



UNIVERSIDAD DE CONCEPCIÓN  
FACULTAD DE CIENCIAS FÍSICAS Y MATEMÁTICAS  
DEPARTAMENTO DE INGENIERÍA MATEMÁTICA

**NEW MIXED FINITE ELEMENT METHODS  
FOR THE COUPLED CONVECTIVE  
BRINKMAN-FORCHHEIMER AND  
DOUBLE-DIFFUSION EQUATIONS**

POR

Sergio Enrique Carrasco Hidalgo

Tesis presentada a la Facultad de Ciencias Físicas y Matemáticas de la  
Universidad de Concepción para optar al título profesional de  
Ingeniero Civil Matemático

Profesores Guía: Gabriel N. Gatica, Sergio Caucao

Agosto 2023,  
Concepción, Chile.

© 2023 Sergio Enrique Carrasco Hidalgo

Se autoriza la reproducción total o parcial, con fines académicos, por cualquier medio o procedimiento, incluyendo la cita bibliográfica del documento.

# NEW MIXED FINITE ELEMENT METHODS FOR THE COUPLED CONVECTIVE BRINKMAN-FORCHHEIMER AND DOUBLE-DIFFUSION EQUATIONS

NUEVOS MÉTODOS DE ELEMENTOS FINITOS MIXTOS  
PARA LAS ECUACIONES ACOPLADAS DE  
BRINKMAN-FORCHHEIMER CONVECTIVA  
Y DE DOBLE DIFUSIÓN

## Comisión evaluadora

GABRIEL N. GATICA [PROFESOR GUÍA]

CI<sup>2</sup>MA y Departamento de Ingeniería Matemática, Universidad de Concepción.

SERGIO CAUCAO [PROFESOR CO-GUÍA]

GIANuC<sup>2</sup> y Departamento de Matemática y Física Aplicadas, Universidad Católica de la Santísima Concepción.

JESSIKA CAMAÑO

GIANuC<sup>2</sup> y Departamento de Matemática y Física Aplicadas, Universidad Católica de la Santísima Concepción.

LUIS GATICA

GIANuC<sup>2</sup> y Departamento de Matemática y Física Aplicadas, Universidad Católica de la Santísima Concepción.

RICARDO OYARZÚA

GIMNAP y Departamento de Matemática, Universidad del Bío-Bío.

**Fecha de Defensa: 03 de Agosto de 2023**

*Para el Bigote.*

# Acknowledgements

*Quiero partir comentando que hay muchos nombres y anécdotas circulando por mi cabeza y las estoy agradeciendo mientras pasan: Si las escribo todas, esta parte sería más larga de lo que el documento se trata. Me disculpo por cualquier omisión.*

*Primeramente, quiero agradecer a mi familia, a mi madre Blanca Hidalgo y padre Leonel Carrasco, quienes me apoyaron férreamente en todo este proceso, por todas las sopas de pollo y las aguitas de orégano que me mantuvieron sano y fuerte hasta esta instancia.*

*A mis amigos del barrio, de onces, del fútbol, de los viajes y de las Bálticas, los cuales se sentirán completamente identificados con esta breve descripción. Gracias por su apoyo, las risas y buenas aventuras. A su vez, a mis amigos a quienes conocí en la universidad: Francisco, Nicolás, Soledad, Juan, Daniela, Jaime, Javier, entre otros, con los cuales compartimos horas y horas, días y días, noches de estudio, historias, almuerzos, chismecitos y desafíos; gracias por todos los buenos momentos.*

*Especialmente, a Paz, quién me ha acompañado en mis momentos altos y bajos, quién me ha visto en mis mejores y peores momentos, y me ha brindado su inconmensurable apoyo. Gracias por cuidarme y abrazarme cuando más lo he necesitado.*

*Al Departamento de Ingeniería Matemática de la Universidad de Concepción: A la profesora Mónica Selva por aclarar mis dudas pertinentemente. Al profesor Manuel Solano por siempre tener disposición para resolver mis cuestionamientos y jugar fútbol. Al profesor Freddy Paiva por preocuparse por mí en lo personal y darme un consejo certero. Y, a Don Victor Venegas, quien durante años me consideró para ser consultor, además de recibirnos siempre con un café bien conversado junto con un trozo de pastel, sin importar la ocasión.*

*Para empezar a terminar: Agradecer al Centro de Investigación en Ingeniería Matemática (C<sup>2</sup>MA) por acogerme durante este largo periodo, por incluirme en su mesa, por emitir ese cariño hogareño con el que me da tanto gusto volver. En especial, agradecer a la sra. Lorena, sra. Paola, profesor Raimund Bürger, Juan Paulo, Romel, entre otros. Gracias por los gratos momentos.*

*A mi profesor Guía, Gabriel Gatica, a quién admiro y agradezco por brindarme la oportunidad de ser su tesista, además del apoyo, la confianza y, por sobre todo, el conocimiento entregado. Gracias por esas palabras y acciones que a uno lo motivan y lo disciplinan; y de reavivar mis ganas de querer hacer matemática. De igual manera, agradecer a mi profesor co-guía, Sergio Caucao, por su gran dedicación, disposición y paciencia para enseñarme el aspecto computacional de este trabajo, es un honor ser alumno de personas tan profesionales. Espero haber dado la talla.*

*Para cerrar, agradecer a mi tío Ernesto, que en paz descanse. Disculpe por no terminar antes, pero gracias por confiar en mi durante toda mi vida.*

*Finalmente agradezco a ANID-Chile por su apoyo a través de los proyectos CENTRO DE MODELAMIENTO MATEMÁTICO (FB210005), y ANILLO OF COMPUTATIONAL MATHEMATICS FOR DESALINATION PROCESSES (ACT210087).*

# Abstract

In this thesis we introduce and analyze new Banach spaces-based mixed finite element methods for the stationary nonlinear problem arising from the coupling of the convective Brinkman-Forchheimer equations with a double diffusion phenomenon. Besides the velocity and pressure variables, the symmetric stress and the skew-symmetric vorticity tensors are introduced as auxiliary unknowns of the fluid. Thus, the incompressibility condition allows to eliminate the pressure, which, along with the velocity gradient and the shear stress, can be computed afterwards via postprocessing formulae depending on the velocity and the aforementioned new tensors. Regarding the diffusive part of the coupled model, and additionally to the temperature and concentration of the solute, their gradients and pseudoheat/pseudodiffusion vectors are incorporated as further unknowns as well. The resulting mixed variational formulation, settled within a Banach spaces framework, consists of a nonlinear perturbation of, in turn, a nonlinearly perturbed saddle-point scheme, coupled with a usual saddle-point system. A fixed-point strategy, combined with classical and recent solvability results for suitable linearizations of the decoupled problems, including in particular, the Banach-Nečas-Babuška theorem and the Babuška-Brezzi theory, are employed to prove, jointly with the Banach fixed-point theorem, the well-posedness of the continuous and discrete formulations. Both PEERS and AFW elements of order  $\ell \geq 0$  for the fluid variables, and piecewise polynomials of degree  $\leq \ell$  together with Raviart-Thomas elements of order  $\ell$  for the unknowns of the diffusion equations, constitute feasible choices for the Galerkin scheme. In turn, optimal a priori error estimates, including those for the postprocessed unknowns, are derived, and corresponding rates of convergence are established. Finally, several numerical experiments confirming the latter and illustrating the good performance of the proposed methods, are reported.

# Resumen

En esta tesis introducimos y analizamos nuevos métodos de elementos finitos mixtos basados en espacios de Banach para el problema estacionario no lineal que surge del acoplamiento de las ecuaciones de Brinkman-Forchheimer convectivas con el fenómeno de doble difusión. Además de la velocidad y la presión, los tensores de esfuerzo y de vorticidad, los cuales son simétrico y antisimétrico respectivamente, se introducen como incógnitas auxiliares del fluido. Así, la condición de incompresibilidad permite eliminar la presión, la cual, junto con el gradiente de velocidad y el esfuerzo de corte, pueden ser calculadas posteriormente mediante fórmulas de postprocesado que dependen de la velocidad y de los nuevos tensores anteriormente mencionados. En cuanto a la parte difusiva del modelo acoplado, y adicionalmente a la temperatura y concentración del soluto, sus gradientes y vectores de pseudocalor/pseudodifusión se incorporan también como incógnitas extra. La formulación variacional mixta resultante, en un marco de espacios de Banach, consiste en una perturbación no lineal de, a su vez, un esquema de punto de silla con una perturbación no lineal, acoplado con un esquema de punto de silla usual. Una estrategia de punto fijo, combinado con resultados de solubilidad clásicas y recientes para linealizaciones adecuadas de los problemas desacoplados, incluyendo en particular, el teorema de Banach-Nečas-Babuška y la teoría de Babuška-Brezzi, se emplean para probar, junto con el teorema de punto fijo de Banach, el buen planteamiento de las formulaciones continua y discreta. Tanto los elementos PEERS como AFW de orden  $\ell \geq 0$  para las variables correspondientes al fluido, y polinomios a trozos de grado  $\leq \ell$ , junto con los elementos de Raviart-Thomas de orden  $\ell$  para las incógnitas de las ecuaciones de difusión, constituyen opciones factibles para el esquema de Galerkin. A su vez, se derivan estimaciones óptimas del error a priori, incluyendo las de las incógnitas postprocesadas, y se establecen las tasas de convergencia correspondientes. Por último, varios experimentos numéricos confirman e ilustran el buen funcionamiento de los métodos propuestos.

# Contents

Acknowledgements	iv
Abstract	vi
Resumen	vii
Contents	viii
List of figures	x
<b>1 Introduction</b>	<b>1</b>
1.1 Preliminary notations . . . . .	5
<b>2 The model problem</b>	<b>7</b>
<b>3 The continuous formulation</b>	<b>11</b>
<b>4 Analysis of the coupled problem</b>	<b>18</b>
4.1 Preliminaries . . . . .	18
4.2 A fixed point strategy . . . . .	20
4.3 Well posedness of the uncoupled problems . . . . .	22
4.4 Solvability analysis of the fixed–point equation . . . . .	26
<b>5 The Galerkin scheme</b>	<b>34</b>
5.1 Preliminaries . . . . .	34
5.2 Discrete fixed point strategy . . . . .	35
5.3 Discrete solvability analysis . . . . .	37

*CONTENTS*

5.4	A priori error analysis . . . . .	45
<b>6</b>	<b>Specific finite element subspaces</b>	<b>50</b>
6.1	Preliminaries . . . . .	50
6.2	Two examples . . . . .	51
6.3	The rates of convergence . . . . .	53
<b>7</b>	<b>Numerical results</b>	<b>56</b>
<b>8</b>	<b>Conclusions and Future Works</b>	<b>67</b>
8.1	Conclusions . . . . .	67
8.2	Future Works . . . . .	68
	<b>Bibliography</b>	<b>69</b>

# List of Figures

7.1	[Example 1, $\ell = 1$ ] Computed pseudostress tensor component, magnitude of the velocity, vorticity component, and pressure field (top plots); temperature field, magnitude of the pseudoheat vector, concentration field, and magnitude of the pseudodiffusion vector (bottom plots). . . . .	64
7.2	[Example 2] Computed pseudostress tensor component, magnitude of the velocity, vorticity component, and pressure field (top plots); temperature field, magnitude of the pseudoheat vector, concentration field, and magnitude of the pseudodiffusion vector (bottom plots). . . . .	65
7.3	[Example 3] Domain configuration, computed magnitude of the pseudostress tensor, velocity, and velocity gradient tensor (top plots); concentration field, magnitude of the temperature gradient, concentration field, and magnitude of the concentration gradient (bottom plots). . . . .	66

---

## Introduction

---

The multiphysics problem of double-diffusive convection in which two scalar fields, such as heat and concentration of a solute, affect the density distribution in a fluid-saturated highly porous medium, has been intensively studied in recent years (see, e.g., [30], [33], [34], [15], [13], and references therein). Applications include predicting and controlling processes arising in geophysics, oceanography, chemical engineering, and energy technology, to name a few. In particular, some of them includes groundwater system in karst aquifers, fast flows in fractured or vuggy aquifers or reservoirs, chemical processing, convective flow of carbon nanotubes, and propagation of biological fluids (see, for instance, [2], [6], [22], and [36]). In this regard, we remark that much of the research in porous medium has been focused on the use of Darcy's law. However, this fundamental equation may be inaccurate for modeling fluid flow through porous media with high Reynolds numbers or through media with high porosity. To overcome this limitation, it is possible to consider the convective Brinkman–Forchheimer equations (see, e.g., [17], [35], [31], [11], and [12]), where terms are added to Darcy's equation in order to take into account the above described physical aspects. Moreover, this fact has motivated the

introduction of the corresponding coupling with a system of advection-diffusion equations (also called double-diffusion equations), through convective terms and the body force.

Concerning literature devoted to studying the coupling of the Brinkman–Forchheimer and double-diffusion equations, we first highlight that, up to the authors’ knowledge, [30] constitutes one of the first works in analyzing the well-posedness and regularity of solution for a velocity-pressure-temperature-concentration variational formulation. Later on, a finite volume method to solve the coupling of the time-dependent Brinkman–Forchheimer and double-diffusion equations was adopted in [34]. The focus of this work was on the validity of the Brinkman–Forchheimer model when various combinations of the thermal Rayleigh number, permeability ratio, inclination angle, thermal conductivity and buoyancy ratio are considered. More recently, an augmented fully-mixed formulation based on the introduction of the fluid pseudostress tensor, and the pseudoheat and pseudodiffusive vectors (besides the velocity, temperature and concentration fields) was analyzed in [15]. Meanwhile, a non-augmented Banach spaces-based fully-mixed formulation was proposed and analyzed in [13]. In particular, this latter scheme is written equivalently as a fixed-point equation, so that the well-known Banach theorem, combined with classical results on nonlinear monotone operators and Babuška-Brezzi’s theory in Banach spaces, are applied to prove the unique solvability of the continuous and discrete systems.

Regarding literature focused on the analysis of the convective Brinkman–Forchheimer (CBF) equations, we start referring to [17], where the authors analyze the continuous dependence of solutions of the CBF equations written in velocity-pressure formulation on the Forchheimer coefficient in  $H^1$  norm. In turn, an approximation of solutions for the incompressible CBF equations via the artificial compressibility method was proposed and developed in [35], where a family of perturbed compressible CBF equations that approximate the incompressible CBF equations is introduced. Furthermore, the well-posedness of the corresponding velocity-pressure variational formulation of the two-dimensional stationary CBF equations was analyzed in [31]. In addition, error estimates for a mixed finite element approximation were obtained, and a one-step Newton iteration algorithm initialized using a fixed-point iteration, was proposed. Recently, an augmented mixed pseudostress-velocity formulation was analyzed in [11]. In there,

the well-posedness of the problem is achieved by combining a fixed-point strategy, the Lax–Milgram theorem, and the well-known Schauder and Banach fixed-point theorems. We also mention [12], where a Banach spaces-based mixed formulation was proposed and analyzed for the CBF problem, but differently from the techniques previously developed in [11], no augmentation procedure was needed for the formulation nor for the solvability analysis. The resulting non-augmented scheme is then written equivalently as a fixed-point equation, so that results recently established in [19] for perturbed saddle-point problems in Banach spaces, along with the well-known Banach–Nečas–Babuška and Banach theorems, are applied to prove the well-posedness of the continuous and discrete systems.

We point out that the motivation of employing an augmented approach, as in [15] and [11], is originated by the wish of performing the respective solvability analysis of the equations within a Hilbertian framework. However, it is well known that the introduction of additional terms into the formulation, while having some advantages, also leads to much more expensive schemes in terms of complexity and computational implementation. In order to overcome this, in recent years there has arisen an increasing development on Banach spaces-based mixed finite element methods to solve a wide family of single and coupled nonlinear problems in continuum mechanics. In particular, we refer to [9], [8], [16], [18], [5], [25], and [10], for the analysis of mixed formulations within a Banach framework of the Poisson, Navier–Stokes, Brinkman–Forchheimer, Boussinesq, coupled flow-transport, Navier–Stokes–Brinkman, and chemotaxis–Navier–Stokes equations. This kind of procedures shows two advantages at least: no augmentation is required, and the spaces to which the unknowns belong are the natural ones arising from the application of the Cauchy–Schwarz and Hölder inequalities to the terms resulting from the testing and integration by parts of the equations of the model. As a consequence, simpler and closer to the original physical model formulations are obtained.

According to the previous discussion, and aiming to continue extending the applicability of the aforementioned framework, the goal of the present paper is to develop and analyze a new Banach spaces-based fully-mixed formulation, augmentation free, for the coupling of the convective Brinkman–Forchheimer and double-diffusion equations, and study its numerical approximation by the associated mixed finite element method. To this end, and unlike [11] and

[12], where only the pseudostress is employed, here we introduce the symmetric stress and the skew-symmetric vorticity tensors as auxiliary unknowns in the CBF equations, and subsequently eliminate the pressure unknown using the incompressibility condition. In turn, we follow [13, 18] and adopt a dual-mixed formulation for the double-diffusion equations making use of the temperature/concentration gradients and the pseudoheat/pseudodiffusion vectors as further unknowns. The resulting mixed formulation is written as a nonlinear perturbation of, in turn, a nonlinearly perturbed saddle-point scheme, coupled with a usual saddle-point system. Then, similarly to [13], [20], [25], and [10], we combine a fixed-point argument, the abstract results provided in [19], the Banach–Nečas–Babuška theorem, Babuška–Brezzi’s theory in Banach spaces, sufficiently small data assumptions, and the Banach theorem, to establish existence and uniqueness of solution of both the continuous and discrete formulations. In this regard, and since the formulation is similar to the ones considered in [13], [19], and [20], our present analysis certainly makes use of the corresponding results available there. In addition, applying an ad-hoc Strang-type lemma in Banach spaces established in [15], we are able to derive the corresponding *a priori* error estimates for arbitrary discrete subspaces. Next, employing PEERS and AFW elements of order  $\ell \geq 0$  for approximating the fluid variables, and piecewise polynomials of degree  $\leq \ell$  together with Raviart–Thomas elements of order  $\ell$  for the unknowns of the double-diffusion equations, we prove that the corresponding discrete methods are convergent with optimal rates.

The paper is organized as follows. The remainder of this section describes standard notation and functional spaces to be employed throughout the paper. The model problem is introduced in Chapter 2, and all the auxiliary variables to be employed in the setting of the formulation are defined there. Next, in Chapter 3 we derive the corresponding fully-mixed variational formulation in Banach spaces, whereas, the well-posedness of this continuous scheme is established in Chapter 4. The corresponding Galerkin system is introduced and analyzed in Chapter 5, where the discrete analogue of the theory used in the continuous case is employed to prove existence and uniqueness of solution. *A priori* error estimates for arbitrary finite element subspaces are also obtained there. In Chapter 6 we establish the corresponding rates of convergence for specific discrete subspaces. Finally, the performance of the method is illustrated in Chapter 7

throughout several numerical examples in 2D and 3D, with and without manufactured solutions, which confirm the accuracy and flexibility of our fully-mixed finite element method.

