^2_{K^e} + \| \underline{\boldsymbol{e_\rho}} \|^2_{K^e}),$$

*for all $e$ in $\mathcal{E}_h$.*

*Proof.* We use the fact that $|(\boldsymbol{x} + s\boldsymbol{n}_e)\boldsymbol{n}_e| \leq |\bar{\boldsymbol{x}} - \boldsymbol{x}| = l(\boldsymbol{x})$ for all $e$ in $\mathcal{E}_h^\partial$, by (3.2g), identity (1.6) and Cauchy-Schwarz inequality

$$\| \tilde{\boldsymbol{g}} - \tilde{\boldsymbol{g}}_h \|^2_{e,l^{-1}} = \left| \int_e \frac{1}{l(x)} \left[ \int_0^{l(x)} \mathcal{A}(\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}_h)(\boldsymbol{x} + s\boldsymbol{n}_e)\boldsymbol{n}_e ds + \int_0^{l(x)} (\underline{\boldsymbol{\rho}} - \underline{\boldsymbol{\rho}}_h)(\boldsymbol{x} + s\boldsymbol{n}_e)\boldsymbol{n}_e ds \right]^2 dx \right|$$

$$= \left| \int_e \frac{1}{l(x)} \left[ \int_0^{l(x)} \mathcal{A}(\underline{\boldsymbol{\delta_\sigma}} + \underline{\boldsymbol{e_\sigma}})(\boldsymbol{x} + s\boldsymbol{n}_e)\boldsymbol{n}_e ds + \int_0^{l(x)} (\underline{\boldsymbol{\delta_\rho}} - \underline{\boldsymbol{e_\rho}})(\boldsymbol{x} + s\boldsymbol{n}_e)\boldsymbol{n}_e ds \right]^2 dx \right|$$

$$\leq \| \mathcal{A}(\underline{\boldsymbol{\delta_\sigma}} + \underline{\boldsymbol{e_\sigma}}) + (\underline{\boldsymbol{\delta_\rho}} + \underline{\boldsymbol{e_\rho}}) \|^2_{K^e_{ext}} .$$

Then, using Lemma 4.3, we obtain

$$\| \tilde{\boldsymbol{g}} - \tilde{\boldsymbol{g}}_h \|^2_{e,l^{-1}} \leq 2C^2_{\mathcal{A}} \| \underline{\boldsymbol{\delta_\sigma}} + \underline{\boldsymbol{e_\sigma}} \|^2_{K^e_{ext}} + 2 \| \underline{\boldsymbol{\delta_\rho}} + \underline{\boldsymbol{e_\rho}} \|^2_{K^e_{ext}}$$

$$\leq 4C^2_{\mathcal{A}} (\| \underline{\boldsymbol{\delta_\sigma}} \|^2_{K^e_{ext}} + \| \underline{\boldsymbol{e_\sigma}} \|^2_{K^e_{ext}}) + 4(\| \underline{\boldsymbol{\delta_\rho}} \|^2_{K^e_{ext}} + \| \underline{\boldsymbol{e_\rho}} \|^2_{K^e_{ext}})$$

$$\leq 4C^2_{\mathcal{A}} (C^e_{ext})^2 r_e (\| \underline{\boldsymbol{\delta_\sigma}} \|^2_{K^e} + \| \underline{\boldsymbol{e_\sigma}} \|^2_{K^e} \; 4C^2_{\mathcal{A}} (C^e_{ext})^2 r_e (\| \underline{\boldsymbol{\delta_\rho}} \|^2_{K^e} + \| \underline{\boldsymbol{e_\rho}} \|^2_{K^e}).$$

☐

**Lemma 4.17.** *Suppose that $k \geq 1$ and Assumption (C.1) holds, then there exist $C_\eta > 0$ independent of $h$ such that*

$$\| \underline{\boldsymbol{e_\rho}} \|_{D_h}^2 \leq 4(C_\eta)^2 C_{\mathcal{A}}^2 \| \underline{\boldsymbol{e_\sigma}} \|_{D_h}^2 + 4 \sum_{e \in \mathcal{E}_h^\partial} 4(C_\eta)^2 (C_{tr}^e)^2 r_e \| \tilde{\boldsymbol{g}} - \tilde{\boldsymbol{g}}_h \|_{e,l^{-1}}^2 + 4(C_\eta)^2 C_{\mathcal{A}}^2 \| \underline{\boldsymbol{\delta_\sigma}} \|_{D_h}^2 + 4(C_\eta)^2 \| \underline{\boldsymbol{\delta_\rho}} \|_{D_h}^2 .$$

*Proof.* We follow the ideas in [5]. We consider the next orthogonal decomposition:

$$\underline{\boldsymbol{e_\rho}} = \underline{\boldsymbol{e_\rho}}^0 + \underline{\boldsymbol{e_\rho}}^c, \quad \underline{\boldsymbol{e_\rho}}^c := \frac{1}{|K^e|} \int_{K^e} \underline{\boldsymbol{e_\rho}}, \quad \underline{\boldsymbol{e_\rho}}^0 = \underline{\boldsymbol{e_\rho}} - \underline{\boldsymbol{e_\rho}}^c,$$

where $\underline{\boldsymbol{e_\rho}}^0 \in \underline{\boldsymbol{A}}_h^0$ and $\underline{\boldsymbol{e_\rho}}^c \in \underline{\boldsymbol{A}}_h^c$ (we recall that $\underline{\boldsymbol{A}}_h^0$ and $\underline{\boldsymbol{A}}_h^c$ have been introduced in Lemmas 4.4 and 4.5). We proceed in two steps to bound the $\underline{\boldsymbol{e_\rho}}^0$ and $\underline{\boldsymbol{e_\rho}}^c$.

**Step 1** Let $\underline{\boldsymbol{\eta}} := \underline{\boldsymbol{e_\rho}}^0$ in Lemma 4.4, then there exists $\underline{\boldsymbol{v}} \in \underline{\mathcal{B}}_h \subset \underline{\boldsymbol{V}}_h$ satisfying (4.3a) and (4.3b). Then we rewrite the Equation (4.14a) as

$$(\mathcal{A}\underline{\boldsymbol{e_\sigma}}, \underline{\boldsymbol{v}})_{\mathcal{T}_h} + (\boldsymbol{e_u}, \nabla \cdot \underline{\boldsymbol{v}})_{\mathcal{T}_h} + (\underline{\boldsymbol{e_\rho}}^0, \underline{\boldsymbol{v}})_{\mathcal{T}_h} + (\underline{\boldsymbol{e_\rho}}^c, \underline{\boldsymbol{v}})_{\mathcal{T}_h} - \langle \boldsymbol{e_{\hat{u}}}, \underline{\boldsymbol{v}}\boldsymbol{n} \rangle_{\partial\mathcal{T}_h} = -(\mathcal{A}\underline{\boldsymbol{\delta_\sigma}}, \underline{\boldsymbol{v}})_{\mathcal{T}_h} - (\underline{\boldsymbol{\delta_\rho}}, \underline{\boldsymbol{v}})_{\mathcal{T}_h}. \tag{4.20}$$

By property (B.1) and (B.2), we have $(\boldsymbol{e_u}, \nabla \cdot \underline{\boldsymbol{v}})_{\mathcal{T}_h} = 0$ and $\langle \boldsymbol{e_{\hat{u}}}, \underline{\boldsymbol{v}}\boldsymbol{n} \rangle_{\partial\mathcal{T}_h} = 0$. Now considering $\underline{\boldsymbol{\gamma}} := \underline{\boldsymbol{e_\rho}}^c$ in Lemma 4.4, we have that $(\underline{\boldsymbol{e_\rho}}^c, \underline{\boldsymbol{v}})_{\mathcal{T}_h} = (\underline{\boldsymbol{e_\rho}}^0, \underline{\boldsymbol{e_\rho}}^c)_{\mathcal{T}_h} = 0$, since the decomposition of $\underline{\boldsymbol{e_\rho}}$ is orthogonal in $\underline{\boldsymbol{L}}^2$. Moreover, by taking $\underline{\boldsymbol{\gamma}} = \underline{\boldsymbol{\eta}} = \underline{\boldsymbol{e_\rho}}^0$ in (4.3a) we have that $(\underline{\boldsymbol{e_\rho}}^0, \underline{\boldsymbol{v}})_{\mathcal{T}_h} = \| \underline{\boldsymbol{e_\rho}}^0 \|_{D_h}^2$.

Thus, replacing the above terms in Equation (4.20) and using the Equation (4.3b) we get

$$\begin{aligned} \| \underline{\boldsymbol{e_\rho}}^0 \|_{D_h}^2 &= (\underline{\boldsymbol{v}}, \underline{\boldsymbol{e_\rho}}^0)_{\mathcal{T}_h} = -(\mathcal{A}\underline{\boldsymbol{e_\sigma}}, \underline{\boldsymbol{v}})_{\mathcal{T}_h} - (\mathcal{A}\underline{\boldsymbol{\delta_\sigma}}, \underline{\boldsymbol{v}})_{\mathcal{T}_h} - (\underline{\boldsymbol{\delta_\rho}}, \underline{\boldsymbol{v}})_{\mathcal{T}_h} \\ &\leq C_\eta^0 (C_{\mathcal{A}} \| \underline{\boldsymbol{e_\sigma}} \|_{D_h} + C_{\mathcal{A}} \| \underline{\boldsymbol{\delta_\sigma}} \|_{D_h} + \| \underline{\boldsymbol{\delta_\rho}} \|_{D_h}) \| \underline{\boldsymbol{e_\rho}}^0 \|_{D_h} . \end{aligned}$$

Then, we obtain

$$\| \underline{\boldsymbol{e_\rho}}^0 \|_{D_h} \leq C_\eta^0 (C_{\mathcal{A}} \| \underline{\boldsymbol{e_\sigma}} \|_{D_h} + C_{\mathcal{A}} \| \underline{\boldsymbol{\delta_\sigma}} \|_{D_h} + \| \underline{\boldsymbol{\delta_\rho}} \|_{D_h}).$$

**Step 2** Let $\underline{\boldsymbol{\eta}} := \underline{\boldsymbol{e_\rho}}^c$ in Lemma 4.5, then there exists $\underline{\boldsymbol{v}} \in \underline{\mathcal{G}}_h$ satisfying (4.4a)-(4.4c). Then $(\boldsymbol{e_u}, \nabla \cdot \underline{\boldsymbol{v}})_{\mathcal{T}_h} = 0$ and $\langle \boldsymbol{e_{\hat{u}}}, \underline{\boldsymbol{v}}\boldsymbol{n} \rangle_{\partial\mathcal{T}_h} = \langle \tilde{\boldsymbol{g}} - \tilde{\boldsymbol{g}}_h, \underline{\boldsymbol{v}}\boldsymbol{n} \rangle_{\Gamma_h}$, thanks to Equation (4.14e) and the fact that $\underline{\boldsymbol{v}} \in \underline{\boldsymbol{H}}(div; D_h)$ (we recall that we are assuming $k \geq 1$). Thus, with the decomposition of $\underline{\boldsymbol{\rho}}$, Equation (4.14a) yields

$$(\mathcal{A}\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{v}})_{\mathcal{T}_h} + (\underline{\boldsymbol{\rho}}_h^0, \underline{\boldsymbol{v}})_{\mathcal{T}_h} + (\underline{\boldsymbol{\rho}}_h^c, \underline{\boldsymbol{v}})_{\mathcal{T}_h} - \langle \tilde{\boldsymbol{g}} - \tilde{\boldsymbol{g}}_h, \underline{\boldsymbol{v}}\boldsymbol{n} \rangle_{\Gamma_h} = -(\mathcal{A}\underline{\boldsymbol{\delta_\sigma}}, \underline{\boldsymbol{v}})_{\mathcal{T}_h} - (\underline{\boldsymbol{\delta_\rho}}, \underline{\boldsymbol{v}})_{\mathcal{T}_h}. \tag{4.21}$$

Moreover, taking $\underline{\boldsymbol{\gamma}} := \underline{\boldsymbol{e_\rho}}^c$ in (4.4b) we have $(\underline{\boldsymbol{e_\rho}}^c, \underline{\boldsymbol{v}})_{\mathcal{T}_h} = \| \underline{\boldsymbol{e_\rho}}^c \|_{D_h}$ and then from Equation (4.21) we obtain

$$\| \underline{\boldsymbol{e_\rho}}^c \|_{D_h}^2 = -(\mathcal{A}\underline{\boldsymbol{e_\sigma}}, \underline{\boldsymbol{v}})_{\mathcal{T}_h} - (\underline{\boldsymbol{e_\rho}}^0, \underline{\boldsymbol{v}})_{\mathcal{T}_h} - (\mathcal{A}\underline{\boldsymbol{\delta_\sigma}}, \underline{\boldsymbol{v}})_{\mathcal{T}_h} + \langle \tilde{\boldsymbol{g}} - \tilde{\boldsymbol{g}}_h, \underline{\boldsymbol{v}}\boldsymbol{n} \rangle_{\Gamma_h}.$$

Using Cauchy-Schwarz inequality, Lemma 4.2, the bound (4.3.2), Equation (4.4c) of Lemma 4.5, the fact that $l(\boldsymbol{x}) \leq H_e^\perp$ and Assumption (C.1) we get

$$\begin{aligned} \| \underline{\boldsymbol{e_\rho}}^c \|_{D_h}^2 \leq & C_\eta^c \Big\{ C_{\mathcal{A}} \| \underline{\boldsymbol{e_\sigma}} \|_{D_h} + \| \underline{\boldsymbol{e_\rho}}^0 \|_{D_h} + C_{\mathcal{A}} \| \underline{\boldsymbol{\delta_\sigma}} \|_{D_h} + \sum_{e \in \mathcal{E}_h^\partial} |l(x)|^{1/2} \| \tilde{\boldsymbol{g}} - \tilde{\boldsymbol{g}}_h \|_{e,l^{-1}} \Big\} \| \underline{\boldsymbol{e_\rho}}^c \|_{D_h} \\ \leq & C_\eta^c \Big\{ (1 + C_\eta^0) C_{\mathcal{A}} \| \underline{\boldsymbol{e_\sigma}} \|_{D_h} + (1 + C_\eta^0) C_{\mathcal{A}} \| \underline{\boldsymbol{\delta_\sigma}} \|_{D_h} + C_\eta^0 \| \underline{\boldsymbol{\delta_\rho}} \|_{D_h} \\ & + \sum_{e \in \mathcal{E}_h^\partial} C_{tr}^e h_e^{-1/2} (H_e^\perp)^{1/2} \| \tilde{\boldsymbol{g}} - \tilde{\boldsymbol{g}}_h \|_{e,l^{-1}} \Big\} \| \underline{\boldsymbol{e_\rho}}^c \|_{D_h} . \end{aligned}$$

Then,

$$\| \underline{\boldsymbol{e}_{\boldsymbol{\rho}}}^c \|_{D_h} \leq (C_\eta^c + C_\eta^c C_\eta^0) C_{\mathcal{A}} \| \underline{\boldsymbol{e}_{\boldsymbol{\rho}}} \|_{D_h} + (C_\eta^c + C_\eta^c C_\eta^0) C_{\mathcal{A}} \| \underline{\boldsymbol{\delta}_{\boldsymbol{\sigma}}} \|_{D_h} + C_\eta^c C_\eta^0 \| \underline{\boldsymbol{\delta}_{\boldsymbol{\rho}}} \|_{D_h} + \sum_{e \in \mathcal{E}_h^\partial} C_\eta^c C_{tr}^e r_e^{1/2} \| \tilde{\boldsymbol{g}} - \tilde{\boldsymbol{g}}_h \|_{e,l^{-1}} .$$

Finally, using Steps 1 and 2, we get

$$\begin{aligned} \| \underline{\boldsymbol{e}_{\boldsymbol{\rho}}} \|_{D_h} &\leq \| \underline{\boldsymbol{e}_{\boldsymbol{\rho}}}^0 \|_{D_h} + \| \underline{\boldsymbol{e}_{\boldsymbol{\rho}}}^c \|_{D_h} \\ &\leq (C_\eta^0 + C_\eta^0 C_\eta^c + C_\eta^c) C_{\mathcal{A}} \| \underline{\boldsymbol{e}_{\boldsymbol{\sigma}}} \|_{D_h} + (C_\eta^0 + C_\eta^0 C_\eta^c + C_\eta^c) C_{\mathcal{A}} \| \underline{\boldsymbol{\delta}_{\boldsymbol{\sigma}}} \|_{D_h} \\ &\quad + (C_\eta^0 + C_\eta^0 C_\eta^c) \| \underline{\boldsymbol{\delta}_{\boldsymbol{\rho}}} \|_{D_h} + \sum_{e \in \mathcal{E}_h^\partial} C_\eta^c C_{tr}^e r_e^{1/2} \| \tilde{\boldsymbol{g}} - \tilde{\boldsymbol{g}}_h \|_{e,l^{-1}} . \end{aligned}$$

We finish the proof recalling the definition of $C_\eta$, the fact that $C_\eta^c \leq C_\eta$ and defining $C := \max\{C_\eta, C_\eta C_{\mathcal{A}}\}$. □

**Corollary 4.4.** *Let us suppose $k \geq 1$ and Assumption (S.1) holds, then*

$$\| \underline{\boldsymbol{e}_{\boldsymbol{\rho}}} \|_{D_h}^2 \leq 9(C_\eta)^2 (C_{\mathcal{A}})^2 \| \underline{\boldsymbol{e}_{\boldsymbol{\sigma}}} \|_{D_h}^2 + 9(C_\eta)^2 (C_{\mathcal{A}})^2 \| \underline{\boldsymbol{\delta}_{\boldsymbol{\sigma}}} \|_{D_h}^2 + 9(C_\eta)^2 \| \underline{\boldsymbol{\delta}_{\boldsymbol{\rho}}} \|_{D_h}^2 .$$