## 1.1 Preliminary notations

Let  $\Omega \subset \mathbb{R}^n$ ,  $n \in \{2, 3\}$ , be a bounded domain with polyhedral boundary  $\Gamma$ , and let  $\boldsymbol{\nu}$  be the outward unit normal vector on  $\Gamma$ . Standard notation will be adopted for Lebesgue spaces  $L^p(\Omega)$  and Sobolev spaces  $W^{s,p}(\Omega)$ , with  $s \in \mathbb{R}$  and  $p > 1$ , whose corresponding norms, either for the scalar, vectorial, or tensorial case, are denoted by  $\|\cdot\|_{0,p;\Omega}$  and  $\|\cdot\|_{s,p;\Omega}$ , respectively. In particular, given a non-negative integer  $m$ ,  $W^{m,2}(\Omega)$  is also denoted by  $H^m(\Omega)$ , and the notations of its norm and seminorm are simplified to  $\|\cdot\|_{m,\Omega}$  and  $|\cdot|_{m,\Omega}$ , respectively. In addition,  $H^{1/2}(\Gamma)$  is the space of traces of functions of  $H^1(\Omega)$ , and  $H^{-1/2}(\Gamma)$  denotes its dual. On the other hand, given any generic scalar functional space  $\mathbb{S}$ , we let  $\mathbf{S}$  and  $\mathbb{S}$  be the corresponding vectorial and tensorial counterparts, whereas  $\|\cdot\|$ , with no subscripts, will be employed for the norm of any element or operator whenever there is no confusion about the space to which they belong. Also,  $|\cdot|$  denotes the Euclidean norm in both  $\mathbb{R}^n$  and  $\mathbb{R}^{n \times n}$ , and as usual,  $\mathbb{I}$  stands for the identity tensor in  $\mathbb{R}^{n \times n}$ . In turn, for any vector fields  $\mathbf{v} = (v_i)_{i=1,n}$  and  $\mathbf{w} = (w_i)_{i=1,n}$ , we set the gradient, divergence, and tensor product operators, as

$$\nabla \mathbf{v} := \left( \frac{\partial v_i}{\partial x_j} \right)_{i,j=1,n}, \quad \operatorname{div}(\mathbf{v}) := \sum_{j=1}^n \frac{\partial v_j}{\partial x_j}, \quad \text{and} \quad \mathbf{v} \otimes \mathbf{w} := (v_i w_j)_{i,j=1,n},$$

whereas for any tensor fields  $\boldsymbol{\tau} = (\tau_{ij})_{i,j=1,n}$  and  $\boldsymbol{\zeta} = (\zeta_{ij})_{i,j=1,n}$ , we let  $\mathbf{div}(\boldsymbol{\tau})$  be the divergence operator  $\operatorname{div}$  acting along the rows of  $\boldsymbol{\tau}$ , and define the transpose, the trace, the deviatoric tensor, and the tensor inner product, respectively, as

$$\boldsymbol{\tau}^t := (\tau_{ji})_{i,j=1,n}, \quad \operatorname{tr}(\boldsymbol{\tau}) := \sum_{i=1}^n \tau_{ii}, \quad \boldsymbol{\tau}^d := \boldsymbol{\tau} - \frac{1}{n} \operatorname{tr}(\boldsymbol{\tau}) \mathbb{I}, \quad \text{and} \quad \boldsymbol{\tau} : \boldsymbol{\zeta} := \sum_{i,j=1}^n \tau_{ij} \zeta_{ij}.$$

Furthermore, for each  $t \in [1, +\infty)$  we introduce the Banach spaces

$$\mathbf{H}(\operatorname{div}_t; \Omega) := \left\{ \boldsymbol{\tau} \in \mathbf{L}^2(\Omega) : \operatorname{div}(\boldsymbol{\tau}) \in L^t(\Omega) \right\}, \quad \text{and}$$

$$\mathbb{H}(\mathbf{div}_t; \Omega) := \left\{ \boldsymbol{\tau} \in \mathbf{L}^2(\Omega) : \mathbf{div}(\boldsymbol{\tau}) \in \mathbf{L}^t(\Omega) \right\},$$

equipped with the natural norms

$$\|\boldsymbol{\tau}\|_{\mathbf{div}_t; \Omega} := \|\boldsymbol{\tau}\|_{0, \Omega} + \|\mathbf{div}(\boldsymbol{\tau})\|_{0, t; \Omega} \quad \forall \boldsymbol{\tau} \in \mathbf{H}(\mathbf{div}_t; \Omega), \quad \text{and}$$

$$\|\boldsymbol{\tau}\|_{\mathbf{div}_t; \Omega} := \|\boldsymbol{\tau}\|_{0, \Omega} + \|\mathbf{div}(\boldsymbol{\tau})\|_{0, t; \Omega} \quad \forall \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_t; \Omega).$$

Additionally, we recall that, proceeding as in [23, eq. (1.43), Section 1.3.4] (see also [9, Section 4.1] and [18, Section 3.1]), one can prove that for  $t \in \begin{cases} (1, +\infty] \text{ in } \mathbb{R}^2, \\ [\frac{6}{5}, +\infty] \text{ in } \mathbb{R}^3, \end{cases}$  there holds

$$\langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}, v \rangle = \int_{\Omega} \left\{ \boldsymbol{\tau} \cdot \nabla v + v \mathbf{div}(\boldsymbol{\tau}) \right\} \quad \forall (\boldsymbol{\tau}, v) \in \mathbf{H}(\mathbf{div}_t; \Omega) \times \mathbf{H}^1(\Omega), \quad (1.1)$$

and

$$\langle \boldsymbol{\tau} \boldsymbol{\nu}, \mathbf{v} \rangle = \int_{\Omega} \left\{ \boldsymbol{\tau} : \nabla \mathbf{v} + \mathbf{v} \cdot \mathbf{div}(\boldsymbol{\tau}) \right\} \quad \forall (\boldsymbol{\tau}, \mathbf{v}) \in \mathbb{H}(\mathbf{div}_t; \Omega) \times \mathbf{H}^1(\Omega), \quad (1.2)$$

where  $\langle \cdot, \cdot \rangle$  denotes in (1.1) (resp. (1.2)) the duality pairing between  $\mathbf{H}^{1/2}(\Gamma)$  (resp.  $\mathbf{H}^{1/2}(\Gamma)$ ) and  $\mathbf{H}^{-1/2}(\Gamma)$  (resp.  $\mathbf{H}^{1/2}(\Gamma)$ ).

---

## The model problem

---

In what follows we consider the steady convective Brinkman–Forchheimer equations introduced in [31] (see also [35, 12, 11]) coupled with double-diffusion equations, similarly as done in [13]. More precisely, we focus on finding a velocity field  $\mathbf{u}$ , a pressure field  $p$ , a temperature field  $\phi_1$ , and a concentration field  $\phi_2$ , the latter two defining a vector unknown  $\boldsymbol{\phi} := (\phi_1, \phi_2)$ , such that

$$\begin{aligned}
 -\operatorname{div}(\mu \mathbf{e}(\mathbf{u})) + (\nabla \mathbf{u})\mathbf{u} + \mathbf{D} \mathbf{u} + \mathbf{F} |\mathbf{u}|^{\rho-2} \mathbf{u} + \nabla p &= \mathbf{f}(\boldsymbol{\phi}) && \text{in } \Omega, \\
 \operatorname{div}(\mathbf{u}) &= 0 && \text{in } \Omega, \\
 -\operatorname{div}(\mathbf{Q}_1 \nabla \phi_1) + \mathbf{R}_1 \mathbf{u} \cdot \nabla \phi_1 &= g_1 && \text{in } \Omega, \\
 -\operatorname{div}(\mathbf{Q}_2 \nabla \phi_2) + \mathbf{R}_2 \mathbf{u} \cdot \nabla \phi_2 &= g_2 && \text{in } \Omega, \\
 \mathbf{u} = \mathbf{u}_D, \quad \phi_1 = \phi_{1,D}, \quad \text{and} \quad \phi_2 = \phi_{2,D} &&& \text{on } \Gamma,
 \end{aligned} \tag{2.1}$$

where  $\mu$  is the Brinkman coefficient (or effective viscosity), which is assumed to be eventually variable, and bounded, that is there exist constants  $\mu_0, \mu_1 > 0$ , such that

$$\mu_0 \leq \mu(\mathbf{x}) \leq \mu_1 \quad \forall \mathbf{x} \in \Omega. \quad (2.2)$$

In addition,  $\mathbf{e}(\mathbf{u}) := \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^\dagger)$  is the symmetric part of  $\nabla \mathbf{u}$ , also named strain rate tensor,  $D > 0$  is the Darcy coefficient,  $F > 0$  is the Forchheimer coefficient,  $\rho$  is a given number in  $[3, 4]$ , and  $\mathbf{f}(\phi)$  is an external force defined by

$$\mathbf{f}(\phi) := -(\phi_1 - \phi_{1,r}) \mathbf{g} + \frac{1}{\varrho} (\phi_2 - \phi_{2,r}) \mathbf{g}, \quad (2.3)$$

where  $\mathbf{g}$  represents the potential type gravitational acceleration,  $\phi_{1,r}$  and  $\phi_{2,r}$  are the reference temperature and concentration of a solute, respectively, and  $\varrho$  is a parameter experimentally valued that can be assumed to be  $\geq 1$  (see [30, Section 2] for details). The spaces to which  $\phi_{1,r}$  and  $\phi_{2,r}$  belong will be specified later on. In turn,  $\mathbf{Q}_1$  and  $\mathbf{Q}_2$  denote the thermal and concentration diffusion tensors, respectively, which are assumed to belong to  $\mathbb{L}^\infty(\Omega)$ , whereas  $R_1$  is the thermal Rayleigh number and  $R_2$  is the solute Rayleigh number. In addition,  $\mathbf{Q}_1$  and  $\mathbf{Q}_2$  are assumed to be uniformly positive definite tensors, which means that there exist positive constants  $C_1$  and  $C_2$ , such that

$$\mathbf{v} \cdot \mathbf{Q}_j(\mathbf{x}) \mathbf{v} \geq C_j |\mathbf{v}|^2 \quad \forall \mathbf{v} \in \mathbb{R}^n, \quad \forall \mathbf{x} \in \Omega, \quad j \in \{1, 2\}, \quad (2.4)$$

and  $g_1$  and  $g_2$  are given source terms in suitable spaces to be specified later on. Finally,  $\mathbf{u}_D \in \mathbf{H}^{1/2}(\Gamma)$  and  $\phi_{i,D} \in H^{1/2}(\Gamma)$ ,  $i \in \{1, 2\}$ , are given Dirichlet data.

Owing to the incompressibility of the fluid and the Dirichlet boundary condition for  $\mathbf{u}$ , the datum  $\mathbf{u}_D$  must satisfy the compatibility condition

$$\int_{\Gamma} \mathbf{u}_D \cdot \boldsymbol{\nu} = 0. \quad (2.5)$$

In addition, due to the pressure gradient in (2.1), and in order to guarantee uniqueness of this

unknown,  $p$  will be sought in the space

$$L_0^2(\Omega) := \left\{ q \in L^2(\Omega) : \int_{\Omega} q = 0 \right\}.$$

Now, in order to derive a fully mixed formulation for (2.1), thus yielding the Dirichlet boundary conditions to become natural, we proceed similarly to [25] (see also [8] for related approaches), and introduce as a further unknown the symmetric tensor  $\boldsymbol{\sigma}$  defined by

$$\boldsymbol{\sigma} := \mu \mathbf{e}(\mathbf{u}) - (\mathbf{u} \otimes \mathbf{u}) - p \mathbb{I}. \quad (2.6)$$

In this way, the first equation of (2.1) can be rewritten as

$$\mathbf{D} \mathbf{u} + F |\mathbf{u}|^{\rho-2} \mathbf{u} - \mathbf{div}(\boldsymbol{\sigma}) = \mathbf{f}(\phi) \quad \text{in } \Omega, \quad (2.7)$$

whereas applying the trace operator to  $\boldsymbol{\sigma}$  and utilizing the incompressibility condition  $\mathbf{div}(\mathbf{u}) = 0$  in  $\Omega$ , we obtain

$$p = -\frac{1}{n} \operatorname{tr}(\boldsymbol{\sigma} + (\mathbf{u} \otimes \mathbf{u})). \quad (2.8)$$

Moreover, applying the deviatoric operator to (2.6) and dividing by  $\mu$ , we find that

$$\frac{1}{\mu} \boldsymbol{\sigma}^{\mathrm{d}} + \frac{1}{\mu} (\mathbf{u} \otimes \mathbf{u})^{\mathrm{d}} = \mathbf{e}(\mathbf{u}) = \nabla \mathbf{u} - \boldsymbol{\gamma}, \quad (2.9)$$

where the vorticity

$$\boldsymbol{\gamma} := \frac{1}{2} \left( \nabla \mathbf{u} - (\nabla \mathbf{u})^{\mathrm{t}} \right) \quad (2.10)$$

is introduced as a further unknown.

Next, for the double-diffusion equations we consider for each  $j \in \{1, 2\}$  the temperature (or concentration) gradient  $\mathbf{t}_j$ , and the corresponding pseudoheat (or pseudodiffusion)  $\boldsymbol{\vartheta}_j$ , as auxiliary unknowns, which are defined, respectively, by

$$\mathbf{t}_j := \nabla \phi_j, \quad \boldsymbol{\vartheta}_j := \mathbf{Q}_j \mathbf{t}_j - \frac{1}{2} \mathbf{R}_j \phi_j \mathbf{u}, \quad \forall j \in \{1, 2\}, \quad \text{in } \Omega, \quad (2.11)$$

whence the third and fourth equations of (2.1) can be rewritten as

$$\frac{1}{2} \mathbb{R}_j \mathbf{u} \cdot \mathbf{t}_j - \operatorname{div}(\boldsymbol{\vartheta}_j) = g_j \quad \text{in } \Omega, \quad j \in \{1, 2\}. \quad (2.12)$$

Consequently, gathering (2.7), (2.9), (2.11), and (2.12), and incorporating the Dirichlet boundary conditions, we find that (2.1) can be rewritten, equivalently, as follows: Find  $(\boldsymbol{\sigma}, \mathbf{u}, \gamma)$  and  $(\phi_j, \mathbf{t}_j, \boldsymbol{\vartheta}_j)$ ,  $j \in \{1, 2\}$ , in suitable spaces to be indicated below, such that

$$\begin{aligned} \frac{1}{\mu} \boldsymbol{\sigma}^{\text{d}} + \frac{1}{\mu} (\mathbf{u} \otimes \mathbf{u})^{\text{d}} + \gamma &= \nabla \mathbf{u} && \text{in } \Omega, \\ \mathbf{D} \mathbf{u} + \mathbf{F} |\mathbf{u}|^{\rho-2} \mathbf{u} - \operatorname{div}(\boldsymbol{\sigma}) &= \mathbf{f}(\phi) && \text{in } \Omega, \\ \mathbf{t}_j &= \nabla \phi_j && \text{in } \Omega, \quad j \in \{1, 2\}, \\ \mathbf{Q}_j \mathbf{t}_j - \frac{1}{2} \mathbb{R}_j \phi_j \mathbf{u} &= \boldsymbol{\vartheta}_j && \text{in } \Omega, \quad j \in \{1, 2\}, \\ \frac{1}{2} \mathbb{R}_j \mathbf{u} \cdot \mathbf{t}_j - \operatorname{div}(\boldsymbol{\vartheta}_j) &= g_j && \text{in } \Omega, \quad j \in \{1, 2\}, \\ \mathbf{u} = \mathbf{u}_{\text{D}}, \quad \phi_1 = \phi_{1,\text{D}}, \quad \text{and} \quad \phi_2 = \phi_{2,\text{D}} &&& \text{on } \Gamma, \\ \int_{\Omega} \operatorname{tr}(\boldsymbol{\sigma} + (\mathbf{u} \otimes \mathbf{u})) &= 0 && \text{in } \Omega. \end{aligned} \quad (2.13)$$

We stress here that, as suggested by (2.8),  $p$  is eliminated from the present formulation and computed afterwards in terms of  $\boldsymbol{\sigma}$  and  $\mathbf{u}$  by using that identity. This fact justifies the last equation in (2.13), which aims to ensure that the resulting  $p$  does belong to  $L_0^2(\Omega)$ . Notice also that further variables of interest, such as the velocity gradient  $\nabla \mathbf{u}$ , and the shear stress tensor  $\tilde{\boldsymbol{\sigma}} := \mu \mathbf{e}(\mathbf{u}) - p \mathbb{I}$ , can be easily computed, respectively, as follows

$$\nabla \mathbf{u} = \frac{1}{\mu} \boldsymbol{\sigma}^{\text{d}} + \frac{1}{\mu} (\mathbf{u} \otimes \mathbf{u})^{\text{d}} + \gamma \quad \text{and} \quad \tilde{\boldsymbol{\sigma}} = \boldsymbol{\sigma} + (\mathbf{u} \otimes \mathbf{u}). \quad (2.14)$$

---

## The continuous formulation

---

In this chapter we follow [8] and [18] (see also [12, 13, 25, 26, 27]) to derive a mixed formulation for (2.13) within a Banach spaces framework. We begin by testing the first equation of (2.13) against a tensor  $\boldsymbol{\tau}$  associated with the unknown  $\boldsymbol{\sigma}$ , so that, using the identity  $\boldsymbol{\sigma}^{\text{d}} : \boldsymbol{\tau} = \boldsymbol{\sigma}^{\text{d}} : \boldsymbol{\tau}^{\text{d}}$ , we formally get

$$\int_{\Omega} \frac{1}{\mu} \boldsymbol{\sigma}^{\text{d}} : \boldsymbol{\tau}^{\text{d}} + \int_{\Omega} \frac{1}{\mu} (\mathbf{u} \otimes \mathbf{u})^{\text{d}} : \boldsymbol{\tau} + \int_{\Omega} \boldsymbol{\gamma} : \boldsymbol{\tau} = \int_{\Omega} \nabla \mathbf{u} : \boldsymbol{\tau}. \quad (3.1)$$

We observe that the first and third expressions on the left hand side of (3.1) make sense for  $\boldsymbol{\sigma}, \boldsymbol{\tau}, \boldsymbol{\gamma} \in \mathbb{L}^2(\Omega)$ . In turn, seeking originally  $\mathbf{u} \in \mathbf{H}^1(\Omega)$ , which is in line with the condition that  $\mathbf{u}_D \in \mathbf{H}^{1/2}(\Gamma)$ , and assuming that  $\boldsymbol{\tau}$  is taken in  $\mathbb{H}(\mathbf{div}_t; \Omega)$ , with  $t$  fitting the ranges for the validity of (1.1) and (1.2), we can apply the latter, and employ the Dirichlet boundary condition on  $\mathbf{u}$ , to obtain

$$\int_{\Omega} \nabla \mathbf{u} : \boldsymbol{\tau} = - \int_{\Omega} \mathbf{u} \cdot \mathbf{div}(\boldsymbol{\tau}) + \langle \boldsymbol{\tau} \boldsymbol{\nu}, \mathbf{u}_D \rangle_{\Gamma}. \quad (3.2)$$

In this way, replacing (3.2) back into (3.1), we arrive at

$$\int_{\Omega} \frac{1}{\mu} \boldsymbol{\sigma}^d : \boldsymbol{\tau}^d + \int_{\Omega} \mathbf{u} \cdot \mathbf{div}(\boldsymbol{\tau}) + \int_{\Omega} \frac{1}{\mu} (\mathbf{u} \otimes \mathbf{u})^d : \boldsymbol{\tau} + \int_{\Omega} \boldsymbol{\gamma} : \boldsymbol{\tau} = \langle \boldsymbol{\tau} \boldsymbol{\nu}, \mathbf{u}_D \rangle_{\Gamma} \quad (3.3)$$

for all  $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_t; \Omega)$ . Now, knowing that  $\mathbf{div}(\boldsymbol{\tau}) \in \mathbf{L}^t(\Omega)$ , and using Hölder's inequality, we conclude from the second term in (3.3) that it suffices to look for  $\mathbf{u}$  in  $\mathbf{L}^{t'}(\Omega)$  instead of  $\mathbf{H}^1(\Omega)$ , where  $t, t' \in (1, +\infty)$  are conjugate to each other. In addition, employing the Cauchy–Schwarz and Hölder inequalities, we readily deduce that the convective nonlinear term is well defined if  $\mathbf{u} \in \mathbf{L}^4(\Omega)$ , which yields to choose  $t' = 4$ , and thus  $t = 4/3$ , whence the test space for  $\boldsymbol{\tau}$  becomes  $\mathbb{H}(\mathbf{div}_{4/3}; \Omega)$ .

On the other hand, linking the spaces to which the unknown  $\boldsymbol{\sigma}$  and its test functions  $\boldsymbol{\tau}$  belong, we impose to look for  $\boldsymbol{\sigma}$  in  $\mathbb{H}(\mathbf{div}_{4/3}; \Omega)$  as well. Hence, testing the second equation of (2.13) against  $\mathbf{v} \in \mathbf{L}^4(\Omega)$ , formally yields

$$\int_{\Omega} \mathbf{v} \cdot \mathbf{div}(\boldsymbol{\sigma}) - \mathbb{D} \int_{\Omega} \mathbf{u} \cdot \mathbf{v} - \mathbb{F} \int_{\Omega} |\mathbf{u}|^{\rho-2} \mathbf{u} \cdot \mathbf{v} = - \int_{\Omega} \mathbf{f}(\boldsymbol{\phi}) \cdot \mathbf{v} \quad (3.4)$$

for all  $\mathbf{v} \in \mathbf{L}^4(\Omega)$ , from which the first term is bounded thanks to the fact that  $\mathbf{div}(\boldsymbol{\sigma}) \in \mathbf{L}^{4/3}(\Omega)$ . Next, noting that for  $\rho \in [3, 4]$  there holds  $2(\rho - 2) \leq 4$ , we consider the continuous injection  $i_{2(\rho-2)} : \mathbf{L}^4(\Omega) \rightarrow \mathbf{L}^{2(\rho-2)}(\Omega)$  and observe that  $\|i_{2(\rho-2)}\| \leq |\Omega|^{(4-\rho)/4(\rho-2)}$ . In this way, applying the Cauchy–Schwarz and Hölder inequalities to the third term on the left-hand side of (3.4), we find that

$$\left| \int_{\Omega} |\mathbf{w}|^{\rho-2} \mathbf{u} \cdot \mathbf{v} \right| \leq \|\mathbf{w}\|_{0,2(\rho-2); \Omega}^{\rho-2} \|\mathbf{u}\|_{0,4; \Omega} \|\mathbf{v}\|_{0,4; \Omega} \leq |\Omega|^{(4-\rho)/4} \|\mathbf{w}\|_{0,4; \Omega}^{\rho-2} \|\mathbf{u}\|_{0,4; \Omega} \|\mathbf{v}\|_{0,4; \Omega},$$

which proves that the aforementioned term is well-defined for  $\mathbf{u}, \mathbf{w}, \mathbf{v} \in \mathbf{L}^4(\Omega)$ . In turn, being  $\mathbf{L}^4(\Omega)$  certainly contained in  $\mathbf{L}^2(\Omega)$  guarantees that the second term in (3.4) is bounded as well, whereas the right hand side of (3.4) becomes well defined if  $\mathbf{f}(\boldsymbol{\phi})$  (cf. (2.3)) belongs to  $\mathbf{L}^{4/3}(\Omega)$ , which is assumed from now on. We will refer again to this issue later on.

Finally, the symmetry of  $\boldsymbol{\sigma}$  (cf. (2.6)) is imposed weakly as

$$\int_{\Omega} \boldsymbol{\delta} : \boldsymbol{\sigma} = 0 \quad \forall \boldsymbol{\delta} \in \mathbb{L}_{\text{skew}}^2(\Omega), \quad (3.5)$$

where

$$\mathbb{L}_{\text{skew}}^2(\Omega) := \left\{ \boldsymbol{\delta} \in \mathbb{L}^2(\Omega) : \boldsymbol{\delta}^t = -\boldsymbol{\delta} \right\}.$$

According to the previous analysis, the weak formulation of the convective Brinkman–Forchheimer problem (2.13) reduces at first instance to: Find  $(\boldsymbol{\sigma}, \boldsymbol{\gamma}, \mathbf{u}) \in \mathbb{H}(\mathbf{div}_{4/3}; \Omega) \times \mathbb{L}_{\text{skew}}^2(\Omega) \times \mathbf{L}^4(\Omega)$  such that (3.3), (3.4) and (3.5) hold for all  $(\boldsymbol{\tau}, \boldsymbol{\delta}, \mathbf{v}) \in \mathbb{H}(\mathbf{div}_{4/3}; \Omega) \times \mathbb{L}_{\text{skew}}^2(\Omega) \times \mathbf{L}^4(\Omega)$ . However, similarly as in [8] (see also [12], [18]), we consider the decomposition

$$\mathbb{H}(\mathbf{div}_{4/3}; \Omega) = \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega) \oplus \mathbb{R} \mathbb{I}, \quad (3.6)$$

where

$$\mathbb{H}_0(\mathbf{div}_{4/3}; \Omega) := \left\{ \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_{4/3}; \Omega) : \int_{\Omega} \text{tr}(\boldsymbol{\tau}) = 0 \right\},$$

which means that each  $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_{4/3}; \Omega)$  can be uniquely decomposed as

$$\boldsymbol{\tau} = \boldsymbol{\tau}_0 + d_0 \mathbb{I} \quad \text{with} \quad \boldsymbol{\tau}_0 \in \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega) \quad \text{and} \quad d_0 := \frac{1}{n |\Omega|} \int_{\Omega} \text{tr}(\boldsymbol{\tau}) \in \mathbb{R}.$$

In particular, using the last equation of (2.13), we obtain

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_0 + c_0 \mathbb{I} \quad \text{with} \quad \boldsymbol{\sigma}_0 \in \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega) \quad \text{and} \quad c_0 := -\frac{1}{n |\Omega|} \int_{\Omega} \text{tr}(\mathbf{u} \otimes \mathbf{u}), \quad (3.7)$$

which says that  $c_0$  is known explicitly in terms of  $\mathbf{u}$ . Therefore, in order to fully determine  $\boldsymbol{\sigma}$ , it only remains to find its  $\mathbb{H}_0(\mathbf{div}_{4/3}; \Omega)$ -component  $\boldsymbol{\sigma}_0$ , which is renamed from now on simply as  $\boldsymbol{\sigma}$ .

Next, using the compatibility condition (2.5), we observe that both sides of (3.3) vanish when  $\boldsymbol{\tau} = \mathbb{I}$ , and hence testing against  $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_{4/3}; \Omega)$  is equivalent to doing it against

$\boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega)$ . Therefore, bearing in mind the foregoing discussion, denoting

$$\mathbf{H} := \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega), \quad \mathbf{Q} := \mathbf{L}^4(\Omega) \times \mathbb{L}_{\text{skew}}^2(\Omega),$$

and setting

$$\vec{\mathbf{u}} = (\mathbf{u}, \boldsymbol{\gamma}), \quad \vec{\mathbf{v}} = (\mathbf{v}, \boldsymbol{\delta}), \quad \vec{\mathbf{z}} = (\mathbf{z}, \boldsymbol{\chi}) \in \mathbf{Q},$$

we arrive at the following mixed formulation for the convective Brinkman–Forchheimer equations: Find  $(\boldsymbol{\sigma}, \vec{\mathbf{u}}) \in \mathbf{H} \times \mathbf{Q}$  such that

$$\begin{aligned} a(\boldsymbol{\sigma}, \boldsymbol{\tau}) + b(\boldsymbol{\tau}, \vec{\mathbf{u}}) + \mathbf{b}(\mathbf{u}; \mathbf{u}, \boldsymbol{\tau}) &= \mathbf{G}(\boldsymbol{\tau}) \quad \forall \boldsymbol{\tau} \in \mathbf{H}, \\ b(\boldsymbol{\sigma}, \vec{\mathbf{v}}) - c(\mathbf{u}; \vec{\mathbf{u}}, \vec{\mathbf{v}}) &= \mathbf{F}_\phi(\vec{\mathbf{v}}) \quad \forall \vec{\mathbf{v}} \in \mathbf{Q}, \end{aligned} \quad (3.8)$$

where the bilinear forms  $a : \mathbf{H} \times \mathbf{H} \rightarrow \mathbb{R}$  and  $b : \mathbf{H} \times \mathbf{Q} \rightarrow \mathbb{R}$  are defined as

$$a(\boldsymbol{\zeta}, \boldsymbol{\tau}) := \int_{\Omega} \frac{1}{\mu} \boldsymbol{\zeta}^{\text{d}} : \boldsymbol{\tau}^{\text{d}} \quad \forall (\boldsymbol{\zeta}, \boldsymbol{\tau}) \in \mathbf{H} \times \mathbf{H}, \quad (3.9)$$

and

$$b(\boldsymbol{\tau}, \vec{\mathbf{v}}) := \int_{\Omega} \mathbf{v} \cdot \mathbf{div}(\boldsymbol{\tau}) + \int_{\Omega} \boldsymbol{\delta} : \boldsymbol{\tau} \quad \forall (\boldsymbol{\tau}, \vec{\mathbf{v}}) \in \mathbf{H} \times \mathbf{Q}, \quad (3.10)$$

whereas, for each  $\mathbf{w} \in \mathbf{L}^4(\Omega)$ , the bilinear forms  $\mathbf{b}(\mathbf{w}; \cdot, \cdot) : \mathbf{L}^4(\Omega) \times \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega) \rightarrow \mathbb{R}$  and  $c(\mathbf{w}; \cdot, \cdot) : \mathbf{Q} \times \mathbf{Q} \rightarrow \mathbb{R}$  are given by

$$\mathbf{b}(\mathbf{w}; \mathbf{v}, \boldsymbol{\tau}) := \int_{\Omega} \frac{1}{\mu} (\mathbf{w} \otimes \mathbf{v}) : \boldsymbol{\tau} \quad \forall (\mathbf{v}, \boldsymbol{\tau}) \in \mathbf{L}^4(\Omega) \times \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega), \quad (3.11)$$

and

$$c(\mathbf{w}; \vec{\mathbf{u}}, \vec{\mathbf{v}}) := \text{D} \int_{\Omega} \mathbf{u} \cdot \mathbf{v} + \text{F} \int_{\Omega} |\mathbf{w}|^{\rho-2} \mathbf{u} \cdot \mathbf{v} \quad \forall (\vec{\mathbf{u}}, \vec{\mathbf{v}}) \in \mathbf{Q} \times \mathbf{Q}. \quad (3.12)$$

Finally, the linear and bounded functionals  $\mathbf{G} : \mathbf{H} \rightarrow \mathbb{R}$  and  $\mathbf{F}_\phi : \mathbf{Q} \rightarrow \mathbb{R}$  reduce to

$$\mathbf{G}(\boldsymbol{\tau}) := \langle \boldsymbol{\tau} \boldsymbol{\nu}, \mathbf{u}_D \rangle \quad \forall \boldsymbol{\tau} \in \mathbf{H} \quad \text{and} \quad \mathbf{F}_\phi(\vec{\mathbf{v}}) := \int_{\Omega} \mathbf{f}(\phi) \cdot \mathbf{v} \quad \forall \vec{\mathbf{v}} \in \mathbf{Q}. \quad (3.13)$$

On the other hand, for the double diffusion equations, which are described by the third up

to the fifth rows of (2.13), we proceed similarly as for the convective Brinkman-Forchheimer equations, and look originally for  $\phi_j \in H^1(\Omega)$ , which, besides yielding  $\mathbf{t}_j \in \mathbf{L}^2(\Omega)$ , is in line as well with the fact that the data  $\phi_{i,D} \in H^{1/2}(\Gamma)$ ,  $i \in \{1, 2\}$ . Then, testing the aforementioned third equation against  $\boldsymbol{\eta}_j \in \mathbf{H}(\text{div}_t; \Omega)$ , with  $t$  as before, applying now (1.1), and using the Dirichlet boundary condition on  $\phi_j$ , we get

$$\int_{\Omega} \mathbf{t}_j \cdot \boldsymbol{\eta}_j + \int_{\Omega} \phi_j \text{div}(\boldsymbol{\eta}_j) = \langle \boldsymbol{\eta}_j \cdot \boldsymbol{\nu}, \phi_{j,D} \rangle \quad j \in \{1, 2\}. \quad (3.14)$$

In this way, knowing that  $\text{div}(\boldsymbol{\eta}_j) \in L^t(\Omega)$ , we realize from the second term of (3.14) and Hölder's inequality that it suffices to look for  $\phi_j$  in  $L^{t'}(\Omega)$ . Needless to say, it is clear that the first term makes sense since both  $\mathbf{t}_j$  and  $\boldsymbol{\eta}_j$  belong to  $\mathbf{L}^2(\Omega)$ . Next, letting  $\mathbf{L}^2(\Omega)$  be as well the space of test functions associated with the unknown  $\mathbf{t}_j$ , the corresponding testing of the fourth row of (2.13) formally gives

$$\int_{\Omega} \mathbf{Q}_j \mathbf{t}_j \cdot \mathbf{r}_j - \frac{1}{2} \text{R}_j \int_{\Omega} \phi_j (\mathbf{u} \cdot \mathbf{r}_j) - \int_{\Omega} \boldsymbol{\vartheta}_j \cdot \mathbf{r}_j = 0 \quad (3.15)$$

for all  $\mathbf{r}_j \in \mathbf{L}^2(\Omega)$ , so that the third term of (3.15) is well-defined if  $\boldsymbol{\vartheta}_j \in \mathbf{L}^2(\Omega)$ . In turn, regarding the second term, and bearing in mind that from the analysis of the Brinkman-Forchheimer equations we know that  $\mathbf{u}$  must be sought in  $\mathbf{L}^4(\Omega)$ , direct applications of the Cauchy-Schwarz and Hölder inequalities imply

$$\left| \int_{\Omega} \phi_j (\mathbf{u} \cdot \mathbf{r}_j) \right| \leq \|\phi_j\|_{0,4;\Omega} \|\mathbf{u}\|_{0,4;\Omega} \|\mathbf{r}_j\|_{0,\Omega}, \quad (3.16)$$

from which it is natural to fix the seeking space for  $\phi_j$  as  $L^4(\Omega)$ , that is  $t' = 4$ , which yields  $t = 4/3$ . In this way, letting  $\mathbf{H}(\text{div}_{4/3}; \Omega)$  and  $L^4(\Omega)$  be as well the spaces where  $\boldsymbol{\vartheta}_j$  is sought and where the test functions associated with  $\phi_j$  belong to, respectively, we can test the fifth row of (2.13) against  $\psi_j \in L^4(\Omega)$  to obtain

$$\frac{1}{2} \text{R}_j \int_{\Omega} \psi_j (\mathbf{u} \cdot \mathbf{t}_j) - \int_{\Omega} \psi_j \text{div}(\boldsymbol{\vartheta}_j) = \int_{\Omega} g_j \psi_j. \quad (3.17)$$

Note that the first and second terms of (3.17) are well-defined thanks to the analogue estimate

(3.16) and the fact that  $\operatorname{div}(\boldsymbol{\vartheta}_j) \in L^{4/3}(\Omega)$ , whereas the expression on the right-hand side makes sense if  $g_j \in L^{4/3}(\Omega)$ , which we assume from now on. Therefore, introducing the spaces

$$\tilde{\mathbf{H}} := L^4(\Omega) \times \mathbf{L}^2(\Omega) \quad \text{and} \quad \tilde{\mathbf{Q}} := \mathbf{H}(\operatorname{div}_{4/3}; \Omega),$$

setting the variables

$$\vec{\phi}_j = (\phi_j, \mathbf{t}_j), \quad \vec{\psi}_j = (\psi_j, \mathbf{r}_j), \quad \vec{\xi}_j = (\xi_j, \mathbf{s}_j) \in \tilde{\mathbf{H}},$$

and grouping conveniently (3.14), (3.15), and (3.17), we arrive at the weak formulation: Find  $(\vec{\phi}_j, \boldsymbol{\vartheta}_j) \in \tilde{\mathbf{H}} \times \tilde{\mathbf{Q}}$ ,  $j \in \{1, 2\}$ , such that

$$\begin{aligned} \tilde{a}_j(\vec{\phi}_j, \vec{\psi}_j) + \tilde{c}_j(\mathbf{u}; \vec{\phi}_j, \vec{\psi}_j) + \tilde{b}(\vec{\psi}_j, \boldsymbol{\vartheta}_j) &= \tilde{F}_j(\vec{\psi}_j) & \forall \vec{\psi}_j \in \tilde{\mathbf{H}}, \\ \tilde{b}(\vec{\phi}_j, \boldsymbol{\eta}_j) &= \tilde{G}_j(\boldsymbol{\eta}_j) & \forall \boldsymbol{\eta}_j \in \tilde{\mathbf{Q}}, \end{aligned} \quad (3.18)$$

where, for  $j \in \{1, 2\}$ , the bilinear forms  $\tilde{a}_j : \tilde{\mathbf{H}} \times \tilde{\mathbf{H}} \rightarrow \mathbb{R}$ ,  $\tilde{b} : \tilde{\mathbf{H}} \times \tilde{\mathbf{Q}} \rightarrow \mathbb{R}$ , and the linear and bounded functionals  $\tilde{F}_j : \tilde{\mathbf{H}} \rightarrow \mathbb{R}$  and  $\tilde{G}_j : \tilde{\mathbf{Q}} \rightarrow \mathbb{R}$  are defined, respectively, as:

$$\tilde{a}_j(\vec{\xi}_j, \vec{\psi}_j) := \int_{\Omega} \mathbf{Q}_j \mathbf{s}_j \cdot \mathbf{r}_j \quad \forall (\vec{\xi}_j, \vec{\psi}_j) \in \tilde{\mathbf{H}} \times \tilde{\mathbf{H}}, \quad (3.19)$$

$$\tilde{b}(\vec{\xi}_j, \boldsymbol{\eta}_j) := - \int_{\Omega} \mathbf{s}_j \cdot \boldsymbol{\eta}_j - \int_{\Omega} \xi_j \operatorname{div}(\boldsymbol{\eta}_j) \quad \forall (\vec{\xi}_j, \boldsymbol{\eta}_j) \in \tilde{\mathbf{H}} \times \tilde{\mathbf{Q}}, \quad (3.20)$$

$$\tilde{F}_j(\vec{\psi}_j) := \int_{\Omega} g_j \psi_j \quad \forall \vec{\psi}_j \in \tilde{\mathbf{H}}, \quad \text{and} \quad (3.21)$$

$$\tilde{G}_j(\boldsymbol{\eta}_j) := - \langle \boldsymbol{\eta}_j \cdot \boldsymbol{\nu}, \phi_{j,D} \rangle \quad \forall \boldsymbol{\eta}_j \in \tilde{\mathbf{Q}}, \quad (3.22)$$

whereas, given  $\mathbf{w} \in L^4(\Omega)$ , the bilinear form  $\tilde{c}_j(\mathbf{w}; \cdot, \cdot) : \tilde{\mathbf{H}} \times \tilde{\mathbf{H}} \rightarrow \mathbb{R}$  is given by

$$\tilde{c}_j(\mathbf{w}; \vec{\xi}_j, \vec{\psi}_j) := \frac{1}{2} \mathbb{R}_j \left\{ \int_{\Omega} \psi_j (\mathbf{w} \cdot \mathbf{s}_j) - \int_{\Omega} \xi_j (\mathbf{w} \cdot \mathbf{r}_j) \right\} \quad \forall \vec{\xi}_j, \vec{\psi}_j \in \tilde{\mathbf{H}}. \quad (3.23)$$

Summarizing, the fully mixed formulation of the Brinkman–Forchheimer equations coupled

with double diffusion equations (cf. (2.13)) reads: Find  $(\boldsymbol{\sigma}, \bar{\mathbf{u}}) \in \mathbf{H} \times \mathbf{Q}$  and  $(\vec{\phi}_j, \boldsymbol{\vartheta}_j) \in \tilde{\mathbf{H}} \times \tilde{\mathbf{Q}}$ ,  $j \in \{1, 2\}$ , such that

$$\begin{aligned}
a(\boldsymbol{\sigma}, \boldsymbol{\tau}) + b(\boldsymbol{\tau}, \bar{\mathbf{u}}) + \mathbf{b}(\mathbf{u}; \mathbf{u}, \boldsymbol{\tau}) &= \mathbf{G}(\boldsymbol{\tau}) & \forall \boldsymbol{\tau} \in \mathbf{H}, \\
b(\boldsymbol{\sigma}, \vec{\mathbf{v}}) - c(\mathbf{u}; \bar{\mathbf{u}}, \vec{\mathbf{v}}) &= \mathbf{F}_\phi(\vec{\mathbf{v}}) & \forall \vec{\mathbf{v}} \in \mathbf{Q}, \\
\tilde{a}_j(\vec{\phi}_j, \vec{\psi}_j) + \tilde{c}_j(\mathbf{u}; \vec{\phi}_j, \vec{\psi}_j) + \tilde{b}(\vec{\psi}_j, \boldsymbol{\vartheta}_j) &= \tilde{F}_j(\vec{\psi}_j) & \forall \vec{\psi}_j \in \tilde{\mathbf{H}}, \\
\tilde{b}(\vec{\phi}_j, \boldsymbol{\eta}_j) &= \tilde{G}_j(\boldsymbol{\eta}_j) & \forall \boldsymbol{\eta}_j \in \tilde{\mathbf{Q}}.
\end{aligned} \tag{3.24}$$

---

## Analysis of the coupled problem

---

In this chapter we combine classical and new results on the solvability of variational formulations in Banach spaces to establish the well-posedness of (3.24).

### 4.1 Preliminaries

The stability properties of the operators and functionals involved in (3.24) are provided first. In fact, direct applications of the Cauchy-Schwarz and Hölder inequalities, along with the upper bounds of  $\mu$  (cf. (2.2)), the continuity of the normal trace operator in  $\mathbb{H}(\mathbf{div}_{4/3}; \Omega)$ , and the continuity of the injection  $i_4 : H^1(\Omega) \rightarrow L^4(\Omega)$  and its vectorial version  $\mathbf{i}_4$ , yield the existence

of positive constants, denoted and given as:

$$\begin{aligned}
\|a\| &:= \frac{1}{\mu_0}, \quad \|b\| := 1, \quad \|\tilde{a}_j\| := \|\mathbf{Q}_j\|_{0,\infty;\Omega}, \quad \|\tilde{b}\| := 2, \\
\|\mathbf{G}\| &:= \|\mathbf{u}_D\|_{1/2,\Gamma} \|\mathbf{i}_4\|, \quad \|\mathbf{F}\boldsymbol{\varphi}\| := \|\mathbf{g}\|_{0,\Omega} \left\{ \|\boldsymbol{\varphi}\|_{0,4;\Omega} + \|\boldsymbol{\phi}_r\|_{0,4;\Omega} \right\}, \\
\|\tilde{F}_j\| &:= \|g_j\|_{0,4/3;\Omega}, \quad \text{and} \quad \|\tilde{G}_j\| := \|\phi_{j,D}\|_{1/2,\Gamma} \|\mathbf{i}_4\|,
\end{aligned} \tag{4.1}$$

where  $j \in \{1, 2\}$  and  $\boldsymbol{\varphi} = (\varphi_1, \varphi_2) \in \mathbf{L}^4(\Omega) \times \mathbf{L}^4(\Omega)$ , such that there hold

$$\begin{aligned}
|a(\boldsymbol{\zeta}, \boldsymbol{\tau})| &\leq \|a\| \|\boldsymbol{\zeta}\|_{\mathbf{H}} \|\boldsymbol{\tau}\|_{\mathbf{H}} && \forall \boldsymbol{\zeta}, \boldsymbol{\tau} \in \mathbf{H}, \\
|b(\boldsymbol{\tau}, \vec{\mathbf{v}})| &\leq \|b\| \|\boldsymbol{\tau}\|_{\mathbf{H}} \|\vec{\mathbf{v}}\|_{\mathbf{Q}} && \forall (\boldsymbol{\tau}, \vec{\mathbf{v}}) \in \mathbf{H} \times \mathbf{Q}, \\
|\tilde{a}_j(\vec{\boldsymbol{\phi}}_j, \vec{\boldsymbol{\psi}}_j)| &\leq \|\tilde{a}_j\| \|\vec{\boldsymbol{\phi}}_j\|_{\tilde{\mathbf{H}}} \|\vec{\boldsymbol{\psi}}_j\|_{\tilde{\mathbf{H}}} && \forall \vec{\boldsymbol{\phi}}_j, \vec{\boldsymbol{\psi}}_j \in \tilde{\mathbf{H}}, \\
|\tilde{b}(\vec{\boldsymbol{\psi}}_j, \boldsymbol{\eta}_j)| &\leq \|\tilde{b}\| \|\vec{\boldsymbol{\psi}}_j\|_{\tilde{\mathbf{H}}} \|\boldsymbol{\eta}_j\|_{\tilde{\mathbf{Q}}} && \forall (\vec{\boldsymbol{\psi}}_j, \boldsymbol{\eta}_j) \in \tilde{\mathbf{H}} \times \tilde{\mathbf{Q}}, \\
|\mathbf{G}(\boldsymbol{\tau})| &\leq \|\mathbf{G}\| \|\boldsymbol{\tau}\|_{\mathbf{H}} && \forall \boldsymbol{\tau} \in \mathbf{H}, \\
|\mathbf{F}\boldsymbol{\varphi}(\vec{\mathbf{v}})| &\leq \|\mathbf{F}\boldsymbol{\varphi}\| \|\vec{\mathbf{v}}\|_{\mathbf{Q}} && \forall \vec{\mathbf{v}} \in \mathbf{Q}, \\
|\tilde{F}_j(\vec{\boldsymbol{\psi}}_j)| &\leq \|\tilde{F}_j\| \|\vec{\boldsymbol{\psi}}_j\|_{\tilde{\mathbf{H}}} && \forall \vec{\boldsymbol{\psi}}_j \in \tilde{\mathbf{H}}, \quad \text{and} \\
|\tilde{G}_j(\boldsymbol{\eta}_j)| &\leq \|\tilde{G}_j\| \|\boldsymbol{\eta}_j\|_{\tilde{\mathbf{Q}}} && \forall \boldsymbol{\eta}_j \in \tilde{\mathbf{Q}}.
\end{aligned} \tag{4.2}$$