*Proof.* We replace the estimate given by Lemma 4.16 in the terms of the right hand side in Lemma 4.17 and obtain

$$\begin{aligned} \| \underline{\boldsymbol{e}_{\boldsymbol{\rho}}} \|_{D_h}^2 \leq & \left\{ 4(C_\eta)^2 C_{\mathcal{A}}^2 + 16 (C_\eta)^2 C_{\mathcal{A}}^2 \max_{e \in \mathcal{E}_h^\partial} (C_{tr}^e)^2 (C_{ext}^e)^2 r_e^2 \right\} (\| \underline{\boldsymbol{e}_{\boldsymbol{\sigma}}} \|_{D_h}^2 + \| \underline{\boldsymbol{\delta}_{\boldsymbol{\sigma}}} \|_{D_h}^2) \\ & + \left\{ 4(C_\eta)^2 + 16 (C_\eta)^2 \max_{e \in \mathcal{E}_h^\partial} (C_{tr}^e)^2 (C_{ext}^e)^2 r_e^2 \right\} \| \underline{\boldsymbol{\delta}_{\boldsymbol{\rho}}} \|_{D_h}^2 \\ & + 16 \max_{e \in \Gamma_h} (C_\eta)^2 (C_{tr}^e)^2 (C_{ext}^e)^2 r_e^2 \| \underline{\boldsymbol{e}_{\boldsymbol{\rho}}} \|_{D_h}^2 . \end{aligned}$$

The result follows from rearranging terms in last expression and considering the Assumption (S.1). □

Let us define the following auxiliary variable related to interpolation errors:

$$\Theta(\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\rho}}) := \| \underline{\boldsymbol{\delta}_{\boldsymbol{\sigma}}} \|_{D_h} + \| \underline{\boldsymbol{\delta}_{\boldsymbol{\rho}}} \|_{D_h} + \| \partial_{\boldsymbol{n}} (\underline{\boldsymbol{\delta}_{\boldsymbol{\sigma}}} \boldsymbol{n}) \|_{D_h^c, (h^\perp)^2} + \| \partial_{\boldsymbol{n}} (\underline{\boldsymbol{\delta}_{\boldsymbol{\rho}}} \boldsymbol{n}) \|_{D_h^c, (h^\perp)^2} . \tag{4.22}$$

By Lemma 3.8 of [4], we can easily show that

$$\begin{aligned} \| \partial_{\boldsymbol{n}} (\underline{\boldsymbol{\delta}_{\boldsymbol{\sigma}}} \boldsymbol{n}) \|_{D_h^c, (h^\perp)^2} &\leq C \left( \| \underline{\boldsymbol{\delta}_{\boldsymbol{\sigma}}} \|_{D_h} + h^{k+1} |\underline{\boldsymbol{\sigma}}|_{\underline{\boldsymbol{H}}^{k+1}(\Omega)} \right), \\ \| \partial_{\boldsymbol{n}} (\underline{\boldsymbol{\delta}_{\boldsymbol{\rho}}} \boldsymbol{n}) \|_{D_h^c, (h^\perp)^2} &\leq C \left( \| \underline{\boldsymbol{\delta}_{\boldsymbol{\rho}}} \|_{D_h} + h^{k+1} |\underline{\boldsymbol{\rho}}|_{\underline{\boldsymbol{H}}^{k+1}(\Omega)} \right). \end{aligned}$$

These estimates, together with (4.12a) and (4.13), allow us to conclude that

$$\Theta(\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\rho}}) \leq C h^{k+1} \left( |\boldsymbol{u}|_{\boldsymbol{H}^{k+1}(\Omega)} + |\underline{\boldsymbol{\sigma}}|_{\underline{\boldsymbol{H}}^{k+1}(\Omega)} + |\underline{\boldsymbol{\rho}}|_{\underline{\boldsymbol{H}}^{k+1}(\Omega)} \right). \tag{4.23}$$

**Proposition 4.1.** *Let us suppose $k \geq 1$ and Assumptions C and S hold. Then, there exists a constant $C > 0$ independent of $h$ such that*

$$|||(\underline{\boldsymbol{e}_{\boldsymbol{\sigma}}}, \boldsymbol{e_u} - \boldsymbol{e_{\hat{u}}}, \tilde{\boldsymbol{g}} - \tilde{\boldsymbol{g}}_h)||| + \| \underline{\boldsymbol{e}_{\boldsymbol{\rho}}} \|_{D_h} \leq C \Theta(\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\rho}}),$$

*where,*

$$|||(\underline{\boldsymbol{e}_{\boldsymbol{\sigma}}}, \boldsymbol{e_u} - \boldsymbol{e_{\hat{u}}}, \tilde{\boldsymbol{g}} - \tilde{\boldsymbol{g}}_h)||| := \left( \| \underline{\boldsymbol{e}_{\boldsymbol{\sigma}}} \|_{D_h}^2 + \| \boldsymbol{e_u} - \boldsymbol{e_{\hat{u}}} \|_{D_h}^2 + \| \tilde{\boldsymbol{g}} - \tilde{\boldsymbol{g}}_h \|_{\Gamma_h, l^{-1}}^2 \right)^{1/2} .$$

*Proof.* We use Cauchy-Schwarz inequality in Lemma 4.13 to get

$$C_{el} \parallel \underline{\boldsymbol{e}_{\boldsymbol{\sigma}}} \parallel_{D_h}^2 + \parallel \boldsymbol{e}_{\boldsymbol{u}} - \boldsymbol{e}_{\hat{\boldsymbol{u}}} \parallel_{\partial\mathcal{T}_h,\alpha}^2 \leq \parallel \underline{\boldsymbol{\delta}_{\boldsymbol{\sigma}}} \parallel_{D_h} \parallel \underline{\boldsymbol{e}_{\boldsymbol{\rho}}} \parallel_{D_h} + C_{\mathcal{A}} \parallel \underline{\boldsymbol{\delta}_{\boldsymbol{\sigma}}} \parallel_{D_h} \parallel \underline{\boldsymbol{e}_{\boldsymbol{\sigma}}} \parallel_{D_h} + \parallel \underline{\boldsymbol{\delta}_{\boldsymbol{\rho}}} \parallel_{D_h} \parallel \underline{\boldsymbol{e}_{\boldsymbol{\sigma}}} \parallel_{D_h} + |\mathbb{T}|.$$

Then, using Young's inequality, we obtain for $\varepsilon > 0$

$$\begin{aligned}
\parallel \underline{\boldsymbol{\delta}_{\boldsymbol{\sigma}}} \parallel_{D_h} \parallel \underline{\boldsymbol{e}_{\boldsymbol{\rho}}} \parallel_{D_h} &\leq \varepsilon \frac{C_{el}}{9(C_\eta)^2 C_{\mathcal{A}}^2} \parallel \underline{\boldsymbol{e}_{\boldsymbol{\rho}}} \parallel_{D_h}^2 + \frac{9(C_\eta)^2 C_{\mathcal{A}}}{4C_{el}\varepsilon} \parallel \underline{\boldsymbol{\delta}_{\boldsymbol{\sigma}}} \parallel_{D_h}^2, \\
C_{\mathcal{A}} \parallel \underline{\boldsymbol{\delta}_{\boldsymbol{\sigma}}} \parallel_{D_h} \parallel \underline{\boldsymbol{e}_{\boldsymbol{\sigma}}} \parallel_{D_h} &\leq \varepsilon C_{el} \parallel \underline{\boldsymbol{e}_{\boldsymbol{\sigma}}} \parallel_{D_h}^2 + \frac{C_{\mathcal{A}}^2}{4C_{el}\varepsilon} \parallel \underline{\boldsymbol{\delta}_{\boldsymbol{\sigma}}} \parallel_{D_h}^2, \\
\parallel \underline{\boldsymbol{\delta}_{\boldsymbol{\rho}}} \parallel_{D_h} \parallel \underline{\boldsymbol{e}_{\boldsymbol{\sigma}}} \parallel_{D_h} &\leq \varepsilon C_{el} \parallel \underline{\boldsymbol{e}_{\boldsymbol{\sigma}}} \parallel_{D_h}^2 + \frac{1}{4C_{el}\varepsilon} \parallel \underline{\boldsymbol{\delta}_{\boldsymbol{\rho}}} \parallel_{D_h}^2 .
\end{aligned}$$

Using this result with $\varepsilon = 1/12$, Corollary 4.4 and Assumptions S, we get

$$\begin{aligned}
C_{el} \parallel \underline{\boldsymbol{e}_{\boldsymbol{\sigma}}} \parallel_{D_h}^2 + \frac{1}{2} \parallel \boldsymbol{e}_{\boldsymbol{u}} - \boldsymbol{e}_{\hat{\boldsymbol{u}}} \parallel_{\partial\mathcal{T}_h,\alpha}^2 + \frac{1}{2} \parallel \tilde{\boldsymbol{g}} - \tilde{\boldsymbol{g}}_h \parallel_{\partial\mathcal{T}_h,\alpha}^2 \leq \frac{C_{el}}{4} \parallel \underline{\boldsymbol{e}_{\boldsymbol{\sigma}}} \parallel_{D_h}^2 + \frac{C_{el}}{20} \parallel \underline{\boldsymbol{e}_{\boldsymbol{\sigma}}} \parallel_{D_h}^2 + \frac{9C_{el}}{20} \parallel \underline{\boldsymbol{e}_{\boldsymbol{\sigma}}} \parallel_{D_h}^2 \\
+ C\Big\{ \parallel \underline{\boldsymbol{\delta}_{\boldsymbol{\sigma}}} \parallel_{D_h} + \parallel \partial_{\boldsymbol{n}}(\underline{\boldsymbol{\delta}_{\boldsymbol{\sigma}}}\boldsymbol{n}) \parallel_{D_h^c,(h^\perp)^2} + \parallel \underline{\boldsymbol{\delta}_{\boldsymbol{\rho}}} \parallel_{D_h} + \parallel \partial_{\boldsymbol{n}}(\underline{\boldsymbol{\delta}_{\boldsymbol{\rho}}}\boldsymbol{n}) \parallel_{D_h^c,(h^\perp)^2} \Big\},
\end{aligned}$$

which implies, after rearranging terms, that $|||(\underline{\boldsymbol{e}_{\boldsymbol{\sigma}}}, \boldsymbol{e}_{\boldsymbol{u}} - \boldsymbol{e}_{\hat{\boldsymbol{u}}}, \tilde{\boldsymbol{g}} - \tilde{\boldsymbol{g}}_h)||| \leq C\Theta(\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\rho}})$. Finally by Corollary 4.4 $\parallel \underline{\boldsymbol{e}_{\boldsymbol{\rho}}} \parallel_{D_h} \leq C\Theta(\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\rho}})$. □

**Theorem 4.18.** *Let us suppose the same assumptions as in Proposition 4.1, then*

$$\parallel \underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}_h \parallel_{D_h} + \parallel \underline{\boldsymbol{\rho}} - \underline{\boldsymbol{\rho}}_h \parallel_{D_h} \leq C\Theta(\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\rho}}).$$

*Proof.* Since $\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}_h = \underline{\boldsymbol{e}_{\boldsymbol{\sigma}}} + \underline{\boldsymbol{\delta}_{\boldsymbol{\sigma}}}$ and $\underline{\boldsymbol{\rho}} - \underline{\boldsymbol{\rho}}_h = \underline{\boldsymbol{e}_{\boldsymbol{\rho}}} + \underline{\boldsymbol{\delta}_{\boldsymbol{\rho}}}$, then the result is a direct consequence of triangle inequality and Proposition 4.1. □

### 4.3.3 Duality argument

Next we use a duality argument to get an estimate for $\boldsymbol{e}_{\boldsymbol{u}}$. Now, we introduce the so-called dual problem:

$$\nabla \cdot \underline{\boldsymbol{\psi}} = \boldsymbol{\theta} \quad \text{in} \quad \Omega, \tag{4.24a}$$

$$\mathcal{A}\underline{\boldsymbol{\psi}} - \nabla\boldsymbol{\phi} + \underline{\boldsymbol{\xi}} = 0 \quad \text{in} \quad \Omega, \tag{4.24b}$$

$$\boldsymbol{\phi} = \boldsymbol{\theta} \quad \text{on} \ \partial\Omega. \tag{4.24c}$$

Here $\underline{\boldsymbol{\xi}} = \frac{1}{2}(\nabla\boldsymbol{\phi} - \nabla^T\boldsymbol{\phi})$. We assume the solution $(\underline{\boldsymbol{\xi}}, \boldsymbol{\phi})$ has the elliptic regularity property

$$\parallel \underline{\boldsymbol{\psi}} \parallel_{H^s(\Omega)} + \parallel \boldsymbol{\phi} \parallel_{H^{1+s}(\Omega)} \leq C_{reg} \parallel \boldsymbol{\theta} \parallel_\Omega \tag{4.24d}$$

for some $s \geqslant 0$. This property holds, for example, with $s = 1$ in the case of planar elasticity with scalar coefficients in a convex domain; see [1].

**Lemma 4.19.** *Suppose the Assumption (C.6) is satisfied and* (4.24d) *holds with* $s = 1$, *then*

$$\parallel (\boldsymbol{\mathcal{I}} - \boldsymbol{\mathcal{P}_M})\boldsymbol{\phi} \parallel_{\Gamma_h,(h^\perp)^{-1}} \leq Ch \parallel \boldsymbol{\theta} \parallel_\Omega, \tag{4.25a}$$

$$\parallel (\boldsymbol{\mathcal{I}} - \boldsymbol{\mathcal{P}_M})\partial_{\boldsymbol{n}}\boldsymbol{\phi} \parallel_{\Gamma_h,l} \leq CRh \parallel \boldsymbol{\theta} \parallel_\Omega, \tag{4.25b}$$

$$\parallel \boldsymbol{\phi} + l\partial_{\boldsymbol{n}}\boldsymbol{\phi} \parallel_{\Gamma_h,l^{-3}} \leq C \parallel \boldsymbol{\theta} \parallel_\Omega, \tag{4.25c}$$

$$\parallel \boldsymbol{\phi} \parallel_{\Gamma_h,l^{-2}} \leq C \parallel \boldsymbol{\theta} \parallel_\Omega . \tag{4.25d}$$

*Proof.* The results follows from Lemma 5.5 in [4] applied to each component of $\boldsymbol{\phi}$. $\square$

**Proposition 4.2.** *We have*

$$(\boldsymbol{e_u}, \boldsymbol{\theta})_{\mathcal{T}_h} = (\mathcal{A}\underline{\boldsymbol{e_\sigma}}, \underline{\boldsymbol{\delta_\psi}})_{\mathcal{T}_h} + (\underline{\boldsymbol{e_\rho}}, \underline{\boldsymbol{\delta_\psi}})_{\mathcal{T}_h} - (\mathcal{A}\underline{\boldsymbol{\delta_\sigma}} + \underline{\boldsymbol{\delta_\rho}}, \underline{\boldsymbol{\Pi}}^{\boldsymbol{D}}\underline{\boldsymbol{\psi}})_{\mathcal{T}_h} + (\underline{\boldsymbol{e_\sigma}}, \underline{\boldsymbol{\delta_\xi}})_{\mathcal{T}_h} - (\underline{\boldsymbol{\delta_\sigma}}, \underline{\boldsymbol{\Pi_A \xi}})_{\mathcal{T}_h} + \mathbb{T}_{u,\sigma}$$

*where* $\underline{\boldsymbol{\delta_\xi}} = \underline{\boldsymbol{\xi}} - \underline{\boldsymbol{\Pi_A \xi}}$, $\underline{\boldsymbol{\delta_\psi}} = \underline{\boldsymbol{\psi}} - \underline{\boldsymbol{\Pi}}^{\boldsymbol{D}}\psi$, *and* $\mathbb{T}_{u,\sigma} := \langle \boldsymbol{e_{\hat{u}}}, \underline{\boldsymbol{\psi}}\boldsymbol{n}\rangle_{\Gamma_h} - \langle \underline{\boldsymbol{e_{\hat{\sigma}}}}\boldsymbol{n}, \boldsymbol{\phi}\rangle_{\Gamma_h}$,

*Proof.* By the dual equation (4.24), we can write

$$\begin{aligned}(\boldsymbol{e_u}, \boldsymbol{\theta})_{\mathcal{T}_h} =& (\boldsymbol{e_u}, \nabla\cdot\underline{\boldsymbol{\psi}})_{\mathcal{T}_h} + (\underline{\boldsymbol{e_\sigma}}, \mathcal{A}\underline{\boldsymbol{\psi}} - \nabla\boldsymbol{\phi} + \underline{\boldsymbol{\xi}})_{\mathcal{T}_h}\\ =& (\boldsymbol{e_u}, \nabla\cdot\underline{\boldsymbol{\psi}})_{\mathcal{T}_h} + (\mathcal{A}\underline{\boldsymbol{e_\sigma}}, \underline{\boldsymbol{\psi}}))_{\mathcal{T}_h} - (\underline{\boldsymbol{e_\sigma}}, \nabla\boldsymbol{\phi})_{\mathcal{T}_h} + (\underline{\boldsymbol{e_\sigma}}, \underline{\boldsymbol{\xi}})_{\mathcal{T}_h}\\ =& (\boldsymbol{e_u}, \nabla\cdot\underline{\boldsymbol{\Pi}}^{\boldsymbol{D}}\underline{\boldsymbol{\psi}})_{\mathcal{T}_h} + (\boldsymbol{e_u}, \nabla\cdot\underline{\boldsymbol{\delta_\psi}})_{\mathcal{T}_h} + (\mathcal{A}\underline{\boldsymbol{e_\sigma}}, \underline{\boldsymbol{\Pi}}^{\boldsymbol{D}}\underline{\boldsymbol{\psi}})_{\mathcal{T}_h} + (\mathcal{A}\underline{\boldsymbol{e_\sigma}}, \underline{\boldsymbol{\delta_\psi}})_{\mathcal{T}_h}\\ &- (\underline{\boldsymbol{e_\sigma}}, \nabla\boldsymbol{\Pi_W\phi})_{\mathcal{T}_h} - (\underline{\boldsymbol{e_\sigma}}, \nabla\boldsymbol{\delta_\phi}) + (\underline{\boldsymbol{e_\sigma}}, \underline{\boldsymbol{\xi}})_{\mathcal{T}_h}.\end{aligned}$$

Next, note by (4.11a) of the projection and the fact that $\boldsymbol{e_u} \in \boldsymbol{W}_h$, we have

$$(\boldsymbol{e_u}, \nabla\cdot\underline{\boldsymbol{\delta_\psi}})_{\mathcal{T}_h} = \langle \boldsymbol{e_u}, \underline{\boldsymbol{\delta_\psi}}\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} - (\nabla\boldsymbol{e_u}, \underline{\boldsymbol{\delta_\psi}})_{\mathcal{T}_h} = \langle \boldsymbol{e_u}, \underline{\boldsymbol{\delta_\psi}}\boldsymbol{n}\rangle_{\partial\mathcal{T}_h}.$$