In turn, given  $\mathbf{w} \in \mathbf{L}^4(\Omega)$ , we apply the Cauchy-Schwarz and Hölder inequalities, similarly as we did in (4.1) - (4.2), and previously in (3.16), to derive the following bounds for  $\mathbf{b}$  (cf. (3.11)),  $c$  (cf. (3.12)), and  $\tilde{c}_j$  (cf. (3.23))

$$\begin{aligned}
|\mathbf{b}(\mathbf{w}; \mathbf{v}, \boldsymbol{\tau})| &\leq \frac{1}{\mu_0} \|\mathbf{w}\|_{0,4;\Omega} \|\mathbf{v}\|_{0,4;\Omega} \|\boldsymbol{\tau}\|_{\text{div}_{4/3};\Omega} && \forall (\mathbf{v}, \boldsymbol{\tau}) \in \mathbf{L}^4(\Omega) \times \mathbb{H}_0(\text{div}_{4/3}; \Omega), \\
|c(\mathbf{w}; \vec{\mathbf{v}}, \vec{\mathbf{z}})| &\leq |\Omega|^{1/2} (\mathbf{D} + \mathbf{F} \|\mathbf{w}\|_{0,4;\Omega}^{\rho-2}) \|\vec{\mathbf{v}}\|_{\mathbf{Q}} \|\vec{\mathbf{z}}\|_{\mathbf{Q}} && \forall \vec{\mathbf{v}}, \vec{\mathbf{z}} \in \mathbf{Q}, \quad \text{and} \\
|\tilde{c}_j(\mathbf{w}; \vec{\boldsymbol{\phi}}_j, \vec{\boldsymbol{\psi}}_j)| &\leq R_j \|\mathbf{w}\|_{0,4;\Omega} \|\vec{\boldsymbol{\phi}}_j\|_{\tilde{\mathbf{H}}} \|\vec{\boldsymbol{\psi}}_j\|_{\tilde{\mathbf{H}}} && \forall \vec{\boldsymbol{\phi}}_j, \vec{\boldsymbol{\psi}}_j \in \tilde{\mathbf{H}}.
\end{aligned} \tag{4.3}$$

Moreover, noting from the definition of  $\tilde{c}_j$  (cf. (3.23)) that  $\tilde{c}_j(\cdot; \vec{\boldsymbol{\phi}}_j, \vec{\boldsymbol{\psi}}_j)$  is linear, we readily

deduce from the third row of (4.3) that

$$|\tilde{c}_j(\mathbf{w}; \vec{\phi}_j, \vec{\psi}_j) - \tilde{c}_j(\mathbf{z}; \vec{\phi}_j, \vec{\psi}_j)| \leq R_j \|\mathbf{w} - \mathbf{z}\|_{0,4;\Omega} \|\vec{\phi}_j\|_{\tilde{\mathbf{H}}} \|\vec{\psi}_j\|_{\tilde{\mathbf{H}}} \quad (4.4)$$

for all  $\mathbf{w}, \mathbf{z} \in \mathbf{L}^4(\Omega)$  and for all  $\vec{\phi}_j, \vec{\psi}_j \in \tilde{\mathbf{H}}$ , and it is also clear from (3.23) that there holds

$$\tilde{c}_j(\mathbf{w}; \vec{\psi}_j, \vec{\psi}_j) = 0 \quad (4.5)$$

for all  $\mathbf{w} \in \mathbf{L}^4(\Omega)$  and for all  $\vec{\psi}_j \in \tilde{\mathbf{H}}$ .

## 4.2 A fixed point strategy

In what follows, we proceed similarly to [13] (see also [12]) and adopt a fixed-point strategy to address the well-posedness of (3.24). We begin by letting  $\mathbf{S} : \mathbf{L}^4(\Omega) \times (\mathbf{L}^4(\Omega) \times \mathbf{L}^4(\Omega)) \rightarrow \mathbf{L}^4(\Omega)$  be the operator defined as

$$\mathbf{S}(\mathbf{w}, \boldsymbol{\varphi}) := \mathbf{u} \quad \forall (\mathbf{w}, \boldsymbol{\varphi}) \in \mathbf{L}^4(\Omega) \times (\mathbf{L}^4(\Omega) \times \mathbf{L}^4(\Omega)), \quad (4.6)$$

where  $(\boldsymbol{\sigma}, \vec{\mathbf{u}}) := (\boldsymbol{\sigma}, (\mathbf{u}, \boldsymbol{\gamma})) \in \mathbf{H} \times \mathbf{Q}$  is the unique solution (to be confirmed below) of the problem arising from the first two rows of (3.24) after replacing  $\mathbf{b}(\mathbf{u}; \cdot, \cdot)$ ,  $c(\mathbf{u}; \cdot, \cdot)$ , and  $\mathbf{F}_\phi$  by  $\mathbf{b}(\mathbf{w}; \cdot, \cdot)$ ,  $c(\mathbf{w}; \cdot, \cdot)$ , and  $\mathbf{F}_\varphi$ , respectively, that is

$$\begin{aligned} a(\boldsymbol{\sigma}, \boldsymbol{\tau}) + b(\boldsymbol{\tau}, \vec{\mathbf{u}}) + \mathbf{b}(\mathbf{w}; \mathbf{u}, \boldsymbol{\tau}) &= \mathbf{G}(\boldsymbol{\tau}) \quad \forall \boldsymbol{\tau} \in \mathbf{H}, \\ b(\boldsymbol{\sigma}, \vec{\mathbf{v}}) - c(\mathbf{w}; \vec{\mathbf{u}}, \vec{\mathbf{v}}) &= \mathbf{F}_\varphi(\vec{\mathbf{v}}) \quad \forall \vec{\mathbf{v}} \in \mathbf{Q}. \end{aligned} \quad (4.7)$$

Equivalently, introducing the bilinear form  $\mathbf{A}_\mathbf{w} : (\mathbf{H} \times \mathbf{Q}) \times (\mathbf{H} \times \mathbf{Q}) \rightarrow \mathbb{R}$  given by

$$\mathbf{A}_\mathbf{w}((\boldsymbol{\sigma}, \vec{\mathbf{u}}), (\boldsymbol{\tau}, \vec{\mathbf{v}})) := a(\boldsymbol{\sigma}, \boldsymbol{\tau}) + b(\boldsymbol{\tau}, \vec{\mathbf{u}}) + b(\boldsymbol{\sigma}, \vec{\mathbf{v}}) - c(\mathbf{w}; \vec{\mathbf{u}}, \vec{\mathbf{v}}) \quad (4.8)$$

for all  $(\boldsymbol{\sigma}, \vec{\mathbf{u}}), (\boldsymbol{\tau}, \vec{\mathbf{v}}) \in \mathbf{H} \times \mathbf{Q}$ , the uncoupled problem (4.7) can be rewritten as

$$\mathbf{A}_{\mathbf{w}}((\boldsymbol{\sigma}, \vec{\mathbf{u}}), (\boldsymbol{\tau}, \vec{\mathbf{v}})) + \mathbf{b}(\mathbf{w}; \mathbf{u}, \boldsymbol{\tau}) = \mathbf{G}(\boldsymbol{\tau}) + \mathbf{F}_{\boldsymbol{\varphi}}(\vec{\mathbf{v}}) \quad \forall (\boldsymbol{\tau}, \vec{\mathbf{v}}) \in \mathbf{H} \times \mathbf{Q}. \quad (4.9)$$

In turn, for each  $j \in \{1, 2\}$  we define the operator  $\tilde{\mathbf{S}}_j : \mathbf{L}^4(\Omega) \rightarrow \mathbf{L}^4(\Omega)$  given by

$$\tilde{\mathbf{S}}_j(\mathbf{w}) := \phi_j \quad \forall \mathbf{w} \in \mathbf{L}^4(\Omega),$$

where  $(\vec{\phi}_j, \boldsymbol{\vartheta}_j) := ((\phi_j, \mathbf{t}_j), \boldsymbol{\vartheta}_j) \in \tilde{\mathbf{H}} \times \tilde{\mathbf{Q}}$  is the unique solution (to be confirmed below) of the problem arising from the third and fourth rows of (3.24) after replacing  $\tilde{c}_j(\mathbf{u}; \vec{\phi}_j, \vec{\psi}_j)$  by  $\tilde{c}_j(\mathbf{w}; \vec{\phi}_j, \vec{\psi}_j)$ , that is

$$\begin{aligned} \tilde{a}_j(\vec{\phi}_j, \vec{\psi}_j) + \tilde{c}_j(\mathbf{w}; \vec{\phi}_j, \vec{\psi}_j) + \tilde{b}(\vec{\psi}_j, \boldsymbol{\vartheta}_j) &= \tilde{F}_j(\vec{\psi}_j) \quad \forall \vec{\psi}_j \in \tilde{\mathbf{H}}, \\ \tilde{b}(\vec{\phi}_j, \boldsymbol{\eta}_j) &= \tilde{G}_j(\boldsymbol{\eta}_j) \quad \forall \boldsymbol{\eta}_j \in \tilde{\mathbf{Q}}. \end{aligned} \quad (4.10)$$

Similarly as for (4.9), for each  $j \in \{1, 2\}$  we define the bilinear form  $\tilde{a}_{j,\mathbf{w}} : \tilde{\mathbf{H}} \times \tilde{\mathbf{H}} \rightarrow \mathbb{R}$  as

$$\tilde{a}_{j,\mathbf{w}}(\vec{\phi}_j, \vec{\psi}_j) := \tilde{a}_j(\vec{\phi}_j, \vec{\psi}_j) + \tilde{c}_j(\mathbf{w}; \vec{\phi}_j, \vec{\psi}_j) \quad \forall \vec{\phi}_j, \vec{\psi}_j \in \tilde{\mathbf{H}}, \quad (4.11)$$

which allows us to restate (4.10) as

$$\begin{aligned} \tilde{a}_{j,\mathbf{w}}(\vec{\phi}_j, \vec{\psi}_j) + \tilde{b}(\vec{\psi}_j, \boldsymbol{\vartheta}_j) &= \tilde{F}_j(\vec{\psi}_j) \quad \forall \vec{\psi}_j \in \tilde{\mathbf{H}}, \\ \tilde{b}(\vec{\phi}_j, \boldsymbol{\eta}_j) &= \tilde{G}_j(\boldsymbol{\eta}_j) \quad \forall \boldsymbol{\eta}_j \in \tilde{\mathbf{Q}}, \end{aligned} \quad (4.12)$$

Hence, defining  $\tilde{\mathbf{S}} : \mathbf{L}^4(\Omega) \rightarrow (\mathbf{L}^4(\Omega) \times \mathbf{L}^4(\Omega))$  as

$$\tilde{\mathbf{S}}(\mathbf{w}) := (\tilde{\mathbf{S}}_1(\mathbf{w}), \tilde{\mathbf{S}}_2(\mathbf{w})) \quad \forall \mathbf{w} \in \mathbf{L}^4(\Omega), \quad (4.13)$$

and letting  $\mathbf{T} : \mathbf{L}^4(\Omega) \rightarrow \mathbf{L}^4(\Omega)$  be the operator defined by

$$\mathbf{T}(\mathbf{w}) := \mathbf{S}(\mathbf{w}, \tilde{\mathbf{S}}(\mathbf{w})) \quad \forall \mathbf{w} \in \mathbf{L}^4(\Omega), \quad (4.14)$$

we see that solving (3.24) is equivalent to seeking a fixed-point of  $\mathbf{T}$ , that is  $\mathbf{u} \in \mathbf{L}^4(\Omega)$  such that

$$\mathbf{T}(\mathbf{u}) = \mathbf{u}. \quad (4.15)$$

### 4.3 Well posedness of the uncoupled problems

In this section we utilize the Banach–Nečas–Babuška Theorem (cf. [21, Theorem 2.6]), along with recent solvability results for perturbed saddle-point problems in Banach spaces (cf. [19], [20]), and the Banach version of the Babuška–Brezzi theory (cf. [21, Theorem 2.34]), to show that the uncoupled problems (4.7) (or (4.9)) and (4.12) are well-posed, which means, equivalently, that the operators  $\mathbf{S}$  (cf. (4.6)) and  $\tilde{\mathbf{S}}$  (cf. (4.13)) are well-defined. We begin by remarking that, being  $L^p(\Omega)$  reflexive for each  $p \in (1, +\infty)$ , all the spaces involved in the formulations (4.9) and (4.12), namely  $\mathbf{L}^2(\Omega)$ ,  $\mathbf{L}^4(\Omega)$ ,  $\mathbb{L}_{\text{skew}}^2(\Omega)$ ,  $\mathbf{H}(\text{div}_{4/3}; \Omega)$ , and  $\mathbb{H}_0(\mathbf{div}_{4/3}; \Omega)$ , are easily shown to be reflexive as well.

In what follows we address the solvability of (4.7), for which we first show that the bilinear forms  $a$  (cf. (3.9)),  $b$  (cf. (3.10)), and  $c(\mathbf{w}; \cdot, \cdot)$  (cf. (3.12)), for each  $\mathbf{w} \in \mathbf{L}^4(\Omega)$ , which define the bilinear form  $\mathbf{A}_{\mathbf{w}}$  (cf. (4.8)), satisfy the hypotheses of [19, Theorem 3.4]. In fact, it is clear from their respective definitions that  $a$  and  $c(\mathbf{w}; \cdot, \cdot)$  are symmetric and positive semi-definite, which confirms the hypothesis **i**) of [19, Theorem 3.4]. Now, letting  $\mathbf{V}$  be the null space of the linear and bounded operator induced by  $b$ , we readily see (cf. (3.10)) that

$$\mathbf{V} = \left\{ \boldsymbol{\zeta} \in \mathbf{H} : \boldsymbol{\zeta} = \boldsymbol{\zeta}^t \quad \text{and} \quad \mathbf{div}(\boldsymbol{\zeta}) = 0 \right\}. \quad (4.16)$$

In addition, it is already well-known that a slight modification of [23, Lemma 2.3] (see also [7, Proposition IV.3.1], [24, Lemma 3.3], and [8, Lemma 3.1]) allows to prove the existence of a positive constant  $c_1$ , depending on  $\Omega$  and the norm of the continuous injection  $i_4 : \mathbb{H}^1(\Omega) \rightarrow \mathbf{L}^4(\Omega)$ , such that

$$c_1 \|\boldsymbol{\zeta}\|_{0,\Omega} \leq \|\boldsymbol{\zeta}^d\|_{0,\Omega} + \|\mathbf{div}(\boldsymbol{\zeta})\|_{0,4/3;\Omega} \quad \forall \boldsymbol{\zeta} \in \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega). \quad (4.17)$$

Thus, thanks to the boundedness of  $\mu$  (cf. (2.2)) and the inequality (4.17), we deduce that

$$a(\boldsymbol{\zeta}, \boldsymbol{\zeta}) \geq \frac{1}{\mu_1} \|\boldsymbol{\zeta}^d\|_{0,\Omega}^2 \geq \alpha \|\boldsymbol{\zeta}\|_{\mathbf{H}}^2 \quad \forall \boldsymbol{\zeta} \in \mathbf{V}, \quad (4.18)$$

with  $\alpha := \frac{c_1^2}{\mu_1}$ , which easily implies the verification of the continuous inf-sup condition for  $a$  required by the hypothesis **ii**) of [19, Theorem 3.4]. On the other hand, letting  $c_P$  be the positive constant yielding Poincaré's inequality, that is such that  $\|\mathbf{v}\|_{1,\Omega}^2 \leq c_P |\mathbf{v}|_{1,\Omega}^2 \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega)$ , and recalling that  $\mathbf{i}_4$  is the continuous injection of  $\mathbf{H}^1(\Omega)$  into  $\mathbf{L}^4(\Omega)$ , it can be proved (cf. [27, Lemma 3.5]) that there exists a positive constant  $\beta$ , depending only on  $c_P$  and  $\|\mathbf{i}_4\|$ , such that

$$\sup_{\substack{\boldsymbol{\tau} \in \mathbf{V} \\ \boldsymbol{\tau} \neq \mathbf{0}}} \frac{b(\boldsymbol{\tau}, \vec{\mathbf{v}})}{\|\boldsymbol{\tau}\|_{\mathbf{H}}} \geq \beta \|\vec{\mathbf{v}}\|_{\mathbf{Q}} \quad \forall \vec{\mathbf{v}} \in \mathbf{Q},$$

which accomplishes the hypothesis **iii**) of [19, Theorem 3.4]. Furthermore, letting  $\delta > 0$  be an arbitrary radius, we introduce the ball

$$\mathbf{W}(\delta) := \left\{ \mathbf{w} \in \mathbf{L}^4(\Omega) : \|\mathbf{w}\|_{0,4;\Omega} \leq \delta \right\}, \quad (4.19)$$

so that for each  $\mathbf{w} \in \mathbf{W}(\delta)$  the boundedness estimate for  $c(\mathbf{w}; \cdot, \cdot)$  becomes (cf. (4.3))

$$|c(\mathbf{w}; \vec{\mathbf{v}}, \vec{\mathbf{z}})| \leq |\Omega|^{1/2} (D + F \delta^{\rho-2}) \|\vec{\mathbf{v}}\|_{\mathbf{Q}} \|\vec{\mathbf{z}}\|_{\mathbf{Q}} \quad \forall \vec{\mathbf{v}}, \vec{\mathbf{z}} \in \mathbf{Q}. \quad (4.20)$$

Hence, bearing also in mind the expression for  $\|a\|$  (cf. (4.1)), a straightforward application of [19, Theorem 3.4] ensures the existence of a positive constant  $\alpha_{\mathbf{A}}$ , depending only on  $\mu_0$ ,  $|\Omega|$ ,  $D$ ,  $F$ ,  $\delta$ ,  $\rho$ ,  $\alpha$ , and  $\beta$ , such that for each  $\mathbf{w} \in \mathbf{W}(\delta)$  there holds

$$\sup_{\substack{(\boldsymbol{\zeta}, \vec{\mathbf{z}}) \in \mathbf{H} \times \mathbf{Q} \\ (\boldsymbol{\zeta}, \vec{\mathbf{z}}) \neq \mathbf{0}}} \frac{\mathbf{A}_{\mathbf{w}}((\boldsymbol{\tau}, \vec{\mathbf{v}}), (\boldsymbol{\zeta}, \vec{\mathbf{z}}))}{\|(\boldsymbol{\zeta}, \vec{\mathbf{z}})\|_{\mathbf{H} \times \mathbf{Q}}} \geq \alpha_{\mathbf{A}} \|(\boldsymbol{\tau}, \vec{\mathbf{v}})\|_{\mathbf{H} \times \mathbf{Q}} \quad \forall (\boldsymbol{\tau}, \vec{\mathbf{v}}) \in \mathbf{H} \times \mathbf{Q}. \quad (4.21)$$

Then, combining (4.21) with the boundedness estimate for  $\mathbf{b}(\mathbf{w}; \cdot, \cdot)$  (cf. (4.3)), we arrive at

$$\sup_{\substack{(\boldsymbol{\zeta}, \vec{\mathbf{z}}) \in \mathbf{H} \times \mathbf{Q} \\ (\boldsymbol{\zeta}, \vec{\mathbf{z}}) \neq \mathbf{0}}} \frac{\mathbf{A}_{\mathbf{w}}((\boldsymbol{\tau}, \vec{\mathbf{v}}), (\boldsymbol{\zeta}, \vec{\mathbf{z}})) + \mathbf{b}(\mathbf{w}; \mathbf{v}, \boldsymbol{\zeta})}{\|(\boldsymbol{\zeta}, \vec{\mathbf{z}})\|_{\mathbf{H} \times \mathbf{Q}}} \geq \left\{ \alpha_{\mathbf{A}} - \frac{1}{\mu_0} \|\mathbf{w}\|_{0,4;\Omega} \right\} \|(\boldsymbol{\tau}, \vec{\mathbf{v}})\|_{\mathbf{H} \times \mathbf{Q}}$$

for all  $(\boldsymbol{\tau}, \vec{\mathbf{v}}) \in \mathbf{H} \times \mathbf{Q}$ , from which, under the additional assumption that  $\|\mathbf{w}\|_{0,4;\Omega} \leq \frac{\mu_0 \alpha_{\mathbf{A}}}{2}$ , we conclude that

$$\sup_{\substack{(\boldsymbol{\zeta}, \vec{\mathbf{z}}) \in \mathbf{H} \times \mathbf{Q} \\ (\boldsymbol{\zeta}, \vec{\mathbf{z}}) \neq \mathbf{0}}} \frac{\mathbf{A}_{\mathbf{w}}((\boldsymbol{\tau}, \vec{\mathbf{v}}), (\boldsymbol{\zeta}, \vec{\mathbf{z}})) + \mathbf{b}(\mathbf{w}; \mathbf{v}, \boldsymbol{\zeta})}{\|(\boldsymbol{\zeta}, \vec{\mathbf{z}})\|_{\mathbf{H} \times \mathbf{Q}}} \geq \frac{\alpha_{\mathbf{A}}}{2} \|(\boldsymbol{\tau}, \vec{\mathbf{v}})\|_{\mathbf{H} \times \mathbf{Q}} \quad (4.22)$$

for all  $(\boldsymbol{\tau}, \vec{\mathbf{v}}) \in \mathbf{H} \times \mathbf{Q}$ . Similarly, noting that  $\mathbf{A}_{\mathbf{w}}$  is symmetric (since  $a$  and  $c$  are), using again the boundedness estimate for  $\mathbf{b}(\mathbf{w}; \cdot, \cdot)$  (cf. (4.3)), and under the same assumption on  $\mathbf{w}$ , we obtain

$$\sup_{\substack{(\boldsymbol{\tau}, \vec{\mathbf{v}}) \in \mathbf{H} \times \mathbf{Q} \\ (\boldsymbol{\tau}, \vec{\mathbf{v}}) \neq \mathbf{0}}} \frac{\mathbf{A}_{\mathbf{w}}((\boldsymbol{\tau}, \vec{\mathbf{v}}), (\boldsymbol{\zeta}, \vec{\mathbf{z}})) + \mathbf{b}(\mathbf{w}; \mathbf{v}, \boldsymbol{\zeta})}{\|(\boldsymbol{\tau}, \vec{\mathbf{v}})\|_{\mathbf{H} \times \mathbf{Q}}} \geq \frac{\alpha_{\mathbf{A}}}{2} \|(\boldsymbol{\zeta}, \vec{\mathbf{z}})\|_{\mathbf{H} \times \mathbf{Q}} \quad (4.23)$$

for all  $(\boldsymbol{\zeta}, \vec{\mathbf{z}}) \in \mathbf{H} \times \mathbf{Q}$ .

We are now in position of establishing next the well-posedness of (4.9), thanks to which the operator  $\mathbf{S}$  is well-defined.

**Theorem 4.1.** Given  $\delta > 0$ , let  $r \in (0, r_0]$ , with

$$r_0 := \min \left\{ \delta, \frac{\mu_0 \alpha_{\mathbf{A}}}{2} \right\}. \quad (4.24)$$

Then, for each  $(\mathbf{w}, \boldsymbol{\varphi}) \in \mathbf{L}^4(\Omega) \times (\mathbf{L}^4(\Omega) \times \mathbf{L}^4(\Omega))$  such that  $\|\mathbf{w}\|_{0,4;\Omega} \leq r$ , (4.9) (equivalently, (4.7)) has a unique solution  $(\boldsymbol{\sigma}, \vec{\mathbf{u}}) := (\boldsymbol{\sigma}, (\mathbf{u}, \boldsymbol{\gamma})) \in \mathbf{H} \times \mathbf{Q}$ , and hence one can define  $\mathbf{S}(\mathbf{w}, \boldsymbol{\varphi}) := \mathbf{u}$ . Moreover, there exists a positive constant  $C_{\mathbf{S}}$ , depending only on  $\|\mathbf{i}_4\|$ ,  $\|\mathbf{g}\|_{0,\Omega}$ , and  $\alpha_{\mathbf{A}}$ , such that

$$\|\mathbf{S}(\mathbf{w}, \boldsymbol{\varphi})\|_{0,4;\Omega} = \|\mathbf{u}\|_{0,4;\Omega} \leq \|(\boldsymbol{\sigma}, \vec{\mathbf{u}})\|_{\mathbf{H} \times \mathbf{Q}} \leq C_{\mathbf{S}} \left\{ \|\mathbf{u}_D\|_{1/2,\Gamma} + \|\boldsymbol{\phi}_r\|_{0,4;\Omega} + \|\boldsymbol{\varphi}\|_{0,4;\Omega} \right\}. \quad (4.25)$$

*Proof.* It is clear from (4.22) and (4.23) that the bilinear form  $\mathbf{A}_{\mathbf{w}} + \mathbf{b}(\mathbf{w}; \cdot, \cdot)$  satisfies the assumptions of the Banach–Nečas–Babuška Theorem (cf. [21, Theorem 2.6]), and hence, knowing from (4.1) and (4.2) that  $\mathbf{G} \in \mathbf{H}'$  and  $\mathbf{F}_{\boldsymbol{\varphi}} \in \mathbf{Q}'$ , the proof reduces to a straightforward application of that theorem. In particular, the a priori estimate (4.25) follows from [21, Theorem 2.6, eq. (2.5)] and the upper bounds for  $\|\mathbf{G}\|_{\mathbf{H}'}$  and  $\|\mathbf{F}_{\boldsymbol{\varphi}}\|_{\mathbf{Q}'}$  (cf. (4.1)).  $\square$

On the other hand, in order to derive the well-posedness of (4.12), equivalently (4.10), we aim to prove that the bilinear forms  $\tilde{a}_{j,\mathbf{w}}$  (cf. (4.11)) and  $\tilde{b}$  (cf. (3.20)) satisfy the hypotheses of [21, Theorem 2.34]. In this way, letting  $\tilde{\mathbf{V}}$  be the null space of the linear and bounded operator induced by the bilinear form  $\tilde{b}$ , we first observe that (cf. [13, eq. (3.35)])

$$\tilde{\mathbf{V}} = \left\{ \vec{\psi} := (\psi, \mathbf{r}) \in \tilde{\mathbf{H}} : \psi \in H_0^1(\Omega) \quad \text{and} \quad \mathbf{r} = \nabla \psi \right\}.$$

Next, according to the definition of  $\tilde{a}_{j,\mathbf{w}}$  (cf. (4.11)), with a given  $\mathbf{w} \in \mathbf{L}^4(\Omega)$ , and employing (4.5) and (2.4), we obtain, similarly as in the proof of [13, Lemma 3.2], that for each  $\vec{\psi}_j := (\psi_j, \mathbf{r}_j) \in \tilde{\mathbf{V}}$  there holds

$$\tilde{a}_{j,\mathbf{w}}(\vec{\psi}_j, \vec{\psi}_j) = \tilde{a}_j(\vec{\psi}_j, \vec{\psi}_j) = \int_{\Omega} \mathbf{Q}_j |\mathbf{r}_j|^2 \geq \tilde{\alpha}_j \|\vec{\psi}_j\|_{\tilde{\mathbf{H}}}^2, \quad (4.26)$$

where  $\tilde{\alpha}_j$  is a positive constant depending only on  $C_j$  (cf. (2.4)),  $\|i_4\|$ , and  $c_P$ . Then, it is easily seen that (4.26) implies the hypotheses on  $\tilde{a}_{j,\mathbf{w}}$  required in [21, Theorem 2.34, eq. (2.28)]. Furthermore, we recall from [18, Lemma 3.3] that  $\tilde{b}$  satisfies the continuous inf-sup condition required in [21, Theorem 2.34, eq. (2.29)], that is, there exists a positive constant  $\tilde{\beta}$ , depending only on  $|\Omega|$ , such that

$$\sup_{\substack{\vec{\psi} \in \tilde{\mathbf{H}} \\ \vec{\psi} \neq \mathbf{0}}} \frac{\tilde{b}(\vec{\psi}, \boldsymbol{\eta})}{\|\vec{\psi}\|_{\tilde{\mathbf{H}}}} \geq \tilde{\beta} \|\boldsymbol{\eta}\|_{\tilde{\mathbf{Q}}} \quad \forall \boldsymbol{\eta} \in \tilde{\mathbf{Q}}.$$

Consequently, the well-posedness of (4.12), and thus the well-definedness of the operator  $\tilde{\mathbf{S}}$  (cf. (4.13)), is stated as follows.

**Theorem 4.2.** For each  $\mathbf{w} \in \mathbf{L}^4(\Omega)$ , and for each  $j \in \{1, 2\}$ , there exists a unique  $(\vec{\phi}_j, \boldsymbol{\vartheta}_j) := ((\phi_j, \mathbf{t}_j), \boldsymbol{\vartheta}_j) \in \tilde{\mathbf{H}} \times \tilde{\mathbf{Q}}$  solution to (4.12) (equivalently, (4.10)), and hence one can define  $\tilde{\mathbf{S}}_j(\mathbf{w}) := \phi_j$ . Moreover, there exists a positive constant  $C_{\tilde{\mathbf{S}}}$ , depending only on  $\tilde{\alpha}_j$ ,  $\tilde{\beta}$ ,  $\|\mathbf{Q}_j\|_{0,\infty;\Omega}$ ,  $\|i_4\|$ , and  $R_j$ ,  $j \in \{1, 2\}$ , such that

$$\begin{aligned} \|\tilde{\mathbf{S}}(\mathbf{w})\|_{0,4;\Omega} &:= \|(\tilde{\mathbf{S}}_1(\mathbf{w}), \tilde{\mathbf{S}}_2(\mathbf{w}))\|_{0,4;\Omega} \\ &\leq C_{\tilde{\mathbf{S}}} \sum_{j=1}^2 \left\{ \|g_j\|_{0,4/3;\Omega} + (1 + \|\mathbf{w}\|_{0,4;\Omega}) \|\phi_{j,D}\|_{1/2,\Gamma} \right\}. \end{aligned} \quad (4.27)$$

*Proof.* Bearing in mind the previous discussion on  $\tilde{a}_{j,\mathbf{w}}$  and  $\tilde{b}$ , for each  $\mathbf{w} \in \mathbf{L}^4(\Omega)$  and for each  $j \in \{1, 2\}$ , and recalling from (4.1) and (4.2) that  $\tilde{F}_j \in \tilde{\mathbf{H}}'$  and  $\tilde{G}_j \in \tilde{\mathbf{Q}}'$ , the proof follows from a direct application of [21, Theorem 2.34]. In this way, the corresponding a priori estimate (cf. [21, Theorem 2.34, eq. (2.30)]) yields

$$\|\tilde{\mathbf{S}}_j(\mathbf{w})\|_{0,4;\Omega} = \|\phi_j\|_{0,4;\Omega} \leq \|\vec{\phi}_j\|_{\tilde{\mathbf{H}}} \leq \frac{1}{\tilde{\alpha}_j} \|\tilde{F}_j\| + \frac{1}{\tilde{\beta}} \left(1 + \frac{\|\tilde{a}_{j,\mathbf{w}}\|}{\tilde{\alpha}_j}\right) \|\tilde{G}_j\|,$$

so that, noting from (4.1) and (4.3) that  $\|\tilde{a}_{j,\mathbf{w}}\| \leq \|\mathbf{Q}_j\|_{0,\infty;\Omega} + R_j \|\mathbf{w}\|_{0,4;\Omega}$ , and employing the expressions for  $\|\tilde{F}_j\|$  and  $\|\tilde{G}_j\|$  provided in (4.1), the foregoing estimate becomes

$$\|\tilde{\mathbf{S}}_j(\mathbf{w})\|_{0,4;\Omega} \leq \|\vec{\phi}_j\|_{\tilde{\mathbf{H}}} \leq \tilde{C}_j \left\{ \|g_j\|_{0,4/3;\Omega} + (1 + \|\mathbf{w}\|_{0,4;\Omega}) \|\phi_{j,D}\|_{1/2,\Gamma} \right\}, \quad (4.28)$$

where  $\tilde{C}_j$  is a positive constant depending on  $\tilde{\alpha}_j$ ,  $\tilde{\beta}$ ,  $\|\mathbf{Q}_j\|_{0,\infty;\Omega}$ ,  $\|i_4\|$ , and  $R_j$ . Finally, summing up in (4.28) over  $j \in \{1, 2\}$ , we arrive at (4.27) with  $C_{\tilde{\mathbf{S}}} = \tilde{C}_1 + \tilde{C}_2$ .  $\square$

For sake of completeness, we provide next the upper bound for the component  $\boldsymbol{\vartheta}_j$  of the solution of (4.12). In fact, according now to the second inequality in [21, Theorem 2.34, eq. (2.30)], we find that

$$\|\boldsymbol{\vartheta}_j\|_{\tilde{\mathbf{Q}}} \leq \frac{1}{\tilde{\beta}} \left(1 + \frac{\|\tilde{a}_{j,\mathbf{w}}\|}{\tilde{\alpha}_j}\right) \|\tilde{F}_j\| + \frac{\|\tilde{a}_{j,\mathbf{w}}\|}{\tilde{\beta}^2} \left(1 + \frac{\|\tilde{a}_{j,\mathbf{w}}\|}{\tilde{\alpha}_j}\right) \|\tilde{G}_j\|,$$

which yields

$$\|\boldsymbol{\vartheta}_j\|_{\tilde{\mathbf{Q}}} \leq \tilde{M}_j (1 + \|\mathbf{w}\|_{0,4;\Omega}) \left\{ \|g_j\|_{0,4/3;\Omega} + (1 + \|\mathbf{w}\|_{0,4;\Omega}) \|\phi_{j,D}\|_{1/2,\Gamma} \right\}, \quad (4.29)$$

with a positive constant  $\tilde{M}_j$  depending as well on  $\tilde{\alpha}_j$ ,  $\tilde{\beta}$ ,  $\|\mathbf{Q}_j\|_{0,\infty;\Omega}$ ,  $\|i_4\|$ , and  $R_j$ .

## 4.4 Solvability analysis of the fixed-point equation

Having proved the well-posedness of the uncoupled problems (4.7) and (4.10), in particular the former under the assumption on  $\mathbf{w}$  specified in Theorem 4.1, thus ensuring that the operators

$\mathbf{S}$  (cf. (4.6)),  $\tilde{\mathbf{S}}$  (cf. (4.13)), and hence  $\mathbf{T}$  (cf. (4.14)), are well-defined, our next goal is to establish the existence of a unique fixed-point of  $\mathbf{T}$ . For this purpose, in what follows we aim to verify the hypotheses of the Banach theorem, starting by providing a suitable condition guaranteeing that  $\mathbf{T}$  maps a ball into itself. Indeed, given  $r \in (0, r_0]$ , with  $r_0$  as in (4.24), we let, as in (4.19),

$$\mathbf{W}(r) := \left\{ \mathbf{w} \in \mathbf{L}^4(\Omega) : \|\mathbf{w}\|_{0,4;\Omega} \leq r \right\}, \quad (4.30)$$

and observe, thanks to the a priori estimates (4.25) and (4.27), that for each  $\mathbf{w} \in \mathbf{W}(r)$  there holds

$$\begin{aligned} \|\mathbf{T}(\mathbf{w})\|_{0,4;\Omega} &= \|\mathbf{S}(\mathbf{w}, \tilde{\mathbf{S}}(\mathbf{w}))\|_{0,4;\Omega} \leq C_{\mathbf{S}} \left\{ \|\mathbf{u}_D\|_{1/2,\Gamma} + \|\boldsymbol{\phi}_r\|_{0,4;\Omega} + \|\tilde{\mathbf{S}}(\mathbf{w})\|_{0,4;\Omega} \right\} \\ &\leq C_{\mathbf{T}} \left\{ \|\mathbf{u}_D\|_{1/2,\Gamma} + \|\boldsymbol{\phi}_r\|_{0,4;\Omega} + \sum_{j=1}^2 \left\{ \|g_j\|_{0,4/3;\Omega} + (1+r) \|\phi_{j,D}\|_{1/2,\Gamma} \right\} \right\}, \end{aligned} \quad (4.31)$$

where  $C_{\mathbf{T}} := C_{\mathbf{S}} \max\{1, C_{\tilde{\mathbf{S}}}\}$ . Then, we have the following result.

**Lemma 4.1.** Given  $r \in (0, r_0]$ , with  $r_0$  as in (4.24), assume that the data satisfy

$$C_{\mathbf{T}} \left\{ \|\mathbf{u}_D\|_{1/2,\Gamma} + \|\boldsymbol{\phi}_r\|_{0,4;\Omega} + \sum_{j=1}^2 \left\{ \|g_j\|_{0,4/3;\Omega} + (1+r) \|\phi_{j,D}\|_{1/2,\Gamma} \right\} \right\} \leq r. \quad (4.32)$$

Then, there holds  $\mathbf{T}(\mathbf{W}(r)) \subseteq \mathbf{W}(r)$ .

*Proof.* It is a direct consequence of the estimate (4.31).  $\square$

Next, we aim to show that the operator  $\mathbf{T}$  is Lipschitz continuous, for which, according to (4.14), it suffices to prove suitable continuity properties for  $\mathbf{S}$  and  $\tilde{\mathbf{S}}$ . In order to derive the corresponding result for  $\mathbf{S}$ , we need the technical estimate for  $c$  provided by the following lemma.

**Lemma 4.2.** For each  $\rho \in [3, 4]$  there exists a positive constant  $L_c$ , depending only on  $F$ ,  $|\Omega|$ , and  $\rho$ , such that

$$|c(\mathbf{w}; \vec{\mathbf{u}}, \vec{\mathbf{v}}) - c(\mathbf{z}; \vec{\mathbf{u}}, \vec{\mathbf{v}})| \leq L_c \left\{ \|\mathbf{w}\|_{0,4;\Omega} + \|\mathbf{z}\|_{0,4;\Omega} \right\}^{\rho-3} \|\mathbf{w} - \mathbf{z}\|_{0,4;\Omega} \|\mathbf{u}\|_{0,4;\Omega} \|\mathbf{v}\|_{0,4;\Omega} \quad (4.33)$$

for all  $\mathbf{w}, \mathbf{z} \in \mathbf{L}^4(\Omega)$ , and for all  $\vec{\mathbf{u}}, \vec{\mathbf{v}} \in \mathbf{Q}$ .

*Proof.* We begin by noticing from the definition of  $c$  (cf. (3.12)) that, given  $\mathbf{w}, \mathbf{z} \in \mathbf{L}^4(\Omega)$ , and  $\vec{\mathbf{u}} := (\mathbf{u}, \gamma), \vec{\mathbf{v}} := (\mathbf{v}, \delta) \in \mathbf{Q}$ , there holds

$$|c(\mathbf{w}; \vec{\mathbf{u}}, \vec{\mathbf{v}}) - c(\mathbf{z}; \vec{\mathbf{u}}, \vec{\mathbf{v}})| \leq F \int_{\Omega} \left| |\mathbf{w}|^{\rho-2} - |\mathbf{z}|^{\rho-2} \right| |\mathbf{u} \cdot \mathbf{v}|. \quad (4.34)$$

Next, defining  $\tilde{\mathbf{w}} := (|\mathbf{w}|, \mathbf{0}), \tilde{\mathbf{z}} := (|\mathbf{z}|, \mathbf{0}) \in \mathbb{R}^n$ , with  $\mathbf{0} \in \mathbb{R}^{n-1}$ , we observe that

$$\left| |\mathbf{w}|^{\rho-2} - |\mathbf{z}|^{\rho-2} \right| = \left| |\tilde{\mathbf{w}}|^{\rho-3} \tilde{\mathbf{w}} - |\tilde{\mathbf{z}}|^{\rho-3} \tilde{\mathbf{z}} \right|, \quad (4.35)$$

and recall from [28, Lemma 5.3] that for each  $t \geq 2$  there exists a positive constant  $C_t$  such that

$$\left| |\mathbf{x}|^{t-2} \mathbf{x} - |\mathbf{y}|^{t-2} \mathbf{y} \right| \leq C_t (|\mathbf{x}| + |\mathbf{y}|)^{t-2} |\mathbf{x} - \mathbf{y}| \quad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n,$$

so that applying the foregoing inequality with  $t = \rho - 1$ , and denoting  $C(\rho) := C_{\rho-1}$ , we deduce that

$$\left| |\tilde{\mathbf{w}}|^{\rho-3} \tilde{\mathbf{w}} - |\tilde{\mathbf{z}}|^{\rho-3} \tilde{\mathbf{z}} \right| \leq C(\rho) (|\tilde{\mathbf{w}}| + |\tilde{\mathbf{z}}|)^{\rho-3} |\tilde{\mathbf{w}} - \tilde{\mathbf{z}}|. \quad (4.36)$$

Thus, replacing (4.36) back into (4.35), and then the resulting estimate back into (4.34), returning to the original variables, and using, in particular, that  $|\tilde{\mathbf{w}} - \tilde{\mathbf{z}}| = \left| |\mathbf{w}| - |\mathbf{z}| \right| \leq |\mathbf{w} - \mathbf{z}|$ , we arrive at

$$|c(\mathbf{w}; \vec{\mathbf{u}}, \vec{\mathbf{v}}) - c(\mathbf{z}; \vec{\mathbf{u}}, \vec{\mathbf{v}})| \leq C(\rho) F \int_{\Omega} (|\mathbf{w}| + |\mathbf{z}|)^{\rho-3} |\mathbf{w} - \mathbf{z}| |\mathbf{u} \cdot \mathbf{v}|,$$

from which, applying Cauchy-Schwarz's inequality, we deduce that

$$|c(\mathbf{w}; \vec{\mathbf{u}}, \vec{\mathbf{v}}) - c(\mathbf{z}; \vec{\mathbf{u}}, \vec{\mathbf{v}})| \leq C(\rho) F \left\| (|\mathbf{w}| + |\mathbf{z}|)^{\rho-3} \right\|_{0,4;\Omega} \|\mathbf{w} - \mathbf{z}\|_{0,4;\Omega} \|\mathbf{u}\|_{0,4;\Omega} \|\mathbf{v}\|_{0,4;\Omega}. \quad (4.37)$$

It remains to estimate the expression  $\left\| (|\mathbf{w}| + |\mathbf{z}|)^{\rho-3} \right\|_{0,4;\Omega}$ . The case  $\rho = 3$  is straightforward since  $\left\| (|\mathbf{w}| + |\mathbf{z}|)^{\rho-3} \right\|_{0,4;\Omega}$  becomes  $|\Omega|^{1/4}$ , which yields (4.33) with  $L_c := C(\rho) F |\Omega|^{1/4}$ . In turn, when  $\rho = 4$ , we get by triangle inequality that  $\left\| (|\mathbf{w}| + |\mathbf{z}|)^{\rho-3} \right\|_{0,4;\Omega} = \|\mathbf{w}| + |\mathbf{z}|\|_{0,4;\Omega} \leq$

$\|\mathbf{w}\|_{0,4;\Omega} + \|\mathbf{z}\|_{0,4;\Omega}$ , which implies (4.33) with  $L_c := C(\rho)F$ . Finally, if  $\rho \in (3, 4)$ , we apply Hölder's inequality with  $r := \frac{1}{4-\rho} \in (1, +\infty)$  and its conjugate  $r' := \frac{1}{\rho-3}$ , to obtain

$$\left\| (|\mathbf{w}| + |\mathbf{z}|)^{\rho-3} \right\|_{0,4;\Omega} \leq |\Omega|^{(4-\rho)/4} \|\mathbf{w} + \mathbf{z}\|_{0,4;\Omega}^{\rho-3} \leq |\Omega|^{(4-\rho)/4} \left( \|\mathbf{w}\|_{0,4;\Omega} + \|\mathbf{z}\|_{0,4;\Omega} \right)^{\rho-3},$$

which, along with (4.37), gives (4.33) with  $L_c := C(\rho)F|\Omega|^{(4-\rho)/4}$ . Summarizing, (4.33) holds with this latter value of  $L_c$  for all  $\rho \in [3, 4]$ .  $\square$

The announced property for  $\mathbf{S}$  is established next.