Similarly, with the fact $\underline{\boldsymbol{e_\sigma}} \in \underline{\boldsymbol{V}}_h$ and (4.11b), we obtain

$$(\underline{\boldsymbol{e_\sigma}}, \nabla\boldsymbol{\delta_\phi})_{\mathcal{T}_h} = \langle \underline{\boldsymbol{e_\sigma}}\boldsymbol{n}, \boldsymbol{\delta_\phi}\rangle_{\partial\mathcal{T}_h} - (\nabla\cdot\underline{\boldsymbol{e_\sigma}}, \boldsymbol{\delta_\phi})_{\mathcal{T}_h} = \langle \underline{\boldsymbol{e_\sigma}}\boldsymbol{n}, \boldsymbol{\delta_\phi}\rangle_{\partial\mathcal{T}_h}.$$

Inserting these two results onto the first equation, we get

$$\begin{aligned}(\boldsymbol{e_u}, \boldsymbol{\theta})_{\mathcal{T}_h} =& (\boldsymbol{e_u}, \nabla\cdot\underline{\boldsymbol{\Pi}}^{\boldsymbol{D}}\underline{\boldsymbol{\psi}})_{\mathcal{T}_h} + (\mathcal{A}\underline{\boldsymbol{e_\sigma}}, \underline{\boldsymbol{\Pi}}^{\boldsymbol{D}}\underline{\boldsymbol{\psi}})_{\mathcal{T}_h} - (\underline{\boldsymbol{e_\sigma}}, \nabla\boldsymbol{\Pi_W\phi})_{\mathcal{T}_h}\\ &+ (\underline{\boldsymbol{e_\sigma}}, \underline{\boldsymbol{\xi}})_{\mathcal{T}_h} + (\mathcal{A}\underline{\boldsymbol{e_\sigma}}, \underline{\boldsymbol{\psi}})_{\mathcal{T}_h} + \langle \boldsymbol{e_u}, \underline{\boldsymbol{\delta_\psi}}\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} - \langle \underline{\boldsymbol{e_\sigma}}\boldsymbol{n}, \boldsymbol{\delta_\phi}\rangle_{\partial\mathcal{T}_h}.\end{aligned} \tag{4.26}$$

Taking $\underline{\boldsymbol{v}} := \underline{\boldsymbol{\Pi}}^{\boldsymbol{D}}\underline{\boldsymbol{\psi}}$, and $\boldsymbol{w} := \boldsymbol{\Pi_W\phi}$, in the error Equations (4.14a) and (4.14b) , respectively, we have

$$\begin{aligned}(\mathcal{A}\underline{\boldsymbol{e_\sigma}}, \underline{\boldsymbol{\Pi}}^{\boldsymbol{D}}\underline{\boldsymbol{\psi}})_{\mathcal{T}_h} + (\boldsymbol{e_u}, \nabla\cdot\underline{\boldsymbol{\Pi}}^{\boldsymbol{D}}\underline{\boldsymbol{\psi}})_{\mathcal{T}_h} &= -(\underline{\boldsymbol{e_\rho}}, \underline{\boldsymbol{\Pi}}^{\boldsymbol{D}}\underline{\boldsymbol{\psi}})_{\mathcal{T}_h} + \langle \boldsymbol{e_{\hat{u}}}, \underline{\boldsymbol{\Pi}}^{\boldsymbol{D}}\underline{\boldsymbol{\psi}}\rangle_{\partial\mathcal{T}_h} - (\mathcal{A}\underline{\boldsymbol{\delta_\sigma}} + \underline{\boldsymbol{\delta_\rho}}, \underline{\boldsymbol{\Pi}}^{\boldsymbol{D}}\underline{\boldsymbol{\psi}})_{\mathcal{T}_h},\\ (\underline{\boldsymbol{e_\sigma}}, \nabla\boldsymbol{\Pi_W\phi})_{\mathcal{T}_h} &= \langle \underline{\boldsymbol{e_{\hat{\sigma}}}}\boldsymbol{n}, \boldsymbol{\Pi_W\phi}\rangle_{\partial\mathcal{T}_h}.\end{aligned}$$

Replacing these last two expression in to (4.26), we obtain

$$\begin{aligned}(\boldsymbol{e_u}, \boldsymbol{\theta})_{\mathcal{T}_h} =& (\underline{\boldsymbol{e_\sigma}}, \underline{\boldsymbol{\xi}})_{\mathcal{T}_h} + (\mathcal{A}\underline{\boldsymbol{e_\sigma}}, \underline{\boldsymbol{\delta_\psi}})_{\mathcal{T}_h} - (\underline{\boldsymbol{e_\rho}}, \underline{\boldsymbol{\Pi}}^{\boldsymbol{D}}\underline{\boldsymbol{\psi}})_{\mathcal{T}_h} - (\mathcal{A}\underline{\boldsymbol{\delta_\sigma}} + \underline{\boldsymbol{\delta_\rho}}, \underline{\boldsymbol{\Pi}}^{\boldsymbol{D}}\underline{\boldsymbol{\psi}})_{\mathcal{T}_h}\\ &- \langle \boldsymbol{e_u}, \underline{\boldsymbol{\delta_\psi}}\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} - \langle \underline{\boldsymbol{e_\sigma}}\boldsymbol{n}, \boldsymbol{\delta_\phi}\rangle_{\partial\mathcal{T}_h} + \langle \boldsymbol{e_{\hat{u}}}, \underline{\boldsymbol{\Pi}}^{\boldsymbol{D}}\underline{\boldsymbol{\psi}}\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} - \langle \underline{\boldsymbol{e_{\hat{\sigma}}}}\boldsymbol{n}, \boldsymbol{\Pi_W\phi}\rangle_{\partial\mathcal{T}_h}.\end{aligned}$$

Next, note that $(\underline{\boldsymbol{e_\rho}}, \underline{\boldsymbol{\psi}})_{\mathcal{T}_h} = 0$ since $\underline{\boldsymbol{e_\rho}} \in \underline{\boldsymbol{AS}}(D_h)$ and $\underline{\boldsymbol{\psi}}$ is symmetric. Also, note that by the regularity assumption, $(\underline{\boldsymbol{\psi}}, \boldsymbol{\phi}) \in \underline{H}^1(\Omega) \times H^1(\Omega)$, so $\underline{\boldsymbol{\psi}}\boldsymbol{n}, \phi$ are single valued on each face $e \in \mathcal{E}_h$. This implies that

$$\begin{aligned}\langle \boldsymbol{e_{\hat{u}}}, \underline{\boldsymbol{\psi}}\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} &= \langle \boldsymbol{e_{\hat{u}}}, \underline{\boldsymbol{\psi}}\boldsymbol{n}\rangle_{\Gamma_h} = \langle \tilde{\boldsymbol{g}} - \tilde{\boldsymbol{g}}_h, \underline{\boldsymbol{\psi}}\boldsymbol{n}\rangle_{\Gamma_h} && \text{by (4.14e)},\\ \langle \underline{\boldsymbol{e_{\hat{\sigma}}}}\boldsymbol{n}, \boldsymbol{\phi}\rangle_{\partial\mathcal{T}_h} &= \langle \underline{\boldsymbol{e_{\hat{\sigma}}}}\boldsymbol{n}, \boldsymbol{\mathcal{P}_M\phi}\rangle_{\partial\mathcal{T}_h} = \langle \underline{\boldsymbol{e_{\hat{\sigma}}}}\boldsymbol{n}, \boldsymbol{\mathcal{P}_M\phi}\rangle_{\Gamma_h} && \text{by (4.14c) and (4.24c)}.\end{aligned}$$

Inserting these three terms onto the previous equation, we can write

$$\begin{aligned}(\underline{\boldsymbol{e_\rho}}, \underline{\boldsymbol{\Pi}}^{\boldsymbol{D}}\underline{\boldsymbol{\psi}})_{\mathcal{T}_h} &= (\underline{\boldsymbol{e_\rho}}, \underline{\boldsymbol{\Pi}}^{\boldsymbol{D}}\underline{\boldsymbol{\psi}} - \underline{\boldsymbol{\psi}})_{\mathcal{T}_h} = -(\underline{\boldsymbol{e_\rho}}, \underline{\boldsymbol{\delta_\psi}})_{\mathcal{T}_h},\\ \langle \boldsymbol{e_{\hat{u}}}, \underline{\boldsymbol{\Pi}}^{\boldsymbol{D}}\underline{\boldsymbol{\psi}}\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} &= \langle \boldsymbol{e_{\hat{u}}}, \underline{\boldsymbol{\Pi}}^{\boldsymbol{D}}\underline{\boldsymbol{\psi}}\boldsymbol{n} - \underline{\boldsymbol{\psi}}\boldsymbol{n} + \underline{\boldsymbol{\psi}}\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} = -\langle \boldsymbol{e_{\hat{u}}}, \underline{\boldsymbol{\delta_\psi}}\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} + \langle \boldsymbol{e_{\hat{u}}}, \underline{\boldsymbol{\psi}}\boldsymbol{n}\rangle_{\Gamma_h},\\ \langle \underline{\boldsymbol{e_{\hat{\sigma}}}}\boldsymbol{n}, \boldsymbol{\Pi_W\phi}\rangle_{\partial\mathcal{T}_h} &= -\langle \underline{\boldsymbol{e_{\hat{\sigma}}}}\boldsymbol{n}, \boldsymbol{\mathcal{P}_M\phi} - \boldsymbol{\Pi_W\phi}\rangle_{\partial\mathcal{T}_h} + \langle \underline{\boldsymbol{e_{\hat{\sigma}}}}\boldsymbol{n}, \boldsymbol{\Pi_W\phi}\rangle_{\Gamma_h} = -\langle \underline{\boldsymbol{e_{\hat{\sigma}}}}\boldsymbol{n}, \boldsymbol{\delta_\phi}\rangle_{\partial\mathcal{T}_h} + \langle \underline{\boldsymbol{e_{\hat{\sigma}}}}\boldsymbol{n}, \boldsymbol{\Pi_W\phi}\rangle_{\Gamma_h}.\end{aligned}$$

Therefore, we have

$$(\boldsymbol{e_u},\boldsymbol{\theta})_{\mathcal{T}_h} = (\underline{\boldsymbol{e_\sigma}},\underline{\boldsymbol{\xi}})_{\mathcal{T}_h} + (\mathcal{A}\underline{\boldsymbol{e_\sigma}},\underline{\boldsymbol{\delta_\psi}})_{\mathcal{T}_h} + (\underline{\boldsymbol{e_\rho}},\underline{\boldsymbol{\delta_\psi}})_{\mathcal{T}_h} - (\mathcal{A}\underline{\boldsymbol{\delta_\sigma}} + \underline{\boldsymbol{\delta_\rho}},\underline{\boldsymbol{\Pi}}^{\boldsymbol{D}}\underline{\boldsymbol{\psi}})_{\mathcal{T}_h} + \tilde{\mathbb{T}} + \mathbb{T}_{u,\sigma},$$

where,

$$\tilde{\mathbb{T}} := \langle \boldsymbol{e_u},\underline{\boldsymbol{\delta_\psi}}\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} - \langle \boldsymbol{e_{\hat{u}}},\underline{\boldsymbol{\delta_\psi}}\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} - \langle \underline{\boldsymbol{e_\sigma}}\boldsymbol{n},\boldsymbol{\delta_\phi}\rangle_{\partial\mathcal{T}_h} + \langle \underline{\boldsymbol{e_{\hat{\sigma}}}}\boldsymbol{n},\boldsymbol{\delta_\phi}\rangle_{\partial\mathcal{T}_h}, \qquad \mathbb{T}_{u,\sigma} := \langle \boldsymbol{e_{\hat{u}}},\underline{\boldsymbol{\psi}}\boldsymbol{n}\rangle_{\Gamma_h} - \langle \underline{\boldsymbol{e_{\hat{\sigma}}}}\boldsymbol{n},\boldsymbol{\phi}\rangle_{\Gamma_h}$$

and $\boldsymbol{\delta_\phi} := \boldsymbol{\phi} - \boldsymbol{\Pi_W}\boldsymbol{\phi}$. We only need to show that

$$(\underline{\boldsymbol{e_\sigma}},\underline{\boldsymbol{\xi}})_{\mathcal{T}_h} = (\underline{\boldsymbol{e_\sigma}},\underline{\boldsymbol{\delta_\xi}})_{\mathcal{T}_h} + (\underline{\boldsymbol{\delta_\sigma}},\underline{\boldsymbol{\Pi_A\xi}})_{\mathcal{T}_h}, \qquad \text{and} \qquad \tilde{\mathbb{T}} = 0.$$

By (4.14c), we have

$$(\underline{\boldsymbol{e_\sigma}},\underline{\boldsymbol{\xi}})_{\mathcal{T}_h} = (\underline{\boldsymbol{e_\sigma}},\underline{\boldsymbol{\delta_\xi}})_{\mathcal{T}_h} + (\underline{\boldsymbol{e_\sigma}},\underline{\boldsymbol{\Pi_A\xi}})_{\mathcal{T}_h} = (\underline{\boldsymbol{e_\sigma}},\underline{\boldsymbol{\delta_\xi}})_{\mathcal{T}_h} - (\underline{\boldsymbol{\delta_\sigma}},\underline{\boldsymbol{\Pi_A\xi}})_{\mathcal{T}_h}.$$

Let us end the proof by showing that $\tilde{\mathbb{T}} = 0$. Thanks by (4.16) we can write

$$\begin{aligned}
\mathbb{T} &= \langle \boldsymbol{e_u} - \boldsymbol{e_{\hat{u}}},\underline{\boldsymbol{\delta_\psi}}\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} - \langle \underline{\boldsymbol{e_\sigma}}\boldsymbol{n} - \underline{\boldsymbol{e_{\hat{\sigma}}}}\boldsymbol{n},\boldsymbol{\delta_\phi}\rangle_{\partial\mathcal{T}_h}\\
&= \langle \boldsymbol{e_u} - \boldsymbol{e_{\hat{u}}},\underline{\boldsymbol{\delta_\psi}}\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} - \langle \underline{\boldsymbol{e_\sigma}}\boldsymbol{n} - \underline{\boldsymbol{e_{\hat{\sigma}}}}\boldsymbol{n},\boldsymbol{\mathcal{P}_M}\boldsymbol{\phi} - \boldsymbol{\Pi_W}\boldsymbol{\phi}\rangle_{\partial\mathcal{T}_h}\\
&= \langle \boldsymbol{e_u} - \boldsymbol{e_{\hat{u}}},\underline{\boldsymbol{\delta_\psi}}\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} - \langle \alpha(\boldsymbol{e_u} - \boldsymbol{e_{\hat{u}}}),\boldsymbol{\mathcal{P}_M}\boldsymbol{\phi} - \boldsymbol{\Pi_W}\boldsymbol{\phi}\rangle_{\partial\mathcal{T}_h}\\
&= \langle \boldsymbol{e_u} - \boldsymbol{e_{\hat{u}}},\underline{\boldsymbol{\delta_\psi}}\boldsymbol{n} - \alpha(\boldsymbol{\mathcal{P}_M}\boldsymbol{\phi} - \boldsymbol{\Pi_W}\boldsymbol{\phi})\rangle_{\partial\mathcal{T}_h},\\
&= 0,
\end{aligned}$$

by property (4.11c) we obtain the result. □

**Lemma 4.20.** *We have*

$$\mathbb{T}_{u,h} = \sum_{i=1}^{14}\mathbb{T}_{u,h}^i,$$

*where,*

$$\begin{aligned}
&\mathbb{T}_{u,h}^1 = \langle(\tilde{\boldsymbol{g}} - \tilde{\boldsymbol{g}}_h)/l,\boldsymbol{\phi} + l\partial_n\boldsymbol{\phi}\rangle_{\Gamma_h}, && \mathbb{T}_{u,h}^2 = -\langle\tilde{\boldsymbol{g}} - \tilde{\boldsymbol{g}}_h,(\boldsymbol{\mathcal{I}} - \boldsymbol{\mathcal{P}_M})\partial_n\boldsymbol{\phi}\rangle_{\Gamma_h}, && \mathbb{T}_{u,h}^3 = \langle\tilde{\boldsymbol{g}} - \tilde{\boldsymbol{g}}_h,(\underline{\boldsymbol{\psi}} - \nabla\boldsymbol{\phi})\boldsymbol{n}\rangle_{\Gamma_h},\\
&\mathbb{T}_{u,h}^4 = \langle\boldsymbol{\Lambda}^{\underline{\boldsymbol{\delta_\rho}}}(\boldsymbol{x}),\boldsymbol{\phi}\rangle_{\Gamma_h}, && \mathbb{T}_{u,h}^5 = \langle\underline{\boldsymbol{\delta_\rho}}\boldsymbol{n},\boldsymbol{\phi}\rangle_{\Gamma_h}, && \mathbb{T}_{u,h}^6 = \langle\boldsymbol{\Lambda}^{\underline{\boldsymbol{e_\rho}}}(\boldsymbol{x}),\boldsymbol{\phi}\rangle_{\Gamma_h},\\
&\mathbb{T}_{u,h}^7 = \langle\underline{\boldsymbol{e_\rho}}\boldsymbol{n},\boldsymbol{\phi}\rangle_{\Gamma_h}, && \mathbb{T}_{u,h}^8 = \langle\boldsymbol{\Lambda}^{\mathcal{A}\underline{\boldsymbol{e_\sigma}}}(\boldsymbol{x}),\boldsymbol{\phi}\rangle_{\Gamma_h}, && \mathbb{T}_{u,h}^9 = -\langle(\underline{\boldsymbol{\delta_\sigma}} - \mathcal{A}\underline{\boldsymbol{\delta_\sigma}})\boldsymbol{n},\boldsymbol{\phi}\rangle_{\Gamma_h},\\
&\mathbb{T}_{u,h}^{10} = \langle\underline{\boldsymbol{\delta_\sigma}}\boldsymbol{n},(\boldsymbol{\mathcal{I}} - \boldsymbol{\mathcal{P}_M})\boldsymbol{\phi}\rangle_{\Gamma_h}, && \mathbb{T}_{u,h}^{11} = -\langle\alpha_e\boldsymbol{\mathcal{P}_M}\boldsymbol{\delta_u},\boldsymbol{\phi}\rangle_{\Gamma_h}, && \mathbb{T}_{u,h}^{12} = \langle\boldsymbol{\Lambda}^{\mathcal{A}(\underline{\boldsymbol{\delta_\sigma}})}(\boldsymbol{x}),\boldsymbol{\phi}\rangle_{\Gamma_h},\\
&\mathbb{T}_{u,h}^{13} = \langle\alpha_e(\boldsymbol{e_u} - \boldsymbol{e_{\hat{u}}}),\boldsymbol{\phi}\rangle_{\Gamma_h}, && \mathbb{T}_{u,h}^{14} = -\langle(\underline{\boldsymbol{e_\sigma}} - \mathcal{A}\underline{\boldsymbol{e_\sigma}})\boldsymbol{n},\boldsymbol{\phi}\rangle_{\Gamma_h}.
\end{aligned}$$