**Lemma 4.3.** Let  $r \in (0, r_0]$ , with  $r_0$  as in (4.24). Then, there exists a positive constant  $L_S$ , depending only on  $\alpha_{\mathbf{A}}$ ,  $\|\mathbf{g}\|_{0,\Omega}$ ,  $\mu_0$ ,  $L_c$ , and  $r$ , such that

$$\|\mathbf{S}(\mathbf{w}, \boldsymbol{\varphi}) - \mathbf{S}(\mathbf{z}, \boldsymbol{\xi})\|_{0,4;\Omega} \leq L_S \left\{ \|\boldsymbol{\varphi} - \boldsymbol{\xi}\|_{0,4;\Omega} + \|\mathbf{S}(\mathbf{z}, \boldsymbol{\xi})\|_{0,4;\Omega} \|\mathbf{w} - \mathbf{z}\|_{0,4;\Omega} \right\} \quad (4.38)$$

for all  $(\mathbf{w}, \boldsymbol{\varphi}), (\mathbf{z}, \boldsymbol{\xi}) \in \mathbf{W}(r) \times (\mathbf{L}^4(\Omega) \times \mathbf{L}^4(\Omega))$ .

*Proof.* Let  $(\mathbf{w}, \boldsymbol{\varphi}), (\mathbf{z}, \boldsymbol{\xi}) \in \mathbf{W}(r) \times (\mathbf{L}^4(\Omega) \times \mathbf{L}^4(\Omega))$  such that  $\mathbf{S}(\mathbf{w}, \boldsymbol{\varphi}) = \mathbf{u}_1$  and  $\mathbf{S}(\mathbf{z}, \boldsymbol{\xi}) = \mathbf{u}_2$ , where, for each  $i \in \{1, 2\}$ ,  $(\boldsymbol{\sigma}_i, \vec{\mathbf{u}}_i) := (\boldsymbol{\sigma}_i, (\mathbf{u}_i, \boldsymbol{\gamma}_i)) \in \mathbf{H} \times \mathbf{Q}$  is the corresponding unique solution of (4.9) (equivalently, (4.7)), that is

$$\mathbf{A}_{\mathbf{w}}((\boldsymbol{\sigma}_1, \vec{\mathbf{u}}_1), (\boldsymbol{\tau}, \vec{\mathbf{v}})) + \mathbf{b}(\mathbf{w}; \mathbf{u}_1, \boldsymbol{\tau}) = \mathbf{F}_{\boldsymbol{\varphi}}(\vec{\mathbf{v}}) + \mathbf{G}(\boldsymbol{\tau}) \quad \forall (\boldsymbol{\tau}, \vec{\mathbf{v}}) \in \mathbf{H} \times \mathbf{Q}, \quad (4.39)$$

$$\mathbf{A}_{\mathbf{z}}((\boldsymbol{\sigma}_2, \vec{\mathbf{u}}_2), (\boldsymbol{\tau}, \vec{\mathbf{v}})) + \mathbf{b}(\mathbf{z}; \mathbf{u}_2, \boldsymbol{\tau}) = \mathbf{F}_{\boldsymbol{\xi}}(\vec{\mathbf{v}}) + \mathbf{G}(\boldsymbol{\tau}) \quad \forall (\boldsymbol{\tau}, \vec{\mathbf{v}}) \in \mathbf{H} \times \mathbf{Q}.$$

Then, applying (4.22) to  $(\boldsymbol{\sigma}_1, \vec{\mathbf{u}}_1) - (\boldsymbol{\sigma}_2, \vec{\mathbf{u}}_2) \in \mathbf{H} \times \mathbf{Q}$ , we obtain

$$\begin{aligned} \|\mathbf{S}(\mathbf{w}, \boldsymbol{\varphi}) - \mathbf{S}(\mathbf{z}, \boldsymbol{\xi})\|_{0,4;\Omega} &= \|\mathbf{u}_1 - \mathbf{u}_2\|_{0,4;\Omega} \leq \|(\boldsymbol{\sigma}_1, \vec{\mathbf{u}}_1) - (\boldsymbol{\sigma}_2, \vec{\mathbf{u}}_2)\|_{\mathbf{H} \times \mathbf{Q}} \\ &\leq \frac{2}{\alpha_{\mathbf{A}}} \sup_{\substack{(\boldsymbol{\tau}, \vec{\mathbf{v}}) \in \mathbf{H} \times \mathbf{Q} \\ (\boldsymbol{\tau}, \vec{\mathbf{v}}) \neq \mathbf{0}}} \frac{\mathbf{A}_{\mathbf{w}}((\boldsymbol{\sigma}_1, \vec{\mathbf{u}}_1) - (\boldsymbol{\sigma}_2, \vec{\mathbf{u}}_2), (\boldsymbol{\tau}, \vec{\mathbf{v}})) + \mathbf{b}(\mathbf{w}; \mathbf{u}_1 - \mathbf{u}_2, \boldsymbol{\tau})}{\|(\boldsymbol{\tau}, \vec{\mathbf{v}})\|_{\mathbf{H} \times \mathbf{Q}}}. \end{aligned} \quad (4.40)$$

Now, adding and subtracting  $\mathbf{A}_{\mathbf{z}}((\boldsymbol{\sigma}_2, \vec{\mathbf{u}}_2), (\boldsymbol{\tau}, \vec{\mathbf{v}}))$ , and using (4.39) and the fact that there holds  $(\mathbf{A}_{\mathbf{z}} - \mathbf{A}_{\mathbf{w}})((\boldsymbol{\sigma}_2, \vec{\mathbf{u}}_2), (\boldsymbol{\tau}, \vec{\mathbf{v}})) = c(\mathbf{w}; \vec{\mathbf{u}}_2, \vec{\mathbf{v}}) - c(\mathbf{z}; \vec{\mathbf{u}}_2, \vec{\mathbf{v}})$ , we get after some algebraic

manipulations

$$\begin{aligned} & \mathbf{A}_{\mathbf{w}}((\boldsymbol{\sigma}_1, \vec{\mathbf{u}}_1) - (\boldsymbol{\sigma}_2, \vec{\mathbf{u}}_2), (\boldsymbol{\tau}, \vec{\mathbf{v}})) + \mathbf{b}(\mathbf{w}; \mathbf{u}_1 - \mathbf{u}_2, \boldsymbol{\tau}) \\ &= (\mathbf{F}_{\boldsymbol{\varphi}} - \mathbf{F}_{\boldsymbol{\xi}})(\vec{\mathbf{v}}) + \mathbf{b}(\mathbf{z} - \mathbf{w}; \mathbf{u}_2, \boldsymbol{\tau}) + c(\mathbf{w}; \vec{\mathbf{u}}_2, \vec{\mathbf{v}}) - c(\mathbf{z}; \vec{\mathbf{u}}_2, \vec{\mathbf{v}}). \end{aligned} \quad (4.41)$$

Next, it readily follows from the definitions of the function  $\mathbf{f}$  (cf. (2.3)) and the functional  $\mathbf{F}_{\boldsymbol{\varphi}}$  (cf. (3.13)) that

$$(\mathbf{F}_{\boldsymbol{\varphi}} - \mathbf{F}_{\boldsymbol{\xi}})(\vec{\mathbf{v}}) \leq \|\mathbf{g}\|_{0,\Omega} \|\boldsymbol{\varphi} - \boldsymbol{\xi}\|_{0,4;\Omega} \|\mathbf{v}\|_{0,4;\Omega}, \quad (4.42)$$

whereas it is clear from the first row of (4.3) that

$$\mathbf{b}(\mathbf{z} - \mathbf{w}; \mathbf{u}_2, \boldsymbol{\tau}) \leq \frac{1}{\mu_0} \|\mathbf{w} - \mathbf{z}\|_{0,4;\Omega} \|\mathbf{u}_2\|_{0,4;\Omega} \|\boldsymbol{\tau}\|_{\mathbf{div}_{4/3};\Omega}. \quad (4.43)$$

In turn, applying Lemma 4.2, and using that both  $\|\mathbf{w}\|_{0,4;\Omega}$  and  $\|\mathbf{z}\|_{0,4;\Omega}$  are bounded by  $r$ , we find that

$$\begin{aligned} c(\mathbf{w}; \vec{\mathbf{u}}_2, \vec{\mathbf{v}}) - c(\mathbf{z}; \vec{\mathbf{u}}_2, \vec{\mathbf{v}}) &\leq L_c \left\{ \|\mathbf{w}\|_{0,4;\Omega} + \|\mathbf{z}\|_{0,4;\Omega} \right\}^{\rho-3} \|\mathbf{w} - \mathbf{z}\|_{0,4;\Omega} \|\mathbf{u}_2\|_{0,4;\Omega} \|\mathbf{v}\|_{0,4;\Omega} \\ &\leq L_c (2r)^{\rho-3} \|\mathbf{w} - \mathbf{z}\|_{0,4;\Omega} \|\mathbf{u}_2\|_{0,4;\Omega} \|\mathbf{v}\|_{0,4;\Omega}. \end{aligned} \quad (4.44)$$

Finally, replacing (4.41) back into (4.40), employing the upper bounds provided by (4.42), (4.43), and (4.44), and recalling that  $\mathbf{u}_2 = \mathbf{S}(\mathbf{z}, \boldsymbol{\xi})$ , we arrive at the required inequality (4.38) with a positive constant  $L_{\mathcal{S}}$  as indicated.  $\square$

The following lemma proves the Lipschitz continuity of the operator  $\tilde{\mathbf{S}}$ .

**Lemma 4.4.** Let  $r \in (0, r_0]$ , with  $r_0$  as in (4.24). Then, there exists a positive constant  $L_{\tilde{\mathcal{S}}}$ , depending only on  $\tilde{C}_j$  (cf. (4.28)),  $R_j$ , and  $\tilde{\alpha}_j$ ,  $j \in \{1, 2\}$ , such that

$$\|\tilde{\mathbf{S}}(\mathbf{w}) - \tilde{\mathbf{S}}(\mathbf{z})\|_{0,4;\Omega} \leq L_{\tilde{\mathcal{S}}} \sum_{j=1}^2 \left\{ \|g_j\|_{0,4/3;\Omega} + (1 + \|\mathbf{w}\|_{0,4;\Omega}) \|\phi_{j,D}\|_{1/2,\Gamma} \right\} \|\mathbf{w} - \mathbf{z}\|_{0,4;\Omega} \quad (4.45)$$

for all  $\mathbf{w}, \mathbf{z} \in \mathbf{W}(r)$ .

*Proof.* We proceed similarly to the proof of [13, Lemma 3.11]. Indeed, given  $r \in (0, r_0]$  and

$\mathbf{w}, \mathbf{z} \in \mathbf{W}(r)$ , we let  $\tilde{\mathbf{S}}(\mathbf{w}) := (\phi_1, \phi_2) \in L^4(\Omega) \times L^4(\Omega)$  and  $\tilde{\mathbf{S}}(\mathbf{z}) := (\xi_1, \xi_2) \in L^4(\Omega) \times L^4(\Omega)$ , where, for each  $j \in \{1, 2\}$ ,  $(\vec{\phi}_j, \boldsymbol{\vartheta}_j) := ((\phi_j, \mathbf{t}_j), \boldsymbol{\vartheta}_j) \in \tilde{\mathbf{H}} \times \tilde{\mathbf{Q}}$  and  $(\vec{\xi}_j, \boldsymbol{\zeta}_j) := ((\xi_j, \mathbf{s}_j), \boldsymbol{\zeta}_j) \in \tilde{\mathbf{H}} \times \tilde{\mathbf{Q}}$  are the unique solutions of (4.10) (equivalently, (4.12)) with  $\tilde{c}_j(\mathbf{w}; \cdot, \cdot)$  and  $\tilde{c}_j(\mathbf{z}; \cdot, \cdot)$  (equivalently, with  $\tilde{a}_{j,\mathbf{w}}$  and  $\tilde{a}_{j,\mathbf{z}}$ ), respectively. It follows from the subtraction of the corresponding second equations of (4.10) that  $\vec{\phi}_j - \vec{\xi}_j \in \tilde{\mathbf{V}}$ . In addition, testing the first equations of (4.10) against  $\vec{\psi}_j = \vec{\phi}_j - \vec{\xi}_j$ , and then subtracting them, we deduce that

$$\tilde{a}_j(\vec{\phi}_j - \vec{\xi}_j, \vec{\phi}_j - \vec{\xi}_j) = \tilde{c}_j(\mathbf{z}; \vec{\xi}_j, \vec{\phi}_j - \vec{\xi}_j) - \tilde{c}_j(\mathbf{w}; \vec{\phi}_j, \vec{\phi}_j - \vec{\xi}_j),$$

from which, subtracting and adding  $\vec{\phi}_j$  in the second component of the first term on the right hand-side, and using the identity (4.5), we get

$$\tilde{a}_j(\vec{\phi}_j - \vec{\xi}_j, \vec{\phi}_j - \vec{\xi}_j) = \tilde{c}_j(\mathbf{z}; \vec{\phi}_j, \vec{\phi}_j - \vec{\xi}_j) - \tilde{c}_j(\mathbf{w}; \vec{\phi}_j, \vec{\phi}_j - \vec{\xi}_j).$$

In this way, employing now the ellipticity of  $\tilde{a}_j$  (cf. (4.26)), the foregoing identity, and the continuity property for  $\tilde{c}_j$  provided by (4.4), we find that

$$\begin{aligned} \tilde{\alpha}_j \|\vec{\phi}_j - \vec{\xi}_j\|_{\tilde{\mathbf{H}}}^2 &\leq \tilde{a}_j(\vec{\phi}_j - \vec{\xi}_j, \vec{\phi}_j - \vec{\xi}_j) = \tilde{c}_j(\mathbf{z}; \vec{\phi}_j, \vec{\phi}_j - \vec{\xi}_j) - \tilde{c}_j(\mathbf{w}; \vec{\phi}_j, \vec{\phi}_j - \vec{\xi}_j) \\ &\leq R_j \|\mathbf{w} - \mathbf{z}\|_{0,4;\Omega} \|\vec{\phi}_j\|_{\tilde{\mathbf{H}}} \|\vec{\phi}_j - \vec{\xi}_j\|_{\tilde{\mathbf{H}}}, \end{aligned}$$

which, along with the a priori estimate for  $\|\vec{\phi}_j\|_{\tilde{\mathbf{H}}}$  given by (4.28), yields

$$\begin{aligned} \|\tilde{\mathbf{S}}_j(\mathbf{w}) - \tilde{\mathbf{S}}_j(\mathbf{z})\|_{0,4;\Omega} &= \|\phi_j - \xi_j\|_{0,4;\Omega} \leq \|\vec{\phi}_j - \vec{\xi}_j\|_{\tilde{\mathbf{H}}} \leq \tilde{\alpha}_j^{-1} R_j \|\vec{\phi}_j\|_{\tilde{\mathbf{H}}} \|\mathbf{w} - \mathbf{z}\|_{0,4;\Omega} \\ &\leq \tilde{\alpha}_j^{-1} \mathbf{R}_j \tilde{C}_j \left\{ \|g_j\|_{0,4/3;\Omega} + (1 + \|\mathbf{w}\|_{0,4;\Omega}) \|\phi_{j,D}\|_{1/2,\Gamma} \right\} \|\mathbf{w} - \mathbf{z}\|_{0,4;\Omega}. \end{aligned} \tag{4.46}$$

Finally, summing up in (4.46) over  $j \in \{1, 2\}$ , we get (4.45) with  $L_{\mathfrak{S}} := \max_{j \in \{1,2\}} \{\tilde{\alpha}_j^{-1} R_j \tilde{C}_j\}$ .  $\square$

As a consequence of Lemmas 4.3 and 4.4, we are able now to prove the Lipschitz continuity of  $\mathbf{T}$ .

**Lemma 4.5.** Let  $r \in (0, r_0]$ , with  $r_0$  as in (4.24). Then, there exists a positive constant  $L_{\mathbf{T}}$ ,

depending only on  $L_{\mathbf{S}}$ ,  $L_{\tilde{\mathbf{S}}}$ ,  $C_{\mathbf{S}}$ , and  $C_{\tilde{\mathbf{S}}}$ , such that

$$\begin{aligned} & \|\mathbf{T}(\mathbf{w}) - \mathbf{T}(\mathbf{z})\|_{0,4;\Omega} \\ & \leq L_{\mathbf{T}} \left\{ \|\mathbf{u}_D\|_{1/2,\Gamma} + \|\phi_r\|_{0,4;\Omega} + \sum_{j=1}^2 \left\{ \|g_j\|_{0,4/3;\Omega} + (1+r) \|\phi_{j,D}\|_{1/2,\Gamma} \right\} \right\} \|\mathbf{w} - \mathbf{z}\|_{0,4;\Omega} \end{aligned} \quad (4.47)$$

for all  $\mathbf{w}, \mathbf{z} \in \mathbf{W}(r)$ .

*Proof.* Given  $\mathbf{w}, \mathbf{z} \in \mathbf{W}(r)$ , we first deduce from the definition of  $\mathbf{T}$  (cf. (4.14)) and the continuity property of  $\mathbf{S}$  (cf. Lemma 4.3) that

$$\begin{aligned} \|\mathbf{T}(\mathbf{w}) - \mathbf{T}(\mathbf{z})\|_{0,4;\Omega} &= \|\mathbf{S}(\mathbf{w}, \tilde{\mathbf{S}}(\mathbf{w})) - \mathbf{S}(\mathbf{z}, \tilde{\mathbf{S}}(\mathbf{z}))\|_{0,4;\Omega} \\ &\leq L_{\mathbf{S}} \left\{ \|\tilde{\mathbf{S}}(\mathbf{w}) - \tilde{\mathbf{S}}(\mathbf{z})\|_{0,4;\Omega} + \|\mathbf{S}(\mathbf{z}, \tilde{\mathbf{S}}(\mathbf{z}))\|_{0,4;\Omega} \|\mathbf{w} - \mathbf{z}\|_{0,4;\Omega} \right\}. \end{aligned} \quad (4.48)$$

In turn, the Lipschitz-continuity of  $\tilde{\mathbf{S}}$  (cf. Lemma 4.4) yields

$$\|\tilde{\mathbf{S}}(\mathbf{w}) - \tilde{\mathbf{S}}(\mathbf{z})\|_{0,4;\Omega} \leq L_{\tilde{\mathbf{S}}} \sum_{j=1}^2 \left\{ \|g_j\|_{0,4/3;\Omega} + (1+r) \|\phi_{j,D}\|_{1/2,\Gamma} \right\} \|\mathbf{w} - \mathbf{z}\|_{0,4;\Omega}, \quad (4.49)$$

whereas the a priori estimates for  $\mathbf{S}$  (cf. (4.25)) and  $\tilde{\mathbf{S}}$  (cf. (4.27)) imply

$$\begin{aligned} & \|\mathbf{S}(\mathbf{z}, \tilde{\mathbf{S}}(\mathbf{z}))\|_{0,4;\Omega} \\ & \leq C_{\mathbf{S}} \left\{ \|\mathbf{u}_D\|_{1/2,\Gamma} + \|\phi_r\|_{0,4;\Omega} + C_{\tilde{\mathbf{S}}} \sum_{j=1}^2 \left\{ \|g_j\|_{0,4/3;\Omega} + (1+r) \|\phi_{j,D}\|_{1/2,\Gamma} \right\} \right\}, \end{aligned} \quad (4.50)$$

where the fact that both  $\|\mathbf{w}\|_{0,4;\Omega}$  and  $\|\mathbf{z}\|_{0,4;\Omega}$  are bounded by  $r$  has been utilized in (4.49) and (4.50), respectively. Finally, replacing the latter estimates back into (4.48), and performing simple algebraic manipulations, we arrive at (4.47) and end the proof.  $\square$

The main result of this section, which refers to the solvability of (4.15) (equivalently, (3.24)), is stated as follows.

**Theorem 4.6.** Given  $r \in (0, r_0]$ , with  $r_0$  as in (4.24), assume that, in addition to the hypothesis

of Lemma 4.1 (cf. (4.32)), the data satisfy

$$L_{\mathbf{T}} \left\{ \|\mathbf{u}_D\|_{1/2,\Gamma} + \|\phi_r\|_{0,4;\Omega} + \sum_{j=1}^2 \left\{ \|g_j\|_{0,4/3;\Omega} + (1+r) \|\phi_{j,D}\|_{1/2,\Gamma} \right\} \right\} < 1. \quad (4.51)$$

Then, there exists a unique  $\mathbf{u} \in \mathbf{W}(r)$  (cf. (4.30)) fixed point of  $\mathbf{T}$  (cf. (4.15)). Equivalently, (3.24) has a unique solution  $(\boldsymbol{\sigma}, \bar{\mathbf{u}}) := (\boldsymbol{\sigma}, (\mathbf{u}, \boldsymbol{\gamma})) \in \mathbf{H} \times \mathbf{Q}$  and  $(\vec{\phi}_j, \boldsymbol{\vartheta}_j) := ((\phi_j, \mathbf{t}_j), \boldsymbol{\vartheta}_j) \in \tilde{\mathbf{H}} \times \tilde{\mathbf{Q}}$ ,  $j \in \{1, 2\}$ , with  $\mathbf{u} \in \mathbf{W}(r)$ . Moreover, there exist positive constants  $\mathcal{C}$ ,  $\mathcal{C}_1$ , and  $\mathcal{C}_2$ , depending on  $\tilde{\alpha}_1$ ,  $\tilde{\alpha}_2$ ,  $\tilde{\beta}$ ,  $\|\mathbf{Q}_1\|_{0,\infty;\Omega}$ ,  $\|\mathbf{Q}_2\|_{0,\infty;\Omega}$ ,  $\|i_4\|$ ,  $R_1$ ,  $R_2$ ,  $r$ ,  $\|\mathbf{i}_4\|$ ,  $\|\mathbf{g}\|_{0,\Omega}$ , and  $\alpha_{\mathbf{A}}$ , such that there hold the following a priori bounds

$$\|(\boldsymbol{\sigma}, \bar{\mathbf{u}})\|_{\mathbf{H} \times \mathbf{Q}} \leq \mathcal{C} \left\{ \|\mathbf{u}_D\|_{1/2,\Gamma} + \|\phi_r\|_{0,4;\Omega} + \sum_{j=1}^2 \left\{ \|g_j\|_{0,4/3;\Omega} + \|\phi_{j,D}\|_{1/2,\Gamma} \right\} \right\}, \quad (4.52)$$

and for each  $j \in \{1, 2\}$

$$\|(\vec{\phi}_j, \boldsymbol{\vartheta}_j)\|_{\tilde{\mathbf{H}} \times \tilde{\mathbf{Q}}} \leq \mathcal{C}_j \left\{ \|g_j\|_{0,4/3;\Omega} + \|\phi_{j,D}\|_{1/2,\Gamma} \right\}. \quad (4.53)$$

*Proof.* It is clear from Lemma 4.1 and the assumption (4.51) that  $\mathbf{T}$  is a contraction that maps the ball  $\mathbf{W}(r)$  into itself, and hence a straightforward application of the classical Banach fixed-point theorem implies the indicated solvabilities of (4.15) and (3.24). Furthermore, since  $\mathbf{u} = \mathbf{T}(\mathbf{u}) = \mathbf{S}(\mathbf{u}, \tilde{\mathbf{S}}(\mathbf{u}))$ , we deduce that  $\boldsymbol{\phi} := (\phi_1, \phi_2) = \tilde{\mathbf{S}}(\mathbf{u})$ , whence (4.53) follows from (4.28) and (4.29), whereas (4.52) is consequence of (4.25) and (4.53). We omit further details.  $\square$

---

## The Galerkin scheme

---

The Galerkin scheme of the fully–mixed formulation (3.24) is introduced and analyzed in this chapter. In particular, regarding the solvability analyses of the discrete versions of the decoupled problems studied in Section 4.3, we now apply [19, Theorem 3.5], [21, Theorem 2.22], and [21, Proposition 2.42], which correspond to the discrete analogues of [19, Theorem 3.4], [21, Theorem 2.6], and [21, Theorem 2.34], respectively.

### 5.1 Preliminaries

We begin by letting  $\{\mathcal{T}_h\}_{h>0}$  be a regular family of triangulations of  $\bar{\Omega}$  made up of triangles  $K$  (when  $n = 2$ ) or tetrahedra  $K$  (when  $n = 3$ ) of diameter  $h_K$ , and set  $h := \max\{h_K : K \in \mathcal{T}_h\}$ . Then, we let  $\tilde{\mathbb{H}}_h^\sigma$ ,  $\mathbf{H}_h^u$ ,  $\mathbb{H}_h^\gamma$ ,  $\mathbf{H}_h^\phi$ ,  $\mathbf{H}_h^t$ , and  $\mathbf{H}_h^\vartheta$  be arbitrary finite element subspaces of  $\mathbb{H}(\mathbf{div}_{4/3}; \Omega)$ ,  $\mathbf{L}^4(\Omega)$ ,  $\mathbb{L}_{\text{skew}}^2(\Omega)$ ,  $\mathbf{L}^4(\Omega)$ ,  $\mathbf{L}^2(\Omega)$ , and  $\mathbf{H}(\mathbf{div}_{4/3}; \Omega)$ , respectively. Specific choices of them, satisfying suitable hypotheses to be introduced along the discussion, will be described later on in Chapter 6. Note that  $h$  stands for both, the size of the triangulation  $\mathcal{T}_h$  and the

sub-index of each subspace. Then, defining

$$\mathbf{H}_h := \tilde{\mathbb{H}}_h^\sigma \cap \mathbb{H}_0(\operatorname{div}_{4/3}; \Omega), \quad \mathbf{Q}_h := \mathbf{H}_h^u \times \mathbb{H}_h^\gamma, \quad \tilde{\mathbf{H}}_h := \mathbf{H}_h^\phi \times \mathbf{H}_h^t, \quad \tilde{\mathbf{Q}}_h := \mathbf{H}_h^\vartheta, \quad (5.1)$$

and setting the notations

$$\vec{\mathbf{u}}_h = (\mathbf{u}_h, \gamma_h), \quad \vec{\mathbf{v}}_h = (\mathbf{v}_h, \delta_h), \quad \vec{\mathbf{z}}_h = (\mathbf{z}_h, \zeta_h) \in \mathbf{Q}_h,$$

and for  $j \in \{1, 2\}$

$$\vec{\phi}_{j,h} = (\phi_{j,h}, \mathbf{t}_{j,h}), \quad \vec{\psi}_{j,h} = (\psi_{j,h}, \mathbf{r}_{j,h}), \quad \vec{\xi}_{j,h} = (\xi_{j,h}, \mathbf{s}_{j,h}) \in \tilde{\mathbf{H}}_h,$$

the Galerkin scheme associated with (3.24) reads: Find  $(\boldsymbol{\sigma}_h, \vec{\mathbf{u}}_h) \in \mathbf{H}_h \times \mathbf{Q}_h$  and  $(\vec{\phi}_{j,h}, \boldsymbol{\vartheta}_{j,h}) \in \tilde{\mathbf{H}}_h \times \tilde{\mathbf{Q}}_h$ ,  $j \in \{1, 2\}$ , such that

$$\begin{aligned} a(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + b(\boldsymbol{\tau}_h, \vec{\mathbf{u}}_h) + \mathbf{b}(\mathbf{u}_h; \mathbf{u}_h, \boldsymbol{\tau}_h) &= \mathbf{G}(\boldsymbol{\tau}_h) & \forall \boldsymbol{\tau}_h \in \mathbf{H}_h, \\ b(\boldsymbol{\sigma}_h, \vec{\mathbf{v}}_h) - c(\mathbf{u}_h; \vec{\mathbf{u}}_h, \vec{\mathbf{v}}_h) &= \mathbf{F}_{\phi_h}(\vec{\mathbf{v}}_h) & \forall \vec{\mathbf{v}}_h \in \mathbf{Q}_h, \\ \tilde{a}_j(\vec{\phi}_{j,h}, \vec{\psi}_{j,h}) + \tilde{c}_j(\mathbf{u}_h; \vec{\phi}_{j,h}, \vec{\psi}_{j,h}) + \tilde{b}(\vec{\psi}_{j,h}, \boldsymbol{\vartheta}_{j,h}) &= \tilde{F}_j(\vec{\psi}_{j,h}) & \forall \vec{\psi}_{j,h} \in \tilde{\mathbf{H}}_h, \\ \tilde{b}(\vec{\phi}_{j,h}, \boldsymbol{\eta}_{j,h}) &= \tilde{G}_j(\boldsymbol{\eta}_{j,h}) & \forall \boldsymbol{\eta}_{j,h} \in \tilde{\mathbf{Q}}_h, \end{aligned} \quad (5.2)$$

where  $\phi_h := (\phi_{1,h}, \phi_{2,h}) \in \mathbf{H}_h^\phi \times \mathbf{H}_h^\phi$ .

## 5.2 Discrete fixed point strategy

In order to address the solvability of (5.2), we adopt the discrete analogue of the fixed point strategy employed in Section 4.2. Indeed, we start by introducing the operator  $\mathbf{S}_h : \mathbf{H}_h^u \times (\mathbf{H}_h^\phi \times \mathbf{H}_h^\phi) \rightarrow \mathbf{H}_h^u$  defined by

$$\mathbf{S}_h(\mathbf{w}_h, \boldsymbol{\varphi}_h) := \mathbf{u}_h \quad \forall (\mathbf{w}_h, \boldsymbol{\varphi}_h) \in \mathbf{H}_h^u \times (\mathbf{H}_h^\phi \times \mathbf{H}_h^\phi), \quad (5.3)$$

where  $(\boldsymbol{\sigma}_h, \vec{\mathbf{u}}_h) := (\boldsymbol{\sigma}_h, (\mathbf{u}_h, \boldsymbol{\gamma}_h)) \in \mathbf{H}_h \times \mathbf{Q}_h$  is the unique solution (to be confirmed below) of the problem arising from the first two rows of (5.2) when  $\mathbf{b}(\mathbf{u}_h; \cdot, \cdot)$ ,  $c(\mathbf{u}_h; \cdot, \cdot)$ , and  $\mathbf{F}_{\phi_h}$ , are replaced by  $\mathbf{b}(\mathbf{w}_h; \cdot, \cdot)$ ,  $c(\mathbf{w}_h; \cdot, \cdot)$ , and  $\mathbf{F}_{\varphi_h}$ , respectively, that is

$$\begin{aligned} a(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + b(\boldsymbol{\tau}_h, \vec{\mathbf{u}}_h) + \mathbf{b}(\mathbf{w}_h; \mathbf{u}_h, \boldsymbol{\tau}_h) &= \mathbf{G}(\boldsymbol{\tau}_h) & \forall \boldsymbol{\tau}_h \in \mathbf{H}_h, \\ b(\boldsymbol{\sigma}_h, \vec{\mathbf{v}}_h) - c(\mathbf{w}_h; \vec{\mathbf{u}}_h, \vec{\mathbf{v}}_h) &= \mathbf{F}_{\varphi_h}(\vec{\mathbf{v}}_h) & \forall \vec{\mathbf{v}}_h \in \mathbf{Q}_h, \end{aligned} \quad (5.4)$$

or, equivalently, as the discrete analogue of (4.9)

$$\mathbf{A}_{\mathbf{w}_h}((\boldsymbol{\sigma}_h, \vec{\mathbf{u}}_h), (\boldsymbol{\tau}_h, \vec{\mathbf{v}}_h)) + \mathbf{b}(\mathbf{w}_h; \mathbf{u}_h, \boldsymbol{\tau}_h) = \mathbf{G}(\boldsymbol{\tau}_h) + \mathbf{F}_{\varphi_h}(\vec{\mathbf{v}}_h) \quad \forall (\boldsymbol{\tau}_h, \vec{\mathbf{v}}_h) \in \mathbf{H}_h \times \mathbf{Q}_h, \quad (5.5)$$

where, given  $\mathbf{w}_h \in \mathbf{H}_h^u$ ,  $\mathbf{A}_{\mathbf{w}_h} : (\mathbf{H}_h \times \mathbf{Q}_h) \times (\mathbf{H}_h \times \mathbf{Q}_h) \rightarrow \mathbb{R}$  is defined according to (4.8).

On the other hand, for each  $j \in \{1, 2\}$  we introduce the operator  $\tilde{\mathbf{S}}_{j,h} : \mathbf{H}_h^u \rightarrow \mathbf{H}_h^\phi$  defined by

$$\tilde{\mathbf{S}}_{j,h}(\mathbf{w}_h) := \phi_{j,h} \quad \forall \mathbf{w}_h \in \mathbf{H}_h^u,$$

where  $(\vec{\phi}_{j,h}, \boldsymbol{\vartheta}_{j,h}) := ((\phi_{j,h}, \mathbf{t}_{j,h}), \boldsymbol{\vartheta}_{j,h}) \in \tilde{\mathbf{H}}_h \times \tilde{\mathbf{Q}}_h$  is the unique solution (to be confirmed below) of the problem that arises from the third and fourth rows of (5.2) when  $\tilde{c}_j(\mathbf{u}_h; \cdot, \cdot)$  is replaced by  $\tilde{c}_j(\mathbf{w}_h; \cdot, \cdot)$ , that is

$$\begin{aligned} \tilde{a}_j(\vec{\phi}_{j,h}, \vec{\psi}_{j,h}) + \tilde{c}_j(\mathbf{w}_h; \vec{\phi}_{j,h}, \vec{\psi}_{j,h}) + \tilde{b}(\vec{\psi}_{j,h}, \boldsymbol{\vartheta}_{j,h}) &= \tilde{F}_j(\vec{\psi}_{j,h}) & \forall \vec{\psi}_{j,h} \in \tilde{\mathbf{H}}_h, \\ \tilde{b}(\vec{\phi}_{j,h}, \boldsymbol{\eta}_{j,h}) &= \tilde{G}_j(\boldsymbol{\eta}_{j,h}) & \forall \boldsymbol{\eta}_{j,h} \in \tilde{\mathbf{Q}}_h. \end{aligned} \quad (5.6)$$

Equivalently, defining  $\tilde{a}_{j,\mathbf{w}_h} : \tilde{\mathbf{H}}_h \times \tilde{\mathbf{H}}_h \rightarrow \mathbb{R}$ , for each  $\mathbf{w}_h \in \mathbf{H}_h$ , as in (4.11), we can restate (5.6) as

$$\begin{aligned} \tilde{a}_{j,\mathbf{w}_h}(\vec{\phi}_{j,h}, \vec{\psi}_{j,h}) + \tilde{b}(\vec{\psi}_{j,h}, \boldsymbol{\vartheta}_{j,h}) &= \tilde{F}_j(\vec{\psi}_{j,h}) & \forall \vec{\psi}_{j,h} \in \tilde{\mathbf{H}}_h, \\ \tilde{b}(\vec{\phi}_{j,h}, \boldsymbol{\eta}_{j,h}) &= \tilde{G}_j(\boldsymbol{\eta}_{j,h}) & \forall \boldsymbol{\eta}_{j,h} \in \tilde{\mathbf{Q}}_h. \end{aligned} \quad (5.7)$$

In this way, defining  $\tilde{\mathbf{S}}_h : \mathbf{H}_h^u \rightarrow (\mathbf{H}_h^\phi \times \mathbf{H}_h^\phi)$  as

$$\tilde{\mathbf{S}}_h(\mathbf{w}_h) := (\tilde{\mathbf{S}}_{1,h}(\mathbf{w}_h), \tilde{\mathbf{S}}_{2,h}(\mathbf{w}_h)) \quad \forall \mathbf{w}_h \in \mathbf{H}_h^u, \quad (5.8)$$

and letting  $\mathbf{T}_h : \mathbf{H}_h^{\mathbf{u}} \rightarrow \mathbf{H}_h^{\mathbf{u}}$  be the operator given by

$$\mathbf{T}_h(\mathbf{w}_h) := \mathbf{S}_h(\mathbf{w}_h, \tilde{\mathbf{S}}_h(\mathbf{w}_h)) \quad \forall \mathbf{w}_h \in \mathbf{H}_h^{\mathbf{u}}, \quad (5.9)$$

we realize that solving (5.2) is equivalent to seeking a fixed-point of  $\mathbf{T}_h$ , that is  $\mathbf{u}_h \in \mathbf{H}_h^{\mathbf{u}}$  such that

$$\mathbf{T}_h(\mathbf{u}_h) = \mathbf{u}_h. \quad (5.10)$$

### 5.3 Discrete solvability analysis

In this section we address the solvability of (5.2) via the corresponding analysis of the equivalent fixed-point equation (5.10), which previously requires to prove that the operators  $\mathbf{S}_h$  (cf. (5.3)) and  $\tilde{\mathbf{S}}_h$  (cf. (5.8)), and hence  $\mathbf{T}_h$ , are well-defined. Equivalently, that the uncoupled problems (5.5) (or (5.4)) and (5.7) (or (5.6)) are well-posed.

We begin with the analysis of (5.5), for which we aim to prove that the bilinear forms  $a$ ,  $b$ , and  $c(\mathbf{w}_h; \cdot, \cdot)$ , for each  $\mathbf{w}_h \in \mathbf{H}_h^{\mathbf{u}}$ , when restricted to the corresponding finite element subspaces, satisfy the assumptions of [19, Theorem 3.5]. In fact, being the hypothesis **i)** of [19, Theorem 3.5] basically the same as the one of [19, Theorem 3.4], namely the symmetry and positive semi-definedness of  $a$  and  $c(\mathbf{w}_h; \cdot, \cdot)$ , which was already clarified in Section 4.3, we only need to concentrate here on **ii)** and **iii)** of [19, Theorem 3.5]. To this end, we first consider the following hypotheses on  $\tilde{\mathbb{H}}_h^{\sigma}$  and  $\mathbf{H}_h^{\mathbf{u}}$ :

**(H.1)**  $\tilde{\mathbb{H}}_h^{\sigma}$  contains the multiples of the identity tensor  $\mathbb{I}$ , and

**(H.2)**  $\operatorname{div}(\tilde{\mathbb{H}}_h^{\sigma}) \subseteq \mathbf{H}_h^{\mathbf{u}}$ .

It follows from **(H.1)** and the decomposition (3.6) that  $\mathbf{H}_h$  (cf. (5.1)) can be redefined as

$$\mathbf{H}_h := \left\{ \zeta_h - \left( \frac{1}{n|\Omega|} \int_{\Omega} \operatorname{tr}(\zeta) \right) \mathbb{I} : \zeta_h \in \tilde{\mathbb{H}}_h^{\sigma} \right\}.$$

In turn, letting  $\mathbf{V}_h$  be the kernel of  $b|_{\mathbf{H}_h \times \mathbf{Q}_h}$ , we readily deduce, thanks to the definition of  $b$

(cf. (3.10)) and **(H.2)**, that

$$\mathbf{V}_h := \left\{ \boldsymbol{\zeta}_h \in \mathbf{H}_h : \operatorname{div}(\boldsymbol{\zeta}_h) = 0 \quad \text{and} \quad \int_{\Omega} \boldsymbol{\delta}_h : \boldsymbol{\zeta}_h = 0 \quad \forall \boldsymbol{\delta}_h \in \mathbb{H}_h^\gamma \right\}.$$

Consequently, while  $\mathbf{V}_h$  is not necessarily contained in  $\mathbf{V}$  (cf. (4.16)), the fact that the elements of  $\mathbf{V}_h$  are still divergence-free, along with the inequality (4.17), suffice to conclude the discrete analogue of (4.18), and with the same constant, namely

$$a(\boldsymbol{\zeta}_h, \boldsymbol{\zeta}_h) \geq \alpha_d \|\boldsymbol{\zeta}_h\|_{\mathbf{H}}^2 \quad \forall \boldsymbol{\zeta}_h \in \mathbf{V}_h, \quad (5.11)$$

with  $\alpha_d := \frac{c_1^2}{\mu_1}$ . Similarly as for the continuous case, it is easily seen that (5.11) yields the discrete inf-sup condition for  $a$  required by the hypothesis **ii)** of [19, Theorem 3.5].