*Proof.* Replacing the term $\underline{\boldsymbol{e_{\hat{\sigma}}}}\boldsymbol{n}$ by (4.17) in our expression of $\mathbb{T}_{u,h}$

$$\mathbb{T}_{u,h} = \langle\boldsymbol{e_{\hat{u}}},\underline{\boldsymbol{\psi}}\boldsymbol{n}\rangle_{\Gamma_h} - \{\langle-(\tilde{\boldsymbol{g}} - \tilde{\boldsymbol{g}}_h)/l - \mathbb{T}^\rho - \boldsymbol{\Lambda}^{\mathcal{A}\underline{\boldsymbol{e_\sigma}}}(\boldsymbol{x}) - \boldsymbol{\Lambda}^{\mathcal{A}(\underline{\boldsymbol{\delta_\sigma}})}(\boldsymbol{x}) - \mathcal{A}\underline{\boldsymbol{\delta_\sigma}}\boldsymbol{n} - \alpha_e(\boldsymbol{e_u} - \boldsymbol{e_{\hat{u}}}) + (\underline{\boldsymbol{e_\sigma}} - \mathcal{A}\underline{\boldsymbol{e_\sigma}})\boldsymbol{n},\boldsymbol{\phi}\rangle_{\Gamma_h}\},$$

Now, the first term here can be rewrite thanks the dual Equation (4.24b)

$$\begin{aligned}
\langle\boldsymbol{e_{\hat{u}}},\underline{\boldsymbol{\psi}}\boldsymbol{n}\rangle_{\Gamma_h} &= \langle\boldsymbol{e_{\hat{u}}},\mathcal{A}\underline{\boldsymbol{\psi}}\boldsymbol{n}\rangle_{\Gamma_h} + \langle\boldsymbol{e_{\hat{u}}},\underline{\boldsymbol{\psi}}\boldsymbol{n} - \mathcal{A}\underline{\boldsymbol{\psi}}\boldsymbol{n}\rangle_{\Gamma_h}\\
&= \langle\boldsymbol{e_{\hat{u}}},(\nabla\boldsymbol{\phi} - \underline{\boldsymbol{\xi}})\boldsymbol{n}\rangle_{\Gamma_h} - \langle\boldsymbol{e_{\hat{u}}},(\underline{\boldsymbol{\psi}} - \nabla\boldsymbol{\phi} + \underline{\boldsymbol{\xi}})\boldsymbol{n}\rangle_{\Gamma_h}\\
&= \langle\boldsymbol{e_{\hat{u}}},\boldsymbol{\mathcal{P}_M}\partial_n\boldsymbol{\phi}\rangle_{\Gamma_h} + \langle\boldsymbol{e_{\hat{u}}},(\underline{\boldsymbol{\psi}} - \nabla\boldsymbol{\phi})\boldsymbol{n}\rangle_{\Gamma_h}.
\end{aligned}$$

Thus, replacing the last equation into the new expression of $\mathbb{T}_{u,h}$, using the fact of $\boldsymbol{e}_{\hat{\boldsymbol{u}}} = P_M(\tilde{\boldsymbol{g}} - \tilde{\boldsymbol{g}}_h)$ and adding and subtract the term $\langle \tilde{\boldsymbol{g}} - \tilde{\boldsymbol{g}}_h, \partial_n \boldsymbol{\phi} \rangle_{\Gamma_h}$, we get

$$\begin{aligned}\mathbb{T}_{u,h} = &- \langle \tilde{\boldsymbol{g}} - \tilde{\boldsymbol{g}}_h, (\boldsymbol{\mathcal{I}} - \boldsymbol{\mathcal{P}_M}\partial_n)\boldsymbol{\phi}\rangle_{\Gamma_h} + \langle \tilde{\boldsymbol{g}} - \tilde{\boldsymbol{g}}_h, (\underline{\boldsymbol{\psi}} - \nabla\boldsymbol{\phi})\boldsymbol{n}\rangle_{\Gamma_h} + \langle (\tilde{\boldsymbol{g}} - \tilde{\boldsymbol{g}}_h)/l, \boldsymbol{\phi} + l\partial_n\boldsymbol{\phi}\rangle_{\Gamma_h} \\ &+ \langle \boldsymbol{\Lambda}^{\underline{\boldsymbol{\delta_\rho}}}(\boldsymbol{x}) + \underline{\boldsymbol{\delta_\rho}}\boldsymbol{n} + \boldsymbol{\Lambda}^{\underline{\boldsymbol{e_\rho}}}(\boldsymbol{x}) + \underline{\boldsymbol{e_\rho}}\boldsymbol{n}, \boldsymbol{\phi}\rangle_{\Gamma_h} + \langle \boldsymbol{\Lambda}^{\mathcal{A}\underline{\boldsymbol{e_\sigma}}}(\boldsymbol{x}) + \mathcal{A}\underline{\boldsymbol{\delta_\sigma}}\boldsymbol{n} + \boldsymbol{\Lambda}^{\mathcal{A}(\underline{\boldsymbol{\delta_\sigma}})}(\boldsymbol{x}), \boldsymbol{\phi}\rangle_{\Gamma_h} \\ &+ \langle \alpha_e(\boldsymbol{e_u} - \boldsymbol{e}_{\hat{\boldsymbol{u}}}), \boldsymbol{\phi}\rangle_{\Gamma_h} - \langle (\underline{\boldsymbol{e_\sigma}} - \mathcal{A}\underline{\boldsymbol{e_\sigma}})\boldsymbol{n}, \boldsymbol{\phi}\rangle_{\Gamma_h}\end{aligned}$$

Now using property (4.11c) we have

$$\langle \underline{\boldsymbol{\delta_\sigma}}\boldsymbol{n}, \boldsymbol{\mu}\rangle_e = -\langle \alpha_e \boldsymbol{\delta_u}, \boldsymbol{\mu}\rangle_e, \ \text{ for all } \boldsymbol{\mu} \in \boldsymbol{M}(e) \text{ and } e \in \mathcal{E}_h.$$

Then, by definition of the projection $\boldsymbol{\mathcal{P}_M}$

$$\begin{aligned}\langle \underline{\boldsymbol{\delta_\sigma}}\boldsymbol{n}, \boldsymbol{\phi}\rangle_{\Gamma_h} &= \langle \underline{\boldsymbol{\delta_\sigma}}\boldsymbol{n}, (\boldsymbol{\mathcal{I}} - \boldsymbol{\mathcal{P}_M})\boldsymbol{\phi}\rangle_{\Gamma_h} + \langle \underline{\boldsymbol{\delta_\sigma}}\boldsymbol{n}, \boldsymbol{\mathcal{P}_M}\boldsymbol{\phi}\rangle_{\Gamma_h} \\ &= \langle \underline{\boldsymbol{\delta_\sigma}}\boldsymbol{n}, (\boldsymbol{\mathcal{I}} - \boldsymbol{\mathcal{P}_M})\boldsymbol{\phi}\rangle_{\Gamma_h} - \langle \alpha_e\boldsymbol{\delta_u}, \boldsymbol{\mathcal{P}_M}\boldsymbol{\phi}\rangle_{\Gamma_h} \\ &= \langle \underline{\boldsymbol{\delta_\sigma}}\boldsymbol{n}, (\boldsymbol{\mathcal{I}} - \boldsymbol{\mathcal{P}_M})\boldsymbol{\phi}\rangle_{\Gamma_h} - \langle \alpha_e\boldsymbol{\mathcal{P}_M}\boldsymbol{\delta_u}, \boldsymbol{\phi}\rangle_{\Gamma_h}.\end{aligned} \tag{4.27}$$

Thus, $\mathbb{T}_{u,h}$ becomes:

$$\begin{aligned}\mathbb{T}_{u,h} =&\langle (\tilde{\boldsymbol{g}} - \tilde{\boldsymbol{g}}_h)/l, \boldsymbol{\phi} + l\partial_n\boldsymbol{\phi}\rangle_{\Gamma_h} - \langle \tilde{\boldsymbol{g}} - \tilde{\boldsymbol{g}}_h, (\boldsymbol{\mathcal{I}} - \boldsymbol{\mathcal{P}_M})\boldsymbol{\phi}\rangle_{\Gamma_h} + \langle \tilde{\boldsymbol{g}} - \tilde{\boldsymbol{g}}_h, (\underline{\boldsymbol{\psi}} - \nabla\boldsymbol{\phi})\boldsymbol{n}\rangle_{\Gamma_h} \\ &+ \langle \boldsymbol{\Lambda}^{\underline{\boldsymbol{\delta_\rho}}}(\boldsymbol{x}) + \underline{\boldsymbol{\delta_\rho}}\boldsymbol{n} + \boldsymbol{\Lambda}^{\underline{\boldsymbol{e_\rho}}}(\boldsymbol{x}) + \underline{\boldsymbol{e_\rho}}\boldsymbol{n}, \boldsymbol{\phi}\rangle_{\Gamma_h} + \langle \boldsymbol{\Lambda}^{\mathcal{A}\underline{\boldsymbol{e_\sigma}}}(\boldsymbol{x}) - (\underline{\boldsymbol{\delta_\sigma}} - \mathcal{A}\underline{\boldsymbol{\delta_\sigma}})\boldsymbol{n} + \underline{\boldsymbol{\delta_\sigma}}\boldsymbol{n} + \boldsymbol{\Lambda}^{\mathcal{A}(\underline{\boldsymbol{\delta_\sigma}})}(\boldsymbol{x}), \boldsymbol{\phi}\rangle_{\Gamma_h} \\ &+ \langle \alpha_e(\boldsymbol{e_u} - \boldsymbol{e}_{\hat{\boldsymbol{u}}}), \boldsymbol{\phi}\rangle_{\Gamma_h} - \langle (\underline{\boldsymbol{e_\sigma}} - \mathcal{A}\underline{\boldsymbol{e_\sigma}})\boldsymbol{n}, \boldsymbol{\phi}\rangle_{\Gamma_h},\end{aligned}$$

Finally using (4.27) we complete the proof. □

**Lemma 4.21.** *Let us suppose Assumption (C.6) is satisfied and* (4.24d) *holds with* $s = 1$*, then*

$$\begin{aligned}|\mathbb{T}_{u,h}| \ \leq \ & Ch\left(R^2h + R + 3R^{1/2}h^{-1/2} + R_c h^{-1/2} + R_\alpha + R_T h^{-1/2}\right) |||(\underline{\boldsymbol{e_\sigma}}, \boldsymbol{e_u} - \boldsymbol{e}_{\hat{\boldsymbol{u}}}, \tilde{\boldsymbol{g}} - \tilde{\boldsymbol{g}}_h)||| \, \| \boldsymbol{\theta} \|_\Omega \\ + \ & Ch^{1/2}(R_c + R_T) \| \underline{\boldsymbol{e_\rho}} \|_{D_h} \| \boldsymbol{\theta} \|_\Omega \\ + \ & C\Big(R^{1/2}h^{1/2} \| \partial_{\boldsymbol{n}}(\underline{\boldsymbol{\delta_\rho}}\boldsymbol{n}) \|_{D_h^c,(h^\perp)^2} + R^{1/2}h^{1/2} \| \partial_{\boldsymbol{n}}(\mathcal{A}\underline{\boldsymbol{\delta_\sigma}}\boldsymbol{n}) \|_{D_h^c,(h^\perp)^2} + C_\Omega Rh \| \underline{\boldsymbol{\delta_\rho}} \|_{D_h} \\ + \ & C_\Omega h \| \underline{\boldsymbol{\delta_\sigma}} \|_{D_h} + C_\Omega R_A h \| \underline{\boldsymbol{\delta_\sigma}} \|_{D_h} + C_\Omega R_\alpha h \| \boldsymbol{\delta_u} \|_{D_h} \Big) \| \boldsymbol{\theta} \|_\Omega \,.\end{aligned}$$

*Proof.* By Lemma 3.6, we can write $\mathbb{T}_{u,h} = \sum_{i=1}^{14} \mathbb{T}^i_{u,h}$. Applying the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
&|\mathbb{T}^1_{u,h}| \leq \| \tilde{\boldsymbol{g}} - \tilde{\boldsymbol{g}}_h \|_{\Gamma_h,l} \| \boldsymbol{\phi}/l + \partial_{\boldsymbol{n}}\boldsymbol{\phi} \|_{\Gamma_h,l^{-1}}, && |\mathbb{T}^2_{u,h}| \leq \| \tilde{\boldsymbol{g}} - \tilde{\boldsymbol{g}}_h \|_{\Gamma_h,l^{-1}} \| (Id - \boldsymbol{\mathcal{P}_M})\partial_{\boldsymbol{n}}\boldsymbol{\phi} \|_{\Gamma_h,l}, \\
&|\mathbb{T}^3_{u,h}| \leq \| \tilde{\boldsymbol{g}} - \tilde{\boldsymbol{g}}_h \|_{\Gamma_h,l^{-1}} \| (\underline{\boldsymbol{\psi}} - \nabla\boldsymbol{\phi})\boldsymbol{n} \|_{\Gamma_h,l}, && |\mathbb{T}^4_{u,h}| \leq \| \boldsymbol{\Lambda}^{\underline{\boldsymbol{\delta_\rho}}} \|_{\Gamma_h,l^2} \| \boldsymbol{\phi} \|_{\Gamma_h,l^{-2}}, \\
&|\mathbb{T}^5_{u,h}| \leq \| \underline{\boldsymbol{\delta_\rho}}\boldsymbol{n} \|_{\Gamma_h,l^2} \| \boldsymbol{\phi} \|_{\Gamma_h,l^{-2}}, && |\mathbb{T}^6_{u,h}| \leq \| \boldsymbol{\Lambda}^{\underline{\boldsymbol{e_\rho}}} \|_{\Gamma_h,l^2} \| \boldsymbol{\phi} \|_{\Gamma_h,l^{-2}}, \\
&|\mathbb{T}^7_{u,h}| \leq \| \underline{\boldsymbol{e_\rho}}\boldsymbol{n} \|_{\Gamma_h,l^2} \| \boldsymbol{\phi} \|_{\Gamma_h,l^{-2}}, && |\mathbb{T}^8_{u,h}| \leq \| \boldsymbol{\Lambda}^{\mathcal{A}\underline{\boldsymbol{e_\sigma}}} \|_{\Gamma_h,l^2} \| \boldsymbol{\phi} \|_{\Gamma_h,l^{-2}}, \\
&|\mathbb{T}^9_{u,h}| \leq \| (\underline{\boldsymbol{\delta_\sigma}} - \mathcal{A}\underline{\boldsymbol{\delta_\sigma}})\boldsymbol{n} \|_{\Gamma_h,l^2} \| \boldsymbol{\phi} \|_{\Gamma_h,l^{-2}}, && |\mathbb{T}^{10}_{u,h}| \leq \| \underline{\boldsymbol{\delta_\sigma}}\boldsymbol{n} \|_{\Gamma_h,(h^\perp)} \| (\boldsymbol{\mathcal{I}} - \boldsymbol{\mathcal{P}_M})\boldsymbol{\phi} \|_{\Gamma_h,(h^\perp)^{-1}}, \\
&|\mathbb{T}^{11}_{u,h}| \leq \| \boldsymbol{\mathcal{P}_M}\boldsymbol{\delta_u} \|_{\Gamma_h,\alpha^2,l^2} \| \boldsymbol{\phi} \|_{\Gamma_h,l^{-2}}, && |\mathbb{T}^{12}_{u,h}| \leq \| \boldsymbol{\Lambda}^{\mathcal{A}(\underline{\boldsymbol{\delta_\sigma}})} \|_{\Gamma_h,l^2} \| \boldsymbol{\phi} \|_{\Gamma_h,l^{-2}}, \\
&|\mathbb{T}^{13}_{u,h}| \leq \| \boldsymbol{e_u} - \boldsymbol{e}_{\hat{\boldsymbol{u}}} \|_{\Gamma_h,\alpha^2,l^2} \| \boldsymbol{\phi} \|_{\Gamma_h,l^{-2}}, && |\mathbb{T}^{14}_{u,h}| \leq \| (\underline{\boldsymbol{e_\sigma}} - \mathcal{A}\underline{\boldsymbol{e_\sigma}})\boldsymbol{n} \|_{\Gamma_h,l^2} \| \boldsymbol{\phi} \|_{\Gamma_h,l^{-2}} \,.
\end{aligned}$$

By (4.25b),(4.25c) and the fact that

$$|l(\boldsymbol{x})| \leq H_e^\perp = r_e h_e^\perp \leq r_e h_e \leq r_e h \leq Rh \qquad \text{for all } \boldsymbol{x}, \tag{4.28}$$

we have

$$|\mathbb{T}^1_{u,h}| \leq CR^2h^2 \parallel \tilde{\boldsymbol{g}} - \tilde{\boldsymbol{g}}_h \parallel_{\Gamma_h,l^{-1}} \parallel \boldsymbol{\theta} \parallel_{\Omega}, \qquad |\mathbb{T}^2_{u,h}| \leq CRh \parallel \tilde{\boldsymbol{g}} - \tilde{\boldsymbol{g}}_h \parallel_{\Gamma_h,l^{-1}} \parallel \boldsymbol{\theta} \parallel_{\Omega} .$$

Now, since

$$\parallel (\underline{\boldsymbol{\psi}} - \nabla\boldsymbol{\phi})\boldsymbol{n} \parallel_{\Gamma_h,l} \leq R^{1/2}h^{1/2} \parallel (\underline{\boldsymbol{\psi}} - \nabla\boldsymbol{\phi})\boldsymbol{n} \parallel_{\Gamma_h} \leq CR^{1/2}h^{1/2} \left( \parallel \underline{\boldsymbol{\psi}} \parallel_{\underline{\boldsymbol{H}}^1(\Omega)} + \parallel \boldsymbol{\phi} \parallel_{\boldsymbol{H}^2(\Omega)} \right) \leq CR^{1/2}h^{1/2},$$

we get $|\mathbb{T}^3_{u,h}| \leq CR^{1/2}h^{1/2} \parallel \tilde{\boldsymbol{g}} - \tilde{\boldsymbol{g}}_h \parallel_{\Gamma_h,l^{-1}} \parallel \boldsymbol{\theta} \parallel_{\Omega}$. On the other hand, we use the estimates in Lemma 4.1, Lemma 4.11, (4.25c) and (4.28) to obtain

$$|\mathbb{T}^4_{u,h}| \leq CR^{1/2}h^{1/2} \parallel \partial_{\boldsymbol{n}}(\underline{\boldsymbol{\delta_\rho}}\boldsymbol{n}) \parallel_{D^c_h,(h^\perp)^2} \parallel \boldsymbol{\theta} \parallel_{\Omega}, \qquad |\mathbb{T}^6_{u,h}| \leq C \max_{e\in\mathcal{E}^\partial_h} r_e^2 C^e_{ext} C^e_{inv} h^{1/2} \parallel \underline{\boldsymbol{e_\rho}} \parallel_{D_h} \parallel \boldsymbol{\theta} \parallel_{\Omega},$$

$$|\mathbb{T}^8_{u,h}| \leq C \max_{e\in\mathcal{E}^\partial_h} r_e^2 C^e_{ext} C^e_{inv} C_{\mathcal{A}} h^{1/2} \parallel \underline{\boldsymbol{e_\sigma}} \parallel_{D_h} \parallel \boldsymbol{\theta} \parallel_{\Omega}, \quad |\mathbb{T}^{12}_{u,h}| \leq CR^{1/2}h^{1/2} \parallel \partial_{\boldsymbol{n}}(\mathcal{A}\underline{\boldsymbol{\delta_\sigma}}\boldsymbol{n}) \parallel_{D^c_h,(h^\perp)^2} \parallel \boldsymbol{\theta} \parallel_{\Omega} .$$

Now, using (4.25c) and (4.28) we see that

$$\begin{aligned}
|\mathbb{T}^5_{u,h}| &\leq CRh^{1/2} \parallel \underline{\boldsymbol{\delta_\rho}}\boldsymbol{n} \parallel_{\Gamma_h,(h^\perp)} \parallel \boldsymbol{\theta} \parallel_{\Omega}, \\
|\mathbb{T}^9_{u,h}| &\leq C(1+C_{\mathcal{A}})Rh^{1/2} \parallel \underline{\boldsymbol{\delta_\sigma}} \parallel_{\Gamma_h,(h^\perp)} \parallel \boldsymbol{\theta} \parallel_{\Omega}, \\
|\mathbb{T}^{10}_{u,h}| &\leq Ch \parallel \underline{\boldsymbol{\delta_\sigma}}\boldsymbol{n} \parallel_{\Gamma_h,(h^\perp)} \parallel \boldsymbol{\theta} \parallel_{\Omega} .
\end{aligned}$$

Applying (4.25c) and (4.28) we get

$$|\mathbb{T}^{11}_{u,h}| \leq C \max_{e\in\mathcal{E}^\partial_h} \alpha_e r_e h^{1/2} \parallel \boldsymbol{\delta_u} \parallel_{\Gamma_h,(h^\perp)} \parallel \boldsymbol{\theta} \parallel_{\Omega}, \qquad |\mathbb{T}^{13}_{u,h}| \leq C \max_{e\in\mathcal{E}^\partial_h} \alpha_e r_e h \parallel \boldsymbol{e_u} - \boldsymbol{e_{\hat{u}}} \parallel_{\Gamma_h,\alpha} \parallel \boldsymbol{\theta} \parallel_{\Omega} .$$

Finally, by (4.25c), (4.28) and discrete trace inequality (Lemma 4.2) we obtain

$$|\mathbb{T}^7_{u,h}| \leq C \max_{e\in\mathcal{E}^\partial_h} C^e_{tr} r_e h^{1/2} \parallel \underline{\boldsymbol{e_\rho}} \parallel_{D_h} \parallel \boldsymbol{\theta} \parallel_{\Omega}, \qquad |\mathbb{T}^{14}_{u,h}| \leq C(1+C_{\mathcal{A}}) \max_{e\in\mathcal{E}^\partial_h} C^e_{tr} r_e h^{1/2} \parallel \underline{\boldsymbol{e_\sigma}} \parallel_{D_h} \parallel \boldsymbol{\theta} \parallel_{\Omega} .$$

Then, by the definition of $|||(\underline{\boldsymbol{e_\sigma}}, \boldsymbol{e_u} - \boldsymbol{e_{\hat{u}}}, \tilde{\boldsymbol{g}} - \tilde{\boldsymbol{g}}_h)|||$, the fact

$$\begin{aligned}
\parallel \underline{\boldsymbol{\delta_\rho}}\boldsymbol{n} \parallel_{\Gamma_h,(h^\perp)} &\leq C_\Omega h^{1/2} \parallel \underline{\boldsymbol{\delta_\sigma}} \parallel_{D_h}, & \parallel \underline{\boldsymbol{\delta_\sigma}}\boldsymbol{n} \parallel_{\Gamma_h,(h^\perp)} &\leq C_\Omega h^{1/2} \parallel \underline{\boldsymbol{\delta_\sigma}} \parallel_{D_h}, \\
\parallel \underline{\boldsymbol{\delta_\sigma}} \parallel_{\Gamma_h,(h^\perp)} &\leq C_\Omega h^{1/2} \parallel \underline{\boldsymbol{\delta_\sigma}} \parallel_{D_h}, & \parallel \boldsymbol{\delta_u} \parallel_{\Gamma_h,(h^\perp)} &\leq C_\Omega h^{1/2} \parallel \boldsymbol{\delta_u} \parallel_{D_h},
\end{aligned}$$

(where $C_\Omega >$ is a constant independent of $h$) and Theorem 4.1, we obtain the result. □

**Proposition 4.3.** *We have that*

$$\parallel \boldsymbol{e_u} \parallel_{D_h} \leq h^{1/2} \Big\{ Ch^{1/2} + H_1(R,h) + H_2(R,h) + H_3(R,h) \Big\} \Theta(\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\rho}}) + CR_\alpha h \parallel \boldsymbol{\delta_u} \parallel_{D_h},$$

*where,*

$$\begin{aligned}
H_1(R,h) &:= C\left( R^2h^{3/2} + R + 3R^{1/2} + R_c + R_\alpha h^{1/2} + R_T \right), \\
H_2(R,h) &:= C\left( R_c + R_T \right), \\
H_3(R,h) &:= \left( 2R^{1/2} + C_\Omega Rh^{1/2} + C_\Omega h + C_\Omega R_A h^{1/2} \right),
\end{aligned}$$

*Moreover, if (C.6) holds, then*

$$\parallel \boldsymbol{e_u} \parallel_{D_h} \leq Ch^{1/2} \left( \{h^{1/2} + R^{1/2} + \alpha h^{1/2} R^{1/2}\} \Theta(\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\rho}}) + h^{1/2} \parallel \boldsymbol{\delta_u} \parallel_{D_h} \right).$$

*Remark.* The estimates for $\| \boldsymbol{\delta_u} \|_{D_h}$ have been obtained in (4.12b). Moreover, proceeding as in the proof of Lemma 3.8 in [4], it is possible to show that

$$\| \boldsymbol{\delta_u} \|_{\Gamma_h,(h^\perp)} \leq C \left( \| \boldsymbol{\delta_u} \|_{D_h} + h^{k+1} |\boldsymbol{u}|_{\boldsymbol{H}^{k+1}(\Omega)} \right).$$

*Proof.* Taking $\boldsymbol{\theta} = \boldsymbol{e_u}$ in Proposition 4.2, we can write

$$\begin{aligned} \| \boldsymbol{e_u} \|_{D_h}^2 =& (\mathcal{A}\underline{\boldsymbol{e_\sigma}}, \underline{\boldsymbol{\delta_\psi}})_{\mathcal{T}_h} + (\underline{\boldsymbol{e_\rho}}, \underline{\boldsymbol{\delta_\psi}})_{\mathcal{T}_h} + (\underline{\boldsymbol{e_\sigma}}, \underline{\boldsymbol{\delta_\xi}})_{\mathcal{T}_h} - (\underline{\boldsymbol{\delta_\sigma}}, \underline{\boldsymbol{\Pi_A \xi}})_{\mathcal{T}_h} - (\mathcal{A}\underline{\boldsymbol{\delta_\sigma}}, \underline{\boldsymbol{\Pi}}^{\boldsymbol{D}}\underline{\boldsymbol{\psi}})_{\mathcal{T}_h} + (\underline{\boldsymbol{\delta_\rho}}, \underline{\boldsymbol{\Pi}}^{\boldsymbol{D}}\underline{\boldsymbol{\psi}})_{\mathcal{T}_h} + \mathbb{T}_{u,h} \\ =& (\mathcal{A}\underline{\boldsymbol{e_\sigma}}, \underline{\boldsymbol{\delta_\psi}})_{\mathcal{T}_h} + (\underline{\boldsymbol{e_\rho}}, \underline{\boldsymbol{\delta_\psi}})_{\mathcal{T}_h} + (\underline{\boldsymbol{e_\sigma}}, \underline{\boldsymbol{\delta_\xi}})_{\mathcal{T}_h} + (\underline{\boldsymbol{\delta_\sigma}}, \underline{\boldsymbol{\delta_\xi}})_{\mathcal{T}_h} + (\mathcal{A}\underline{\boldsymbol{\delta_\sigma}}, \underline{\boldsymbol{\delta_\psi}})_{\mathcal{T}_h} + (\underline{\boldsymbol{\delta_\rho}}, \underline{\boldsymbol{\delta_\psi}})_{\mathcal{T}_h} \\ &- (\underline{\boldsymbol{\delta_\sigma}}, \underline{\boldsymbol{\xi}})_{\mathcal{T}_h} - (\mathcal{A}\underline{\boldsymbol{\delta_\sigma}}, \underline{\boldsymbol{\psi}})_{\mathcal{T}_h} - (\underline{\boldsymbol{\delta_\rho}}, \underline{\boldsymbol{\psi}})_{\mathcal{T}_h} + \mathbb{T}_{u,h}. \end{aligned}$$

Using the dual Equation (4.24b), and the fact that $\underline{\boldsymbol{\delta_\rho}}$ is antisymmetric and $\underline{\boldsymbol{\psi}}$ is symmetric, we have $(\underline{\boldsymbol{\delta_\rho}}, \underline{\boldsymbol{\psi}})_{\mathcal{T}_h} = 0$. Next, note that $(\mathcal{A}\underline{\boldsymbol{\delta_\sigma}}, \underline{\boldsymbol{\psi}})_{\mathcal{T}_h} + (\underline{\boldsymbol{\delta_\sigma}}, \underline{\boldsymbol{\xi}})_{\mathcal{T}_h} = \overline{(\underline{\boldsymbol{\delta_\sigma}}, \mathcal{A}\underline{\boldsymbol{\psi}} + \underline{\boldsymbol{\xi}})_{\mathcal{T}_h}} = (\underline{\boldsymbol{\delta_\sigma}}, \nabla\boldsymbol{\phi})_{\mathcal{T}_h}$.Then, by the property (4.11a) with $\underline{\boldsymbol{v}} := \underline{\boldsymbol{P}}_0 \nabla\boldsymbol{\phi}$ (since $k \geq 1$) we get $(\underline{\boldsymbol{\delta_\sigma}}, \underline{\boldsymbol{P}}_0 \nabla\boldsymbol{\phi})_{\mathcal{T}_h} = 0$. Here, $\underline{\boldsymbol{P}}_0$ is the $L^2$ projection onto $\underline{\boldsymbol{P}}_0(K)$ on each $K \in \mathcal{T}_h$, then

$$\begin{aligned} \| \boldsymbol{e_u} \|_{D_h}^2 &= (\mathcal{A}\underline{\boldsymbol{e_\sigma}}, \underline{\boldsymbol{\delta_\psi}})_{\mathcal{T}_h} + (\underline{\boldsymbol{e_\rho}}, \underline{\boldsymbol{\delta_\psi}})_{\mathcal{T}_h} + (\underline{\boldsymbol{e_\sigma}}, \underline{\boldsymbol{\delta_\xi}})_{\mathcal{T}_h} + (\underline{\boldsymbol{\delta_\sigma}}, \underline{\boldsymbol{\delta_\xi}})_{\mathcal{T}_h} \\ &\quad + (\mathcal{A}\underline{\boldsymbol{\delta_\sigma}}, \underline{\boldsymbol{\delta_\psi}})_{\mathcal{T}_h} + (\underline{\boldsymbol{\delta_\rho}}, \underline{\boldsymbol{\delta_\psi}})_{\mathcal{T}_h} - (\underline{\boldsymbol{\delta_\sigma}}, \nabla\boldsymbol{\phi})_{\mathcal{T}_h} + \mathbb{T}_{u,h} \\ &= (\mathcal{A}\underline{\boldsymbol{e_\sigma}}, \underline{\boldsymbol{\delta_\psi}})_{\mathcal{T}_h} + (\underline{\boldsymbol{e_\rho}}, \underline{\boldsymbol{\delta_\psi}})_{\mathcal{T}_h} + (\underline{\boldsymbol{e_\sigma}}, \underline{\boldsymbol{\delta_\xi}})_{\mathcal{T}_h} + (\underline{\boldsymbol{\delta_\sigma}}, \underline{\boldsymbol{\delta_\xi}})_{\mathcal{T}_h} \\ &\quad + (\mathcal{A}\underline{\boldsymbol{\delta_\sigma}}, \underline{\boldsymbol{\delta_\psi}})_{\mathcal{T}_h} + (\underline{\boldsymbol{\delta_\rho}}, \underline{\boldsymbol{\delta_\psi}})_{\mathcal{T}_h} - (\underline{\boldsymbol{\delta_\sigma}}, \nabla\boldsymbol{\phi} - \underline{\boldsymbol{P}}_0 \nabla\boldsymbol{\phi})_{\mathcal{T}_h} + \mathbb{T}_{u,h}. \end{aligned}$$

Applying the Cauchy-Schwarz inequality, we obtain

$$\| \boldsymbol{e_u} \|_{D_h}^2 \leq C \left( \| \underline{\boldsymbol{e_\sigma}} \|_{D_h} + \| \underline{\boldsymbol{e_\rho}} \|_{D_h} + \| \underline{\boldsymbol{\delta_\sigma}} \|_{D_h} + \| \underline{\boldsymbol{\delta_\rho}} \|_{D_h} \right) \left( \| \underline{\boldsymbol{\delta_\psi}} \|_{\Omega} + \| \underline{\boldsymbol{\delta_\xi}} \|_{\Omega} + \| \nabla\boldsymbol{\phi} - P_0 \nabla\boldsymbol{\phi} \|_{\Omega} \right) + |\mathbb{T}_{u,h}|.$$

we note that, by (4.12a)

$$\| \underline{\boldsymbol{\delta_\xi}} \|_{\Omega} \leq Ch |\underline{\boldsymbol{\xi}}|_{H^1(\Omega)} \qquad \text{and} \qquad \| \underline{\boldsymbol{\delta_\psi}} \|_{\Omega} \leq Ch |\underline{\boldsymbol{\psi}}|_{H^1(\Omega)}.$$

Then, using Lemma 4.21 with $\boldsymbol{\theta} = \boldsymbol{e_u}$,

$$\begin{aligned} \| \boldsymbol{e_u} \|_{D_h}^2 &\leq Ch \left( \| \underline{\boldsymbol{e_\sigma}} \|_{D_h} + \| \underline{\boldsymbol{e_\rho}} \|_{D_h} + \| \underline{\boldsymbol{\delta_\sigma}} \|_{D_h} + \| \underline{\boldsymbol{\delta_\rho}} \|_{D_h} \right) (|\underline{\boldsymbol{\psi}}|_{H^1(\Omega)} + |\underline{\boldsymbol{\xi}}|_{H^1(\Omega)}) + |\mathbb{T}_{u,h}| \\ &\leq Ch \left( \| \underline{\boldsymbol{e_\sigma}} \|_{D_h} + \| \underline{\boldsymbol{e_\rho}} \|_{D_h} + \| \underline{\boldsymbol{\delta_\sigma}} \|_{D_h} + \| \underline{\boldsymbol{\delta_\rho}} \|_{D_h} \right) \| \boldsymbol{e_u} \|_{D_h} \\ &+ Ch \left( R^2 h + R + 3R^{1/2} h^{-1/2} + R_c h^{-1/2} + R_\alpha + R_T h^{-1/2} \right) |||(\underline{\boldsymbol{e_\sigma}}, \boldsymbol{e_u} - \boldsymbol{e_{\hat{u}}}, \tilde{\boldsymbol{g}} - \tilde{\boldsymbol{g}}_h)||| \, \| \boldsymbol{e_u} \|_{D_h} \\ &+ Ch^{1/2} (R_c + R_T) \| \underline{\boldsymbol{e_\rho}} \|_{D_h} \| \boldsymbol{e_u} \|_{D_h} \\ &+ C \Big( R^{1/2} h^{1/2} \| \partial_{\boldsymbol{n}} (\underline{\boldsymbol{\delta_\rho}} \boldsymbol{n}) \|_{D_h^c,(h^\perp)^2} + R^{1/2} h^{1/2} \| \partial_{\boldsymbol{n}} (\mathcal{A}\underline{\boldsymbol{\delta_\sigma}} \boldsymbol{n}) \|_{D_h^c,(h^\perp)^2} + C_\Omega R h \| \underline{\boldsymbol{\delta_\rho}} \|_{D_h} \\ &+ C_\Omega h \| \underline{\boldsymbol{\delta_\sigma}} \|_{D_h} + C_\Omega R_A h \| \underline{\boldsymbol{\delta_\sigma}} \|_{D_h} + C_\Omega R_\alpha h \| \boldsymbol{\delta_u} \|_{D_h} \Big) \| \boldsymbol{e_u} \|_{D_h}. \end{aligned}$$

Finally using the definition $|||(\underline{\boldsymbol{e_\sigma}}, \boldsymbol{e_u} - \boldsymbol{e_{\hat{u}}}, \tilde{\boldsymbol{g}} - \tilde{\boldsymbol{g}}_h)|||$ and Theorem 4.1, we get

$$\begin{aligned} \| \boldsymbol{e_u} \|_{D_h}^2 &\leq Ch\Theta(\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\rho}}) \| \boldsymbol{e_u} \|_{D_h} + H_1(R,h)\Theta(\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\rho}}) \| \boldsymbol{e_u} \|_{D_h} + H_2(R,h)\Theta(\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\rho}}) \| \boldsymbol{e_u} \|_{D_h} \\ &+ H_3(R,h)\Theta(\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\rho}}) \| \boldsymbol{e_u} \|_{D_h} + C_\Omega R_\alpha h \| \boldsymbol{\delta_u} \|_{D_h} \| \boldsymbol{e_u} \|_{D_h}. \end{aligned}$$

By a simple rearrangement we obtain the result. □

**Theorem 4.22.** *Let us suppose the same assumptions as in Proposition 4.3, then*

$$\| \boldsymbol{u} - \boldsymbol{u}_h \|_{D_h} \leq C \left( \Theta(\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\rho}}) + \| \boldsymbol{\delta_u} \|_{\Gamma_h,(h^\perp)} \right).$$

*Moreover, if* $r_e = \mathcal{O}(h)$*, then*

$$\| \boldsymbol{e_{\hat{u}}} \|_h \leq Ch \left( \Theta(\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\rho}}) + \| \boldsymbol{\delta_u} \|_{D_h} \right)$$

*and if* $r_e = \mathcal{O}(1)$*, then*

$$\| \boldsymbol{e_{\hat{u}}} \|_h \leq Ch^{1/2} \left( \Theta(\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\rho}}) + \| \boldsymbol{\delta_u} \|_{D_h} \right).$$

*Here,* $\| \boldsymbol{e_{\hat{u}}} \|_h := \left( \sum_{K \in \mathcal{T}_h} h_K \| \boldsymbol{e_{\hat{u}}} \|^2_{\partial K} \right)^{1/2}$.

*Proof.* Since $\boldsymbol{u} - \boldsymbol{u}_h = \boldsymbol{e_u} + \boldsymbol{\delta_u}$ , then the result is a direct consequence of triangle inequality and Proposition 4.3:

$$\begin{aligned} \| \boldsymbol{u} - \boldsymbol{u}_h \|_{D_h} \leq \| \boldsymbol{e_u} \|_{D_h} + \| \boldsymbol{\delta_u} \|_{D_h} &\leq Ch^{1/2} \Theta(\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\rho}}) + (Ch + 1) \| \boldsymbol{\delta_u} \|_{D_h} \\ &\leq C \left( \{ h^{1/2} + R^{1/2} + \alpha h^{1/2} R^{1/2} \} \Theta(\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\rho}}) + \| \boldsymbol{\delta_u} \|_{D_h} \right). \end{aligned}$$

Moreover, if $k \geq 1$, the estimate of $\| \boldsymbol{e_{\hat{u}}} \|_h$ follows from standard arguments in HDG. See, for instance [2]. □

Chapter 5

# Numerical experiments

In this section we present numerical experiments for method HDG method (3.2) in the two-dimensional case. For all the computations we consider the spaces defined in (2.2) with $k \in \{1, 2, 3\}$ and consider the exact solution as

$$\boldsymbol{u} = \begin{pmatrix} \sin(\pi x)\cos(\pi y) \\ \cos(\pi x)\sin(\pi y) \end{pmatrix}.$$

We fix $E = 1$ and take $\nu \in \{0.3, 0.4999\}$ in order to see the effect of the nearly incompressible case. Here, we obtain the following values for $\mu$ and $\lambda$ in the next table:

| $\nu$ | $\mu$ | $\lambda$ |
|---|---|---|
| 0.300 | 0.3846 | 0.5769 |
| 0.4999 | 0.3333 | 1666.444 |

Let us discuss the choice of the stabilization parameter $\alpha$. From the estimates we know that it should be of order one, so in some of the experiments we will take $\alpha = 1$. On the other hand, let $L$ be a characteristic length. Since $\underline{\boldsymbol{\sigma}} = \mathcal{A}^{-1}(\underline{\boldsymbol{\epsilon}}(\boldsymbol{u}))$, (3.2g) suggests that $L\alpha$ should be of the same order of $\| \mathcal{A}^{-1} \|$. In this case, $\| \mathcal{A}^{-1} \|_{\underline{\boldsymbol{L}}^2(\Omega)} \leq C(\mu + \lambda)$. We consider for the examples using three different domains: square domain, circular domain and kidney-shaped domain.

## 5.1 Polygonal Domain

In order to validate the code, we begin by considering a square domain $\Omega :=] - 1, 1[^2$ and a uniformly refined family of triangulations as Figure 5.1 shows. In this case, the Assumptions C and S are all satisfied because $r_e = 0$. Table 5.1 displays the history of convergence of the method considering $k \in \{1, 2, 3\}$, $N \in \{4, 16, 64, 256, 1024\}$, $\nu = 0.3$ and $\alpha = 1$. We see that the $L^2$-errors in $\boldsymbol{u}$ and $\underline{\boldsymbol{\sigma}}$ goes to zero with rate of order $k + 1$ and $\hat{\boldsymbol{u}}_h$ converges to $\boldsymbol{\mathcal{P}_M u}$ with order $k + 2$, as Theorems 4.3 and 4.1 predict. We show in Figure 5.2 the approximation of the first of component of $\boldsymbol{u}$ obtain in the meshes of 5.1 and $k = 1, 2, 3$.

We take now $\nu = 0.4999$. We consider $\alpha = 1$ (Table 5.2) and $\alpha = \lambda$ (Table 5.3). Here, we observe optimal order of converges for all the variables. However, the errors in $\boldsymbol{u}$ and $\hat{\boldsymbol{u}}_h$ considering $\alpha = \lambda$ are

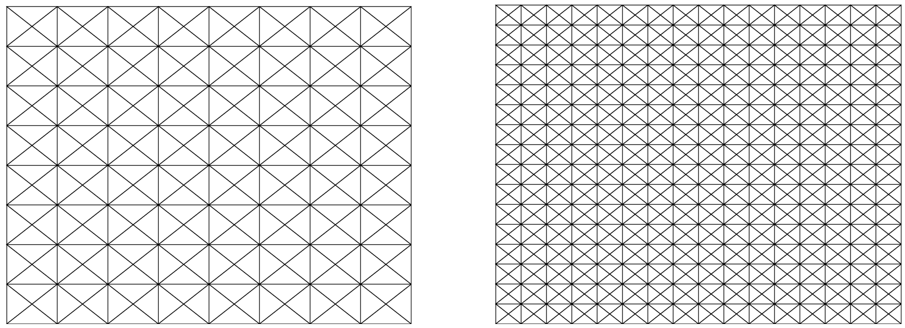


Figure 5.1: Meshes with $N = 256$ and $1024$ elements.

| $k$ | $N$ | $\Vert \boldsymbol{u} - \boldsymbol{u}_h \Vert_{D_h}$ | $r(\boldsymbol{u})$ | $\Vert \underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}_h \Vert_{D_h}$ | $r(\underline{\boldsymbol{\sigma}})$ | $\Vert \underline{\boldsymbol{\rho}} - \underline{\boldsymbol{\rho}}_h \Vert_{D_h}$ | $r(\underline{\boldsymbol{\rho}})$ | $\Vert \boldsymbol{\mathcal{P}_M u} - \hat{\boldsymbol{u}}_h \Vert_h$ | $r(\hat{\boldsymbol{u}}_h)$ |
|---|---|---|---|---|---|---|---|---|---|
| | 64 | $3.61E-01$ | – | $8.12E-01$ | – | $4.02E-01$ | – | $7.70E-02$ | – |
| | 256 | $9.36E-02$ | 1.95 | $2.11E-01$ | 1.94 | $1.09E-01$ | 1.89 | $8.54E-03$ | 3.17 |
| 1 | 1024 | $2.35E-02$ | 1.99 | $5.37E-02$ | 1.98 | $2.72E-02$ | 2.00 | $8.18E-04$ | 3.38 |
| | 4096 | $5.87E-03$ | 2.00 | $1.35E-02$ | 1.99 | $6.71E-03$ | 2.02 | $7.45E-05$ | 3.46 |
| | 16384 | $1.47E-03$ | 2.00 | $3.39E-03$ | 2.00 | $1.66E-03$ | 2.01 | $6.66E-06$ | 3.48 |
| | 64 | $4.63E-02$ | – | $1.06E-01$ | – | $5.19E-02$ | – | $6.96E-03$ | – |
| | 256 | $5.96E-03$ | 2.96 | $1.35E-02$ | 2.97 | $6.49E-03$ | 3.00 | $3.09E-04$ | 4.49 |
| 2 | 1024 | $7.50E-04$ | 2.99 | $1.70E-03$ | 2.99 | $8.32E-04$ | 2.96 | $1.38E-05$ | 4.49 |
| | 4096 | $9.39E-05$ | 3.00 | $2.13E-04$ | 3.00 | $1.06E-04$ | 2.97 | $6.13E-07$ | 4.49 |
| | 16384 | $1.17E-05$ | 3.00 | $2.66E-05$ | 3.00 | $1.34E-05$ | 2.98 | $2.73E-08$ | 4.49 |
| | 64 | $4.53E-03$ | – | $1.02E-02$ | – | $5.59E-03$ | – | $5.43E-04$ | – |
| | 256 | $2.90E-04$ | 3.97 | $6.54E-04$ | 3.97 | $3.75E-04$ | 3.90 | $1.35E-05$ | 5.33 |
| 3 | 1024 | $1.83E-05$ | 3.99 | $4.11E-05$ | 3.99 | $2.38E-05$ | 3.98 | $3.12E-07$ | 5.43 |
| | 4096 | $1.14E-06$ | 4.00 | $2.58E-06$ | 4.00 | $1.49E-06$ | 4.00 | $7.02E-09$ | 5.48 |
| | 16384 | $7.14E-08$ | 4.00 | $1.61E-07$ | 4.00 | $9.30E-08$ | 4.00 | $1.56E-10$ | 5.49 |

Table 5.1: History of convergence of polygonal domain with $\nu = 0.3$ and $\alpha = 1$.

smaller to that of $\alpha = 1$. This choice of taking the stabilization parameter of the order $\Vert \mathcal{A}^{-1} \Vert_{\underline{\boldsymbol{L}}^2(\Omega)}$ seems to improve the approximation of $\boldsymbol{u}$.

| $k$ | $N$ | $\Vert \boldsymbol{u} - \boldsymbol{u}_h \Vert_{D_h}$ | $r(\boldsymbol{u})$ | $\Vert \underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}_h \Vert_{D_h}$ | $r(\underline{\boldsymbol{\sigma}})$ | $\Vert \underline{\boldsymbol{\rho}} - \underline{\boldsymbol{\rho}}_h \Vert_{D_h}$ | $r(\underline{\boldsymbol{\rho}})$ | $\Vert \boldsymbol{\mathcal{P}_M u} - \hat{\boldsymbol{u}}_h \Vert_h$ | $r(\hat{\boldsymbol{u}}_h)$ |
|---|---|---|---|---|---|---|---|---|---|
| | 64 | $4.42E+02$ | – | $1.20E+03$ | – | $4.70E+02$ | – | $1.23E+02$ | – |
| | 256 | $1.13E+02$ | 1.96 | $3.14E+02$ | 1.93 | $1.33E+02$ | 1.82 | $1.42E+01$ | 3.12 |
| 1 | 1024 | $2.83E+01$ | 2.00 | $8.02E+01$ | 1.97 | $3.23E+01$ | 2.04 | $1.42E+00$ | 3.32 |
| | 4096 | $7.03E+00$ | 2.01 | $2.02E+01$ | 1.99 | $7.58E+00$ | 2.09 | $1.31E-01$ | 3.43 |
| | 16384 | $1.75E+00$ | 2.01 | $5.06E+00$ | 2.00 | $1.80E+00$ | 2.08 | $1.18E-02$ | 3.47 |
| | 64 | $5.62E+01$ | – | $1.56E+02$ | – | $5.97E+01$ | – | $1.14E+01$ | – |
| | 256 | $7.16E+00$ | 2.97 | $1.94E+01$ | 3.00 | $5.15E+00$ | 3.53 | $4.66E-01$ | 4.61 |
| 2 | 1024 | $8.97E-01$ | 3.00 | $2.43E+00$ | 3.00 | $5.38E-01$ | 3.26 | $1.97E-02$ | 4.56 |
| | 4096 | $1.12E-01$ | 3.00 | $3.04E-01$ | 3.00 | $6.27E-02$ | 3.10 | $8.61E-04$ | 4.52 |
| | 16384 | $1.40E-02$ | 3.00 | $3.80E-02$ | 3.00 | $7.62E-03$ | 3.04 | $3.79E-05$ | 4.50 |
| | 64 | $5.48E+00$ | – | $1.52E+01$ | – | $5.97E+00$ | – | $8.78E-01$ | – |
| | 256 | $3.48E-01$ | 3.98 | $9.78E-01$ | 3.96 | $4.17E-01$ | 3.84 | $2.34E-02$ | 5.23 |
| 3 | 1024 | $2.18E-02$ | 4.00 | $6.16E-02$ | 3.99 | $2.57E-02$ | 4.02 | $5.59E-04$ | 5.39 |
| | 4096 | $1.36E-03$ | 4.00 | $3.86E-03$ | 4.00 | $1.56E-03$ | 4.04 | $1.27E-05$ | 5.46 |
| | 16384 | $8.49E-05$ | 4.00 | $2.41E-04$ | 4.00 | $9.51E-05$ | 4.03 | $2.82E-07$ | 5.49 |

Table 5.2: History of convergence of polygonal domain with $\nu = 0.4999$ and $\alpha = 1$.

| $k$ | $N$ | $\Vert \boldsymbol{u}-\boldsymbol{u}_h \Vert_{D_h}$ | $r(\boldsymbol{u})$ | $\Vert \underline{\boldsymbol{\sigma}}-\underline{\boldsymbol{\sigma}}_h \Vert_{D_h}$ | $r(\underline{\boldsymbol{\sigma}})$ | $\Vert \underline{\boldsymbol{\rho}}-\underline{\boldsymbol{\rho}}_h \Vert_{D_h}$ | $r(\underline{\boldsymbol{\rho}})$ | $\Vert \boldsymbol{\mathcal{P}_M u}-\hat{\boldsymbol{u}}_h \Vert_h$ | $r(\hat{\boldsymbol{u}}_h)$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 64 | $9.80E-01$ | – | $1.09E+03$ | – | $6.22E+00$ | – | $2.32E+00$ | – |
| | 256 | $1.67E-01$ | 2.55 | $2.78E+02$ | 1.97 | $1.80E+00$ | 1.79 | $2.50E-01$ | 3.21 |
| | 1024 | $4.16E-02$ | 2.00 | $6.93E+01$ | 2.00 | $7.82E-01$ | 1.21 | $4.45E-02$ | 2.49 |
| | 4096 | $1.03E-02$ | 2.01 | $1.72E+01$ | 2.01 | $3.48E-01$ | 1.17 | $7.83E-03$ | 2.50 |
| | 16384 | $2.49E-03$ | 2.05 | $4.26E+00$ | 2.01 | $1.53E-01$ | 1.18 | $1.32E-03$ | 2.57 |
| 2 | 64 | $1.15E-01$ | – | $1.44E+02$ | – | $1.41E+00$ | – | $2.79E-01$ | – |
| | 256 | $1.49E-02$ | 2.95 | $1.75E+01$ | 3.04 | $3.19E-01$ | 2.14 | $2.39E-02$ | 3.54 |
| | 1024 | $1.66E-03$ | 3.17 | $2.16E+00$ | 3.01 | $6.35E-02$ | 2.33 | $1.79E-03$ | 3.74 |
| | 4096 | $1.78E-04$ | 3.22 | $2.70E-01$ | 3.00 | $1.24E-02$ | 2.36 | $1.31E-04$ | 3.77 |
| | 16384 | $1.87E-05$ | 3.25 | $3.37E-02$ | 3.00 | $2.31E-03$ | 2.42 | $9.34E-06$ | 3.81 |
| 3 | 64 | $9.97E-03$ | – | $1.38E+01$ | – | $1.65E-01$ | – | $2.54E-02$ | – |
| | 256 | $5.93E-04$ | 4.07 | $8.81E-01$ | 3.97 | $1.91E-02$ | 3.10 | $1.09E-03$ | 4.54 |
| | 1024 | $3.60E-05$ | 4.04 | $5.50E-02$ | 4.00 | $2.17E-03$ | 3.14 | $4.84E-05$ | 4.50 |
| | 4096 | $2.09E-06$ | 4.10 | $3.41E-03$ | 4.01 | $2.31E-04$ | 3.23 | $2.00E-06$ | 4.60 |
| | 16384 | $1.16E-07$ | 4.17 | $2.12E-04$ | 4.01 | $2.28E-05$ | 3.34 | $7.55E-08$ | 4.72 |

Table 5.3: History of convergence of polygonal domain with $\nu = 0.4999$ and $\alpha = \lambda$.

## 5.2 Non polygonal domain

### 5.2.1 Example 1

We consider the domain as $\Omega := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ and the computational domain is constructed by linearly interpolating the boundary of $\Omega$ as Figure 5.3 shows. In this case, $r_e$ is of order $h$ , then Assumptions S and C are satisfied for $h$ small enough even in the nearly incompressible case. In Tables 5.4 and 5.5 we displays the history of convergences for $\nu = 0.3$ $(\alpha = 1)$ and $\nu = 0.4999$ $(\alpha = \lambda)$. We observe that the $L^2$-errors of $\boldsymbol{u}$, $\underline{\boldsymbol{\sigma}}$, $\underline{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{u}}_h$ behave as Theorems 4.3 and 4.1 predict. We show in Figure 5.4 the approximation the first component of $\boldsymbol{u}$ obtained with the meshes of 5.3 and $k = 1, 2, 3$.

### 5.2.2 Example 2

We consider the same domain as in Example 1, i.e., $\Omega := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$, but we construct the computational domain and transferring path according to the procedure described in Section 2.1 and 2.2. Figure 5.5 shows two consecutive meshes. In this case $r_e$ is of order one, then there is no guaranty that Assumptions C and S hold. However, we want to explore the capabilities of the method in this setting. For $\nu = 0.3$ and $\alpha = 1$ we see in Table 5.6 that order of convergence for $\boldsymbol{u}, \underline{\boldsymbol{\sigma}}$ and $\underline{\boldsymbol{\rho}}$ is of order $k + 1$ as predicted by Theorems 4.1 and 4.3. In addition, the error of $\hat{\boldsymbol{u}}_h$ converges with order $k + 2$ which is half a power more than estimate in Theorem 4.3. In Figure 5.5 we show two consecutive uniformly refined meshes and in Figure 5.6 we display the corresponding approximation of the first component of $\boldsymbol{u}$ for different polynomial degree.

We repeated the experiments (not reported here) with $\nu = 0.4999$ and $\alpha = \lambda$ but it not was possible to draw any conclusion about the convergence of the method. We point out that the ellipticity constant $C_{el}$ is small in this case, then (S.2),(S.3),(C.3) and (C.4) are not satisfied, which explains the bad behaviour observed when $\nu = 0.4999$.

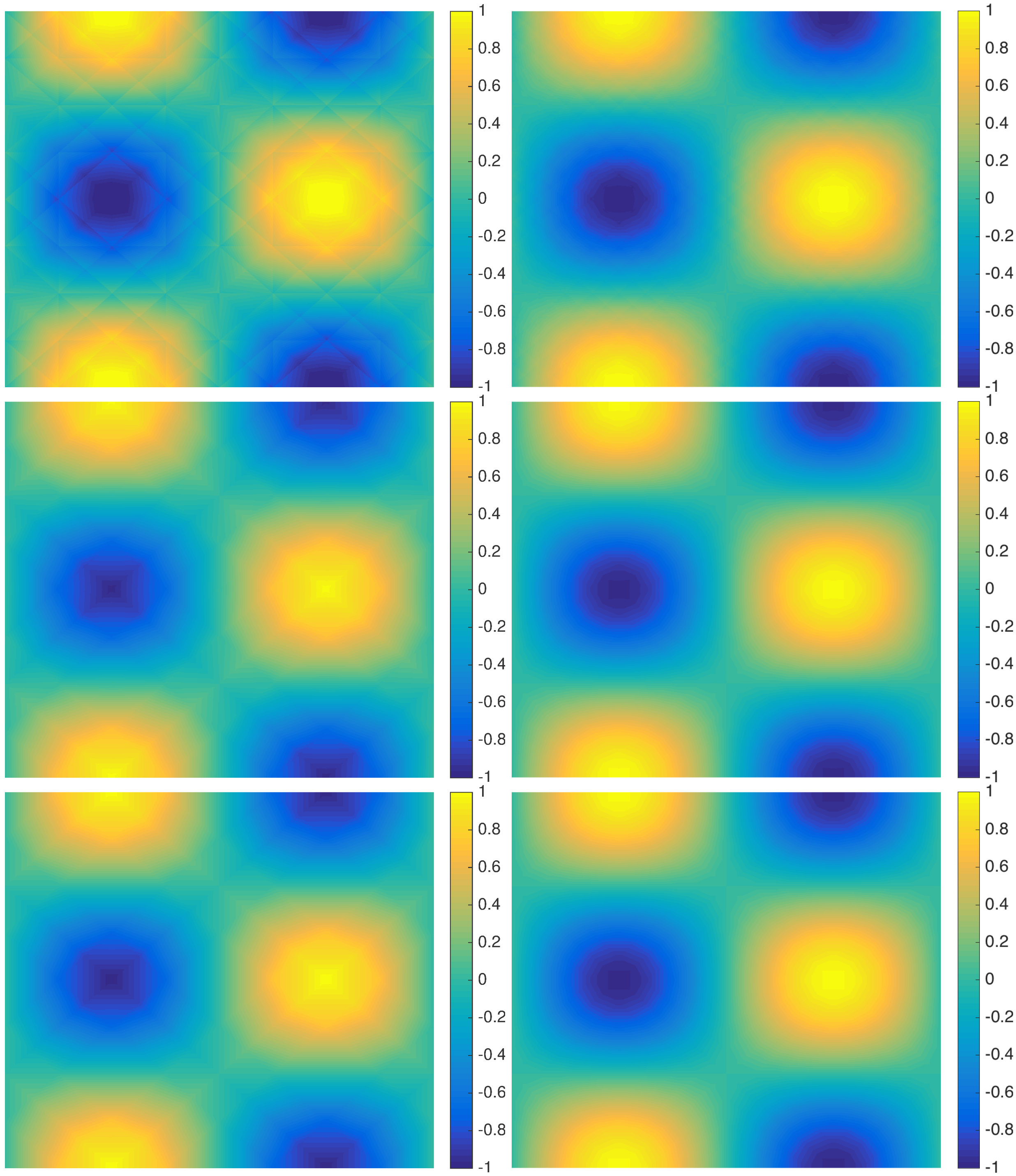



Figure 5.2: Approximation of the first component of $\boldsymbol{u}$. Columns: meshes with $N = 256$ and $1024$ elements. Rows: Polynomial degree $k = 1, 2$ and $3$.

### 5.2.3 Example 3

Finally we consider a kidney-shape domain where its level set is defined by

$$2\left((x+(1/2))^2+y^2\right)-x-(1/2)\big)^2-\left((x+(1/2)^2+y^2)\right)+0.1=0.$$

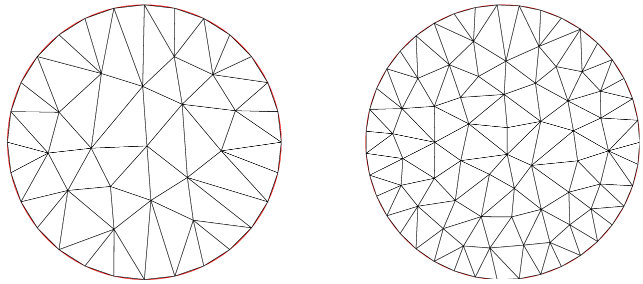


Figure 5.3: Mesh of Example 1 with $N = 60$ and $234$ elements.

| $k$ | $N$ | $\| \boldsymbol{u} - \boldsymbol{u}_h \|_{D_h}$ | $r(\boldsymbol{u})$ | $\| \underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}_h \|_{D_h}$ | $r(\underline{\boldsymbol{\sigma}})$ | $\| \underline{\boldsymbol{\rho}} - \underline{\boldsymbol{\rho}}_h \|_{D_h}$ | $r(\underline{\boldsymbol{\rho}})$ | $\| \boldsymbol{\mathcal{P}_M u} - \hat{\boldsymbol{u}}_h \|_h$ | $r(\hat{\boldsymbol{u}}_h)$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 60 | $1.29E-01$ | – | $2.89E-01$ | – | $1.26E-01$ | – | $2.16E-02$ | – |
| | 234 | $3.18E-02$ | 1.78 | $6.38E-02$ | 1.92 | $2.87E-02$ | 1.87 | $1.37E-03$ | 3.50 |
| | 918 | $8.10E-03$ | 2.39 | $1.62E-02$ | 2.39 | $7.53E-03$ | 2.33 | $1.26E-04$ | 4.16 |
| | 3546 | $2.13E-03$ | 1.96 | $4.25E-03$ | 1.97 | $1.94E-03$ | 1.99 | $1.20E-05$ | 3.46 |
| | 14291 | $5.24E-04$ | 2.10 | $1.05E-03$ | 2.09 | $4.80E-04$ | 2.09 | $1.07E-06$ | 3.62 |
| | 56687 | $1.32E-04$ | 2.08 | $2.65E-04$ | 2.08 | $1.22E-04$ | 2.06 | $9.49E-08$ | 3.65 |
| 2 | 60 | $1.40E-02$ | – | $3.65E-02$ | – | $1.58E-02$ | – | $1.87E-03$ | – |
| | 234 | $1.67E-03$ | 2.70 | $3.32E-03$ | 3.04 | $1.22E-03$ | 3.25 | $4.32E-05$ | 4.79 |
| | 918 | $2.09E-04$ | 3.62 | $4.27E-04$ | 3.58 | $1.56E-04$ | 3.58 | $2.12E-06$ | 5.26 |
| | 3546 | $2.76E-05$ | 2.97 | $5.71E-05$ | 2.95 | $2.33E-05$ | 2.79 | $1.07E-07$ | 4.37 |
| | 14291 | $3.48E-06$ | 3.09 | $7.36E-06$ | 3.07 | $3.05E-06$ | 3.04 | $5.01E-09$ | 4.58 |
| | 56687 | $4.35E-07$ | 3.14 | $9.12E-07$ | 3.15 | $3.78E-07$ | 3.15 | $2.17E-10$ | 4.73 |
| 3 | 60 | $1.30E-03$ | – | $3.34E-03$ | – | $1.33E-03$ | – | $1.38E-04$ | – |
| | 234 | $6.44E-05$ | 3.82 | $1.40E-04$ | 4.03 | $5.63E-05$ | 4.02 | $1.63E-06$ | 5.64 |
| | 918 | $4.12E-06$ | 4.79 | $8.99E-06$ | 4.78 | $3.85E-06$ | 4.67 | $3.94E-08$ | 6.49 |
| | 3546 | $2.81E-07$ | 3.94 | $6.08E-07$ | 3.95 | $2.53E-07$ | 3.99 | $9.55E-10$ | 5.46 |
| | 14291 | $1.74E-08$ | 4.17 | $3.82E-08$ | 4.14 | $1.66E-08$ | 4.08 | $2.21E-11$ | 5.64 |
| | 56687 | $1.10E-09$ | 4.16 | $2.44E-09$ | 4.15 | $1.05E-09$ | 4.16 | $5.01E-13$ | 5.71 |

Table 5.4: History of convergence of Example 1 with $\nu = 0.3$ and $\alpha = 1$.

Figure 5.7 shows two consecutively refined meshes constructed with the procedure in Chapter 2. For these two meshes we display in Figure 5.8 the approximation of the first component of the displacement $\boldsymbol{u}$ obtained with different polynomial degrees , $\nu = 0.3$ and $\alpha = 1$.

In Table 5.7 we observe optimal convergence for $k = 1$. For $k = 2$, the rate of convergence on the fifth mesh is higher than expected but after that it seems to recover the optimal order. Finally, for $k = 3$ we also observe a higher order of convergence of the fifth mesh. Anyway, the rates of convergence of the errors in $\boldsymbol{u}$ and $\hat{\boldsymbol{u}}_h$ seems to be optimal. On the other hand, the order of convergence of the errors of $\underline{\boldsymbol{\sigma}}$ and $\underline{\boldsymbol{\rho}}$ are higher than three but lower than four (except on the fifth mesh).

| $k$ | $N$ | $\| \boldsymbol{u} - \boldsymbol{u}_h \|_{D_h}$ | $r(\boldsymbol{u})$ | $\| \underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}_h \|_{D_h}$ | $r(\underline{\boldsymbol{\sigma}})$ | $\| \underline{\boldsymbol{\rho}} - \underline{\boldsymbol{\rho}}_h \|_{D_h}$ | $r(\underline{\boldsymbol{\rho}})$ | $\| \boldsymbol{\mathcal{P}_M u} - \hat{\boldsymbol{u}}_h \|_h$ | $r(\hat{\boldsymbol{u}}_h)$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 60 | $2.55E-01$ | – | $4.34E+02$ | – | $1.65E+00$ | – | $4.71E-01$ | – |
| | 234 | $6.70E-02$ | 1.70 | $9.48E+01$ | 1.93 | $7.73E-01$ | 0.96 | $8.04E-02$ | 2.25 |
| | 918 | $1.58E-02$ | 2.51 | $2.37E+01$ | 2.41 | $3.34E-01$ | 1.46 | $1.33E-02$ | 3.14 |
| | 3546 | $3.56E-03$ | 2.19 | $6.14E+00$ | 1.98 | $1.38E-01$ | 1.30 | $2.06E-03$ | 2.73 |
| | 14291 | $7.81E-04$ | 2.27 | $1.51E+00$ | 2.10 | $6.03E-02$ | 1.24 | $3.10E-04$ | 2.83 |
| | 56687 | $1.73E-04$ | 2.27 | $3.77E-01$ | 2.09 | $2.52E-02$ | 1.31 | $4.52E-05$ | 2.91 |
| 2 | 60 | $5.15E-02$ | – | $5.65E+01$ | – | $6.47E-01$ | – | $1.11E-01$ | – |
| | 234 | $3.40E-03$ | 3.45 | $5.13E+00$ | 3.05 | $8.43E-02$ | 2.59 | $4.64E-03$ | 4.04 |
| | 918 | $3.66E-04$ | 3.89 | $6.42E-01$ | 3.62 | $1.65E-02$ | 2.85 | $3.44E-04$ | 4.54 |
| | 3546 | $4.44E-05$ | 3.10 | $8.47E-02$ | 2.97 | $3.66E-03$ | 2.21 | $2.95E-05$ | 3.60 |
| | 14291 | $5.05E-06$ | 3.25 | $1.08E-02$ | 3.08 | $7.53E-04$ | 2.37 | $2.26E-06$ | 3.84 |
| | 56687 | $5.50E-07$ | 3.34 | $1.35E-03$ | 3.14 | $1.51E-04$ | 2.43 | $1.56E-07$ | 4.03 |
| 3 | 60 | $4.08E-03$ | – | $5.14E+00$ | – | $8.02E-02$ | – | $8.85E-03$ | – |
| | 234 | $1.36E-04$ | 4.32 | $2.16E-01$ | 4.03 | $5.26E-03$ | 3.46 | $2.05E-04$ | 4.78 |
| | 918 | $8.18E-06$ | 4.90 | $1.38E-02$ | 4.79 | $5.91E-04$ | 3.81 | $9.10E-06$ | 5.43 |
| | 3546 | $4.90E-07$ | 4.13 | $9.15E-04$ | 3.98 | $6.56E-05$ | 3.22 | $3.76E-07$ | 4.67 |
| | 14291 | $2.76E-08$ | 4.30 | $5.72E-05$ | 4.15 | $6.94E-06$ | 3.36 | $1.43E-08$ | 4.89 |
| | 56687 | $1.49E-09$ | 4.41 | $3.68E-06$ | 4.14 | $6.77E-07$ | 3.51 | $4.90E-10$ | 5.09 |

Table 5.5: History of convergence of Example 1 with $\nu = 0.4999$ and $\alpha = \lambda$.

| $k$ | $N$ | $\| \boldsymbol{u} - \boldsymbol{u}_h \|_{D_h}$ | $r(\boldsymbol{u})$ | $\| \underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}_h \|_{D_h}$ | $r(\underline{\boldsymbol{\sigma}})$ | $\| \underline{\boldsymbol{\rho}} - \underline{\boldsymbol{\rho}}_h \|_{D_h}$ | $r(\underline{\boldsymbol{\rho}})$ | $\| \boldsymbol{\mathcal{P}_M u} - \hat{\boldsymbol{u}}_h \|_h$ | $r(\hat{\boldsymbol{u}}_h)$ |
|---|---|---|---|---|---|---|---|---|---|
| | 96 | $5.46E-02$ | – | $1.38E-01$ | – | $1.13E-01$ | – | $1.87E-02$ | – |
| | 400 | $1.42E-02$ | 1.89 | $4.19E-02$ | 1.67 | $5.27E-02$ | 1.07 | $5.73E-03$ | 1.66 |
| 1 | 1680 | $3.67E-03$ | 1.88 | $1.06E-02$ | 1.91 | $1.76E-02$ | 1.53 | $1.49E-03$ | 1.87 |
| | 7000 | $9.39E-04$ | 1.91 | $2.65E-03$ | 1.94 | $5.23E-03$ | 1.70 | $1.77E-04$ | 2.99 |
| | 28504 | $2.37E-04$ | 1.96 | $6.68E-04$ | 1.96 | $1.43E-03$ | 1.85 | $2.02E-05$ | 3.09 |
| | 96 | $4.08E-03$ | – | $1.58E-02$ | – | $7.17E-03$ | – | $5.18E-03$ | – |
| | 400 | $5.39E-04$ | 2.84 | $2.15E-03$ | 2.80 | $8.75E-04$ | 2.95 | $7.05E-04$ | 2.80 |
| 2 | 1680 | $6.40E-05$ | 2.97 | $1.89E-04$ | 3.39 | $8.43E-05$ | 3.26 | $4.97E-05$ | 3.69 |
| | 7000 | $7.93E-06$ | 2.93 | $2.86E-05$ | 2.65 | $2.34E-05$ | 1.79 | $3.70E-06$ | 3.64 |
| | 28504 | $9.86E-07$ | 2.97 | $3.64E-06$ | 2.94 | $3.12E-06$ | 2.87 | $1.90E-07$ | 4.22 |
| | 96 | $2.05E-04$ | – | $1.46E-03$ | – | $1.36E-03$ | – | $3.48E-04$ | – |
| | 400 | $1.84E-05$ | 3.38 | $1.82E-04$ | 2.92 | $1.32E-04$ | 3.27 | $4.30E-05$ | 2.93 |
| 3 | 1680 | $1.93E-06$ | 3.15 | $1.38E-05$ | 3.60 | $9.13E-06$ | 3.72 | $5.10E-06$ | 2.97 |
| | 7000 | $6.80E-08$ | 4.69 | $5.97E-07$ | 4.40 | $5.86E-07$ | 3.85 | $1.42E-07$ | 5.01 |
| | 28504 | $3.17E-09$ | 4.36 | $2.25E-08$ | 4.67 | $2.39E-08$ | 4.56 | $3.67E-09$ | 5.21 |

Table 5.6: History of convergence of Example 2 with $\nu = 0.3$ and $\alpha = 1$.

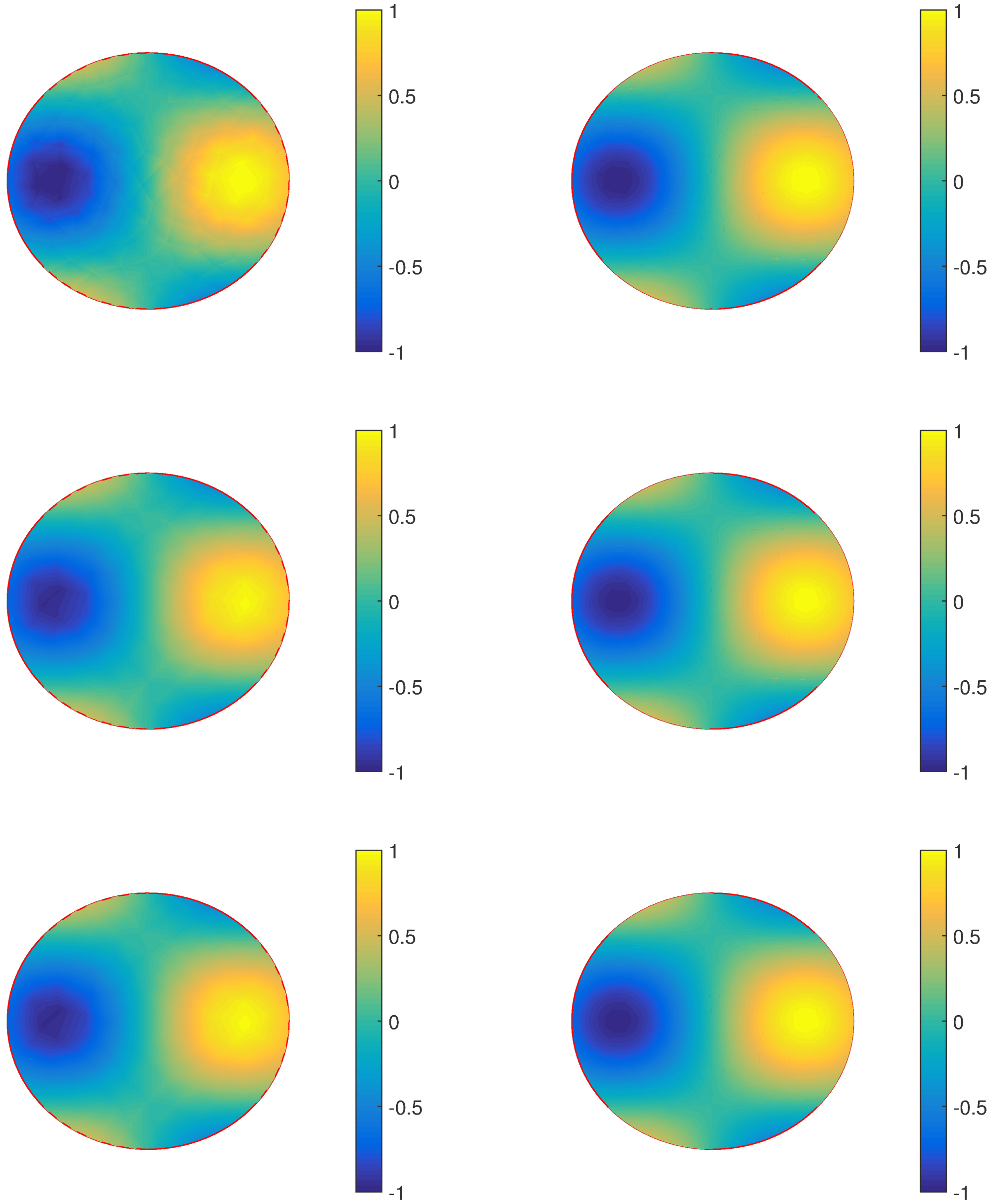



Figure 5.4: Approximation of the first component of $\boldsymbol{u}$ in Example 1. Columns: meshes with $N = 256$ and $1024$ elements. Rows: Polynomial degree $k = 1, 2$ and $3$.

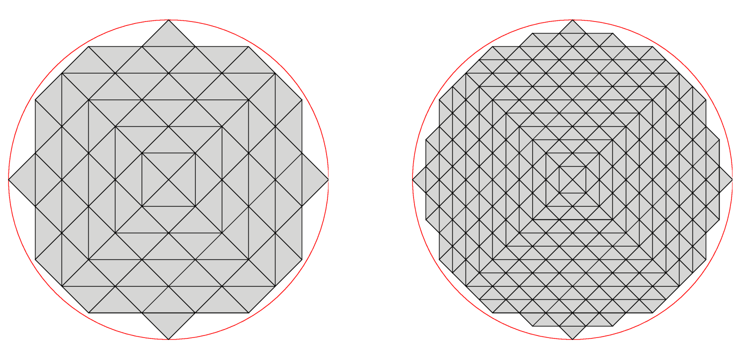


Figure 5.5: Meshes of the Example 2 with $N = 96$ and $400$ elements.

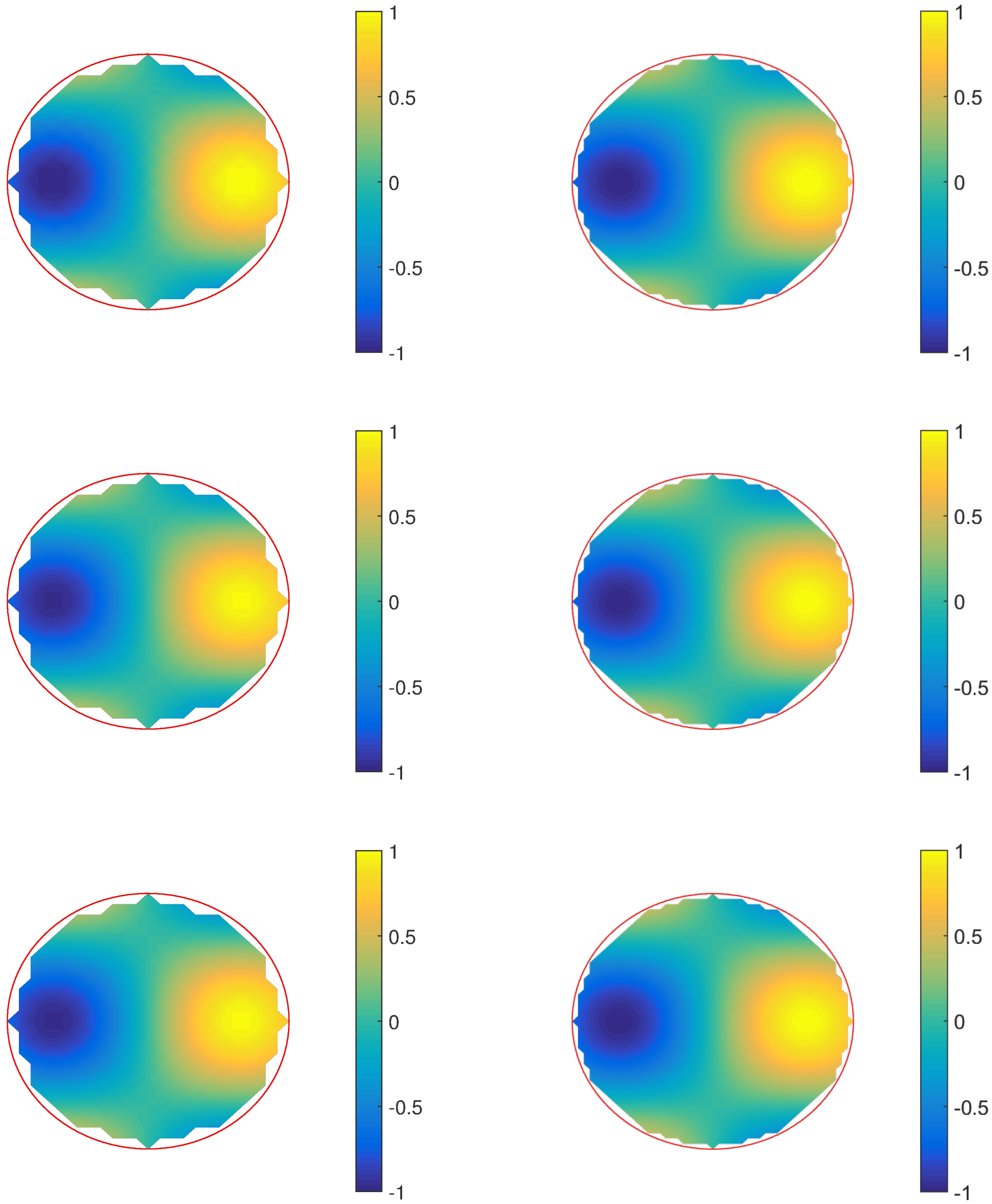



Figure 5.6: Approximation of the first component of $\boldsymbol{u}$ in Example 2. Columns: meshes with $N = 400$ and $1680$ elements. Rows: Polynomial degree $k = 1, 2$ and $3$.

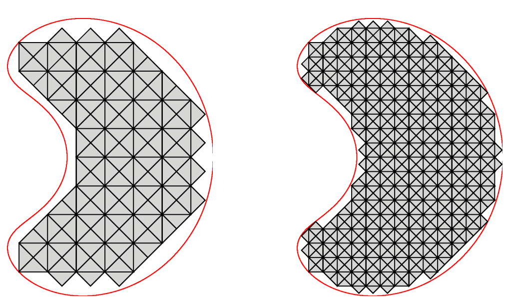


Figure 5.7: Meshes of Example 3 with $N = 154$ and $712$ elements.

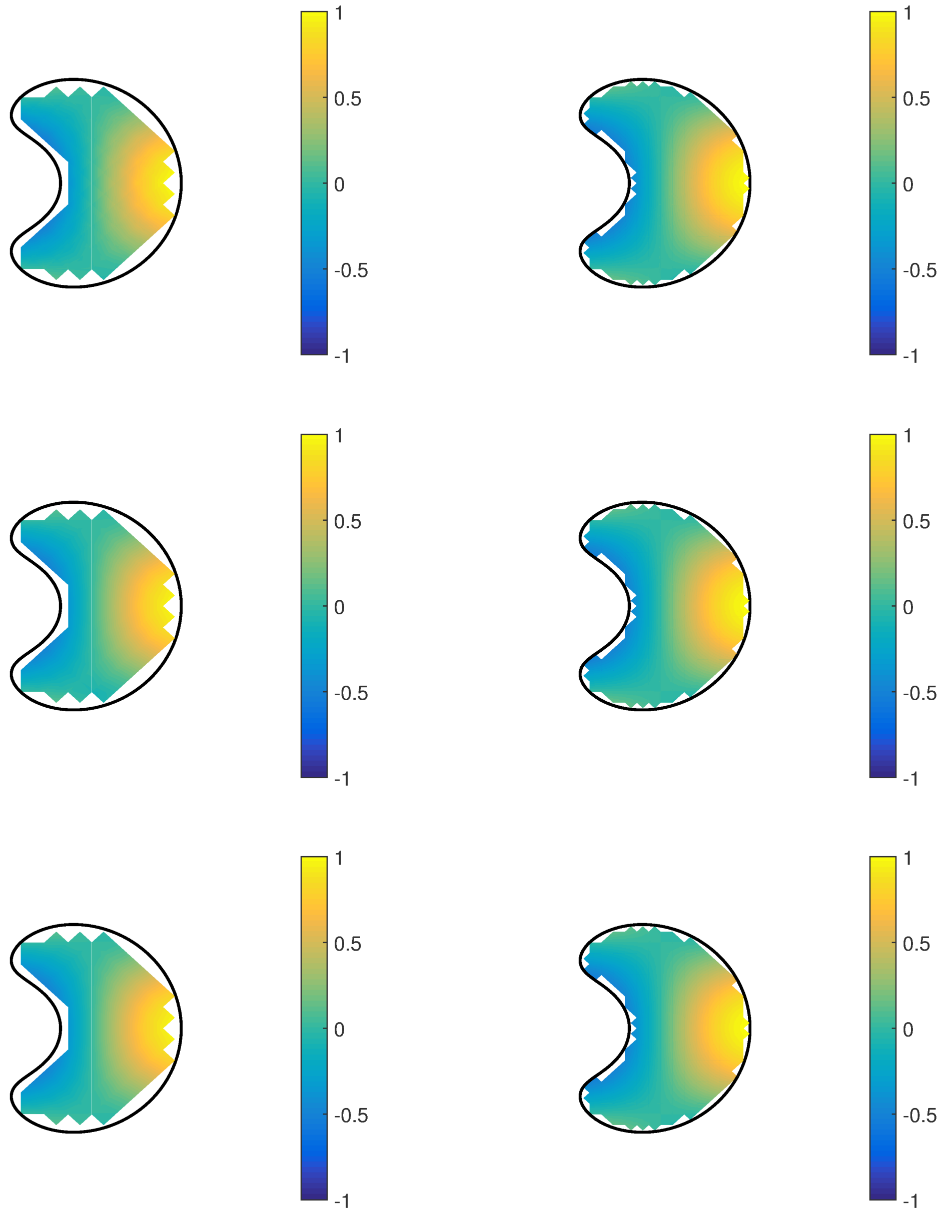



Figure 5.8: Approximation of the first component of $\boldsymbol{u}$ in Example 3. Columns: meshes with $N = 154$ and $712$ elements. Rows: Polynomial degree $k = 1, 2$ and $3$.

| $k$ | $N$ | $\| \boldsymbol{u} - \boldsymbol{u}_h \|_{D_h}$ | $r(\boldsymbol{u})$ | $\| \underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}_h \|_{D_h}$ | $r(\underline{\boldsymbol{\sigma}})$ | $\| \underline{\boldsymbol{\rho}} - \underline{\boldsymbol{\rho}}_h \|_{D_h}$ | $r(\underline{\boldsymbol{\rho}})$ | $\| \boldsymbol{\mathcal{P}_M} \boldsymbol{u} - \hat{\boldsymbol{u}}_h \|_h$ | $r(\hat{\boldsymbol{u}}_h)$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 28 | $5.39E-02$ | $-$ | $1.66E-01$ | $-$ | $1.16E-01$ | $-$ | $2.97E-02$ | $-$ |
| | 154 | $1.49E-02$ | 1.85 | $2.94E-02$ | 2.50 | $2.02E-02$ | 2.53 | $1.96E-03$ | 3.92 |
| | 712 | $3.99E-03$ | 1.90 | $1.00E-02$ | 1.56 | $8.34E-03$ | 1.27 | $3.78E-04$ | 2.38 |
| | 3054 | $1.04E-03$ | 1.94 | $4.56E-03$ | 1.13 | $4.29E-03$ | 0.96 | $8.86E-05$ | 2.09 |
| | 12579 | $2.60E-04$ | 2.00 | $4.99E-04$ | 3.19 | $1.16E-03$ | 1.89 | $1.63E-06$ | 5.77 |
| | 50877 | $6.54E-05$ | 1.99 | $1.28E-04$ | 1.96 | $3.06E-04$ | 1.92 | $2.04E-07$ | 2.99 |
| 2 | 28 | $1.18E-02$ | $-$ | $6.88E-02$ | $-$ | $4.64E-02$ | $-$ | $1.64E-02$ | $-$ |
| | 154 | $9.14E-03$ | 0.37 | $1.00E-01$ | $-0.54$ | $8.53E-02$ | $-0.88$ | $9.33E-03$ | 0.81 |
| | 712 | $1.74E-04$ | 5.72 | $2.62E-03$ | 5.26 | $2.21E-03$ | 5.27 | $1.14E-04$ | 6.35 |
| | 3054 | $2.71E-05$ | 2.68 | $6.67E-04$ | 1.98 | $4.90E-04$ | 2.17 | $1.29E-05$ | 3.15 |
| | 12579 | $1.13E-06$ | 4.58 | $4.84E-06$ | 7.11 | $4.33E-06$ | 6.82 | $3.11E-08$ | 8.69 |
| | 50877 | $1.41E-07$ | 3.00 | $2.21E-07$ | 4.45 | $1.79E-07$ | 4.60 | $6.32E-10$ | 5.62 |
| 3 | 28 | $1.21E-03$ | $-$ | $8.88E-03$ | $-$ | $4.94E-03$ | $-$ | $1.72E-03$ | $-$ |
| | 154 | $6.11E-05$ | 4.30 | $6.17E-04$ | 3.85 | $5.69E-04$ | 3.12 | $6.27E-05$ | 4.78 |
| | 712 | $1.46E-05$ | 2.07 | $1.97E-04$ | 1.65 | $1.81E-04$ | 1.65 | $1.04E-05$ | 2.59 |
| | 3054 | $6.28E-07$ | 4.54 | $1.61E-05$ | 3.61 | $1.10E-05$ | 4.04 | $3.14E-07$ | 5.05 |
| | 12579 | $3.97E-09$ | 7.30 | $1.36E-07$ | 6.89 | $1.41E-07$ | 6.29 | $8.54E-10$ | 8.52 |
| | 50877 | $2.35E-10$ | 4.08 | $1.30E-08$ | 3.38 | $1.39E-08$ | 3.33 | $3.14E-11$ | 4.77 |

Table 5.7: History of convergence of Example 3 with $\nu = 0.3$ and $\alpha = 1$.

Chapter 6

# Conclusion

This work proposed and analyzed an HDG method applied to the linear elasticity problem in curved domains. In fact, in Theorem 4.2 we showed that the scheme is well-posed under certain Assumptions on quantities related to the length of the transferring segments. In addition, under similar assumptions, we proved that the method is optimal. In particular, in Theorem 4.18 we showed that, under regularity assumtpions,

$$\| \underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}_h \|_{D_h} + \| \underline{\boldsymbol{\rho}} - \underline{\boldsymbol{\rho}}_h \|_{D_h} \leq Ch^{k+1}.$$

Moreover, Theorem 4.22 will state that

$$\| \boldsymbol{u} - \boldsymbol{u}_h \|_{D_h} \leq Ch^{k+1}$$

and, if $k \geq 1$,

$$\| \boldsymbol{e}_{\hat{\boldsymbol{u}}} \|_h \leq Ch^{k+3/2}.$$

In HDG method for the linear elasticity problem in polygonal domains [5], the error in $\boldsymbol{e}_{\hat{\boldsymbol{u}}}$ is order $h^{k+2}$(if $k \geq 1$), instead of $h^{k+3/2}$ as we obtained here. However, our numerical experiments suggest that the experimental order of convergences of $\boldsymbol{e}_{\hat{\boldsymbol{u}}}$ is indeed of order $h^{k+2}$. This phenomenon has been also observed in HDG scheme applied to Poisson [4] and Stokes [13] problems in curved domain.

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