Furthermore, in order to continue the analysis, we introduce the discrete inf-sup condition for  $b$  as a third hypothesis, that is:

**(H.3)** there exists a positive constant  $\beta_d$ , independent of  $h$ , such that

$$\sup_{\substack{\boldsymbol{\tau}_h \in \mathbf{H}_h \\ \boldsymbol{\tau}_h \neq \mathbf{0}}} \frac{b(\boldsymbol{\tau}_h, \vec{\mathbf{v}}_h)}{\|\boldsymbol{\tau}_h\|_{\mathbf{H}}} \geq \beta_d \|\vec{\mathbf{v}}_h\|_{\mathbf{Q}} \quad \forall \vec{\mathbf{v}}_h \in \mathbf{Q}_h.$$

Next, proceeding analogously to the continuous case, we consider the same radius  $\delta$  employed in (4.19), and introduce now the discrete ball

$$\mathbf{W}_h(\delta) := \left\{ \mathbf{w}_h \in \mathbf{H}_h^u : \|\mathbf{w}_h\|_{0,4;\Omega} \leq \delta \right\}, \quad (5.12)$$

so that the boundedness for  $c(\mathbf{w}_h; \cdot, \cdot)|_{\mathbf{Q}_h \times \mathbf{Q}_h}$  becomes exactly as in (4.20), that is

$$|c(\mathbf{w}_h; \vec{\mathbf{v}}_h, \vec{\mathbf{z}}_h)| \leq |\Omega|^{1/2} (D + F \delta^{\rho-2}) \|\vec{\mathbf{v}}_h\|_{\mathbf{Q}} \|\vec{\mathbf{z}}_h\|_{\mathbf{Q}} \quad \forall \vec{\mathbf{v}}_h, \vec{\mathbf{z}}_h \in \mathbf{Q}_h.$$

Hence, having satisfied all the hypotheses of [19, Theorem 3.5], a straightforward application of this result implies the existence of a positive constant  $\alpha_{\mathbf{A},d}$ , depending only on  $\mu_0$ ,  $|\Omega|$ ,  $D$ ,  $F$ ,  $\delta$ ,  $\rho$ ,  $\alpha_d$ , and  $\beta_d$ , such that for each  $\mathbf{w}_h \in \mathbf{W}_h(\delta)$  there holds the discrete analogue of (4.21),

that is

$$\sup_{\substack{(\boldsymbol{\zeta}_h, \vec{\boldsymbol{z}}_h) \in \mathbf{H}_h \times \mathbf{Q}_h \\ (\boldsymbol{\zeta}_h, \vec{\boldsymbol{z}}_h) \neq \mathbf{0}}} \frac{\mathbf{A}_{\mathbf{w}_h}((\boldsymbol{\tau}_h, \vec{\boldsymbol{v}}_h), (\boldsymbol{\zeta}_h, \vec{\boldsymbol{z}}_h))}{\|(\boldsymbol{\zeta}_h, \vec{\boldsymbol{z}}_h)\|_{\mathbf{H} \times \mathbf{Q}}} \geq \alpha_{\mathbf{A}, \mathbf{d}} \|(\boldsymbol{\tau}_h, \vec{\boldsymbol{v}}_h)\|_{\mathbf{H} \times \mathbf{Q}} \quad \forall (\boldsymbol{\tau}_h, \vec{\boldsymbol{v}}_h) \in \mathbf{H}_h \times \mathbf{Q}_h. \quad (5.13)$$

Thus, using (5.13) and the boundedness property of  $\mathbf{b}(\mathbf{w}; \cdot, \cdot)$  (cf. (4.3)), similarly as for the derivation of (4.22), we deduce that for each  $\mathbf{w}_h \in \mathbf{W}_h(\delta)$  such that  $\|\mathbf{w}_h\|_{0,4;\Omega} \leq \frac{\mu_0 \alpha_{\mathbf{A}, \mathbf{d}}}{2}$ , there holds

$$\sup_{\substack{(\boldsymbol{\zeta}_h, \vec{\boldsymbol{z}}_h) \in \mathbf{H}_h \times \mathbf{Q}_h \\ (\boldsymbol{\zeta}_h, \vec{\boldsymbol{z}}_h) \neq \mathbf{0}}} \frac{\mathbf{A}_{\mathbf{w}_h}((\boldsymbol{\tau}_h, \vec{\boldsymbol{v}}_h), (\boldsymbol{\zeta}_h, \vec{\boldsymbol{z}}_h)) + \mathbf{b}(\mathbf{w}_h; \mathbf{v}_h, \boldsymbol{\zeta}_h)}{\|(\boldsymbol{\zeta}_h, \vec{\boldsymbol{z}}_h)\|_{\mathbf{H} \times \mathbf{Q}}} \geq \frac{\alpha_{\mathbf{A}, \mathbf{d}}}{2} \|(\boldsymbol{\tau}_h, \vec{\boldsymbol{v}}_h)\|_{\mathbf{H} \times \mathbf{Q}}$$

for all  $(\boldsymbol{\tau}_h, \vec{\boldsymbol{v}}_h) \in \mathbf{H}_h \times \mathbf{Q}_h$ .

Therefore, the well-posedness of (5.5) is established as follows.

**Theorem 5.1.** Given  $\delta > 0$ , let  $r \in (0, r_{0, \mathbf{d}}]$ , with

$$r_{0, \mathbf{d}} := \min \left\{ \delta, \frac{\mu_0 \alpha_{\mathbf{A}, \mathbf{d}}}{2} \right\}. \quad (5.14)$$

Then, for each  $(\mathbf{w}_h, \boldsymbol{\varphi}_h) \in \mathbf{H}_h^{\mathbf{u}} \times (\mathbf{H}_h^\phi \times \mathbf{H}_h^\phi)$  such that  $\|\mathbf{w}_h\|_{0,4;\Omega} \leq r$ , (5.5) (equivalently, (5.4)) has a unique solution  $(\boldsymbol{\sigma}_h, \vec{\mathbf{u}}_h) := (\boldsymbol{\sigma}_h, (\mathbf{u}_h, \boldsymbol{\gamma}_h)) \in \mathbf{H}_h \times \mathbf{Q}_h$ , and hence one can define  $\mathbf{S}_h(\mathbf{w}_h, \boldsymbol{\varphi}_h) := \mathbf{u}_h$ . Moreover, there exists a positive constant  $C_{\mathbf{S}, \mathbf{d}}$ , depending only on  $\|\dot{\mathbf{i}}_4\|$ ,  $\|\mathbf{g}\|_{0,\Omega}$ , and  $\alpha_{\mathbf{A}, \mathbf{d}}$ , such that

$$\begin{aligned} \|\mathbf{S}_h(\mathbf{w}_h, \boldsymbol{\varphi}_h)\|_{0,4;\Omega} &= \|\mathbf{u}_h\|_{0,4;\Omega} \leq \|(\boldsymbol{\sigma}_h, \vec{\mathbf{u}}_h)\|_{\mathbf{H} \times \mathbf{Q}} \\ &\leq C_{\mathbf{S}, \mathbf{d}} \left\{ \|\mathbf{u}_D\|_{1/2, \Gamma} + \|\boldsymbol{\phi}_r\|_{0,4;\Omega} + \|\boldsymbol{\varphi}_h\|_{0,4;\Omega} \right\}. \end{aligned} \quad (5.15)$$

*Proof.* Similarly as for the proof of Theorem 4.1, we observe now that the bilinear form  $\mathbf{A}_{\mathbf{w}_h} + \mathbf{b}(\mathbf{w}_h; \cdot, \cdot)$  satisfies the hypotheses of [21, Theorem 2.22], so that, noting in this case that  $\mathbf{G}|_{\mathbf{H}_h} \in \mathbf{H}'_h$  and  $\mathbf{F}\boldsymbol{\varphi}_h|_{\mathbf{Q}_h} \in \mathbf{Q}'_h$ , an application of that theorem proves the present result.  $\square$

On the other hand, in order to establish the well-posedness of (5.7) (equivalently, (5.6)), in what follows we show that the bilinear forms  $\tilde{a}_{j, \mathbf{w}_h}|_{\tilde{\mathbf{H}}_h \times \tilde{\mathbf{H}}_h}$  and  $\tilde{b}|_{\tilde{\mathbf{H}}_h \times \tilde{\mathbf{Q}}_h}$  satisfy the hypotheses of [21, Proposition 2.42]. To this end, we proceed as in [18, Section 5.5] (see also [5, Section 4.3, Lemma 4.2] and [13, Section 4.2, Lemmas 4.1 and 4.5]), and introduce first the kernel of

$\tilde{b}|_{\tilde{\mathbf{H}}_h \times \tilde{\mathbf{Q}}_h}$ , that is

$$\tilde{\mathbf{V}}_h := \left\{ \vec{\psi}_h = (\psi_h, \mathbf{r}_h) \in \tilde{\mathbf{H}}_h : \tilde{b}(\vec{\psi}_h, \boldsymbol{\eta}_h) = 0 \quad \forall \boldsymbol{\eta}_h \in \tilde{\mathbf{Q}}_h \right\},$$

and

$$Z_{0,h} := \left\{ \boldsymbol{\eta}_h \in \tilde{\mathbf{Q}}_h : \tilde{b}(\vec{\psi}_h, \boldsymbol{\eta}_h) = 0 \quad \forall \vec{\psi}_h = (\psi_h, 0) \in \tilde{\mathbf{H}}_h \right\},$$

which become, respectively,

$$\tilde{\mathbf{V}}_h := \left\{ \vec{\psi}_h = (\psi_h, \mathbf{r}_h) \in \tilde{\mathbf{H}}_h : \int_{\Omega} \mathbf{r}_h \cdot \boldsymbol{\eta}_h + \int_{\Omega} \psi_h \operatorname{div}(\boldsymbol{\eta}_h) = 0 \quad \forall \boldsymbol{\eta}_h \in \mathbf{H}_h^{\vartheta} \right\},$$

and

$$Z_{0,h} := \left\{ \boldsymbol{\eta}_h \in \mathbf{H}_h^{\vartheta} : \int_{\Omega} \psi_h \operatorname{div}(\boldsymbol{\eta}_h) = 0 \quad \forall \psi_h \in \mathbf{H}_h^{\phi} \right\}. \quad (5.16)$$

Next, we consider the following assumptions on the subspaces  $\mathbf{H}_h^{\phi}$ ,  $\mathbf{H}_h^{\mathbf{t}}$ , and  $\mathbf{H}_h^{\vartheta}$ :

**(H.4)**  $\operatorname{div}(\mathbf{H}_h^{\vartheta}) \subseteq \mathbf{H}_h^{\phi}$ ,

**(H.5)**  $Z_{0,h} \subseteq \mathbf{H}_h^{\mathbf{t}}$ , and

**(H.6)** there exists a positive constant  $\beta_{1,d}$ , independent of  $h$ , such that

$$\sup_{\substack{\boldsymbol{\eta}_h \in \mathbf{H}_h^{\vartheta} \\ \boldsymbol{\eta}_h \neq \mathbf{0}}} \frac{\int_{\Omega} \psi_h \operatorname{div}(\boldsymbol{\eta}_h)}{\|\boldsymbol{\eta}_h\|_{\tilde{\mathbf{Q}}}} \geq \beta_{1,d} \|\psi_h\|_{0,4;\Omega} \quad \forall \psi_h \in \mathbf{H}_h^{\phi}.$$

As a consequence of **(H.4)** we easily deduce from (5.16) that

$$Z_{0,h} := \left\{ \boldsymbol{\eta}_h \in \mathbf{H}_h^{\vartheta} : \operatorname{div}(\boldsymbol{\eta}_h) = 0 \quad \text{in } \Omega \right\}, \quad (5.17)$$

and thus, given  $\boldsymbol{\eta}_h \in Z_{0,h}$ , and using **(H.5)**, we bound the supremum by below with  $\mathbf{r}_h :=$

$\boldsymbol{\eta}_h \in \mathbf{H}_h^{\mathbf{t}}$ , to deduce that

$$\sup_{\substack{\mathbf{r}_h \in \mathbf{H}_h^{\mathbf{t}} \\ \mathbf{r}_h \neq \mathbf{0}}} \frac{\int_{\Omega} \mathbf{r}_h \cdot \boldsymbol{\eta}_h}{\|\mathbf{r}_h\|_{0,\Omega}} \geq \|\boldsymbol{\eta}_h\|_{0,\Omega} = \beta_{2,d} \|\boldsymbol{\eta}_h\|_{\tilde{\mathbf{Q}}} \quad \forall \boldsymbol{\eta}_h \in Z_{0,h}, \quad (5.18)$$

with  $\beta_{2,d} = 1$ . Consequently, invoking [18, Lemma 5.1] with local notation there given by  $X = \mathbf{H}_h^{\phi}$ ,  $Y = Y_1 = \mathbf{H}_h^{\mathbf{t}}$ ,  $Y_2 = \{\mathbf{0}\}$ ,  $V = \tilde{\mathbf{V}}_h$ ,  $Z = \mathbf{H}_h^{\boldsymbol{\vartheta}}$ , and  $Z_0 = Z_{0,h}$ , we conclude that (H.6) and (5.18) are equivalent to the existence of positive constants  $\tilde{\beta}_d$  and  $\tilde{C}_d$  such that

$$\sup_{\substack{\vec{\psi}_h \in \tilde{\mathbf{H}}_h \\ \vec{\psi}_h \neq \mathbf{0}}} \frac{\tilde{b}(\vec{\psi}_h, \boldsymbol{\eta}_h)}{\|\vec{\psi}_h\|_{\tilde{\mathbf{H}}}} \geq \tilde{\beta}_d \|\boldsymbol{\eta}_h\|_{\tilde{\mathbf{Q}}} \quad \forall \boldsymbol{\eta}_h \in \tilde{\mathbf{Q}}_h, \quad (5.19)$$

and

$$\|\mathbf{r}_h\|_{0,\Omega} \geq \tilde{C}_d \|\psi_h\|_{0,4;\Omega} \quad \forall \vec{\psi}_h = (\psi_h, \mathbf{r}_h) \in \tilde{\mathbf{V}}_h. \quad (5.20)$$

Note that (5.19) constitutes the discrete inf-sup condition for  $\tilde{b}$  required in [21, Proposition 2.42, eq. (2.36)]. In turn, given  $\vec{\psi}_{j,h} = (\psi_{j,h}, \mathbf{r}_{j,h}) \in \tilde{\mathbf{V}}_h$ , we use (4.5) and (2.4), similarly to the first part of the derivation of (4.26), but then, differently from there, employ (5.20) to conclude that

$$\begin{aligned} \tilde{a}_{j,\mathbf{w}_h}(\vec{\psi}_{j,h}, \vec{\psi}_{j,h}) &= \tilde{a}_j(\vec{\psi}_{j,h}, \vec{\psi}_{j,h}) = \int_{\Omega} \mathbf{Q}_j |\mathbf{r}_{j,h}|^2 \geq C_j \|\mathbf{r}_{j,h}\|_{0,\Omega}^2, \\ &\geq \frac{C_j}{2} \left\{ \tilde{C}_d^2 \|\psi_{j,h}\|_{0,4;\Omega}^2 + \|\mathbf{r}_{j,h}\|_{0,\Omega}^2 \right\} \geq \tilde{\alpha}_{j,d} \|\vec{\psi}_{j,h}\|_{\tilde{\mathbf{H}}}^2, \end{aligned} \quad (5.21)$$

with  $\tilde{\alpha}_{j,d} := \frac{C_j}{2} \min \{ \tilde{C}_d^2, 1 \}$ . Then, analogously to the continuous case, it is readily seen that (5.21) yields the discrete inf-sup condition for  $\tilde{a}_{j,\mathbf{w}_h}$  required in [21, Proposition 2.42, eq. (2.35)].

We are now in position to state the discrete analogue of Theorem 4.2.

**Theorem 5.2.** For each  $\mathbf{w}_h \in \mathbf{H}_h^{\mathbf{u}}$ , and for each  $j \in \{1, 2\}$ , (5.7) (equivalently, (5.6)) has a unique solution  $(\vec{\phi}_{j,h}, \boldsymbol{\vartheta}_{j,h}) := ((\phi_{j,h}, \mathbf{t}_{j,h}), \boldsymbol{\vartheta}_{j,h}) \in \tilde{\mathbf{H}}_h \times \tilde{\mathbf{Q}}_h$ , and hence one can define  $\tilde{\mathbf{S}}_{j,h}(\mathbf{w}_h) := \phi_{j,h}$ . Moreover, there exists a positive constant  $C_{\tilde{\mathbf{S}},d}$ , depending only on  $\tilde{\alpha}_{j,d}$ ,  $\tilde{\beta}_d$ ,  $\|\mathbf{Q}_j\|_{0,\infty;\Omega}$ ,  $\|i_4\|$ , and

$R_j$ ,  $j \in \{1, 2\}$ , such that

$$\begin{aligned} \|\tilde{\mathbf{S}}_h(\mathbf{w})\|_{0,4;\Omega} &:= \|(\tilde{\mathbf{S}}_{1,h}(\mathbf{w}_h), \tilde{\mathbf{S}}_{2,h}(\mathbf{w}_h))\|_{0,4;\Omega} \\ &\leq C_{\tilde{\mathbf{S}},d} \sum_{j=1}^2 \left\{ \|g_j\|_{0,4/3;\Omega} + (1 + \|\mathbf{w}_h\|_{0,4;\Omega}) \|\phi_{j,D}\|_{1/2,\Gamma} \right\}. \end{aligned} \quad (5.22)$$

*Proof.* According to the previous discussion on  $\tilde{b}$  and  $\tilde{a}_{j,\mathbf{w}_h}$ , for each  $\mathbf{w}_h \in \mathbf{H}_h^{\mathbf{u}}$ , and bearing in mind that  $\tilde{a}_{j,\mathbf{w}_h}|_{\tilde{\mathbf{H}}_h \times \tilde{\mathbf{H}}_h}$ ,  $\tilde{b}|_{\tilde{\mathbf{H}}_h \times \tilde{\mathbf{Q}}_h}$ ,  $\tilde{F}_j|_{\tilde{\mathbf{H}}_h}$ , and  $\tilde{G}_j|_{\tilde{\mathbf{Q}}_h}$  are all bounded, the existence of a unique solution of (5.7), for each  $j \in \{1, 2\}$ , follows from a direct application of [21, Proposition 2.42]. In turn, employing the discrete version of the first inequality in [21, Theorem 2.34, eq. (2.30)], we get the a priori estimate for  $\|\tilde{\mathbf{S}}_{j,h}(\mathbf{w}_h)\|_{0,4;\Omega}$ , from which, summing up over  $j \in \{1, 2\}$ , we arrive at (5.22).  $\square$

At this point we remark that, similarly as for the continuous case, the component  $\boldsymbol{\vartheta}_{j,h}$  of the solution of (5.7) can be bounded employing the discrete version of the second inequality in [21, Theorem 2.34, eq. (2.30)], which yields

$$\|\boldsymbol{\vartheta}_{j,h}\|_{\tilde{\mathbf{Q}}} \leq \tilde{M}_{j,d} (1 + \|\mathbf{w}_h\|_{0,4;\Omega}) \left\{ \|g_j\|_{0,4/3;\Omega} + (1 + \|\mathbf{w}_h\|_{0,4;\Omega}) \|\phi_{j,D}\|_{1/2,\Gamma} \right\}, \quad (5.23)$$

where  $\tilde{M}_{j,d}$  is a positive constant depending on  $\tilde{\alpha}_{j,d}$ ,  $\tilde{\beta}_d$ ,  $\|\mathbf{Q}_j\|_{0,\infty;\Omega}$ ,  $\|i_4\|$ , and  $R_j$ .

Having established, thanks to Theorems 5.1 and 5.2, that  $\mathbf{S}_h$  (cf. (5.3)),  $\tilde{\mathbf{S}}_h$  (cf. (5.8)), and hence  $\mathbf{T}_h$  (cf. (5.9)), are well-defined, we now aim to show that  $\mathbf{T}_h$  has a unique fixed-point. More precisely, analogously to the continuous case, in what follows we prove that  $\mathbf{T}_h$  verifies the hypotheses of the Banach theorem. For this purpose, given  $r \in (0, r_{0,d}]$ , with  $r_{0,d}$  as in (5.14), we first follow (5.12) and define

$$\mathbf{W}_h(r) := \left\{ \mathbf{w}_h \in \mathbf{H}_h^{\mathbf{u}} : \|\mathbf{w}_h\|_{0,4;\Omega} \leq r \right\}. \quad (5.24)$$

Then, using now the a priori estimates (5.15) and (5.22), we easily deduce the existence of a positive constant  $C_{\mathbf{T},d}$ , depending only on  $C_{\mathbf{S},d}$  and  $C_{\tilde{\mathbf{S}},d}$ , such that for each  $\mathbf{w}_h \in \mathbf{W}_h(r)$  there

holds

$$\|\mathbf{T}_h(\mathbf{w}_h)\|_{0,4;\Omega} \leq C_{\mathbf{T},d} \left\{ \|\mathbf{u}_D\|_{1/2,\Gamma} + \|\phi_r\|_{0,4;\Omega} + \sum_{j=1}^2 \left\{ \|g_j\|_{0,4/3;\Omega} + (1+r) \|\phi_{j,D}\|_{1/2,\Gamma} \right\} \right\},$$

which constitutes the discrete version of (4.31). Hence, we are able to state next the discrete analogue of Lemma 4.1

**Lemma 5.3.** Given  $r \in (0, r_{0,d}]$ , with  $r_{0,d}$  as in (5.14), assume that the data satisfy

$$C_{\mathbf{T},d} \left\{ \|\mathbf{u}_D\|_{1/2,\Gamma} + \|\phi_r\|_{0,4;\Omega} + \sum_{j=1}^2 \left\{ \|g_j\|_{0,4/3;\Omega} + (1+r) \|\phi_{j,D}\|_{1/2,\Gamma} \right\} \right\} \leq r. \quad (5.25)$$

Then, there holds  $\mathbf{T}_h(\mathbf{W}_h(r)) \subseteq \mathbf{W}_h(r)$ .

In turn, employing similar arguments to those yielding Lemmas 4.3, 4.4, and 4.5, we are able to show their discrete counterparts, that is the continuity properties of  $\mathbf{S}_h$ ,  $\tilde{\mathbf{S}}_h$ , and  $\mathbf{T}_h$ . However, being the respective proofs almost verbatim to the continuous ones, we omit the details and just state the corresponding results as follows.

**Lemma 5.4.** Let  $r \in (0, r_{0,d}]$ , with  $r_{0,d}$  as in (5.14). Then, there exists a positive constant  $L_{\mathbf{S},d}$ , depending only on  $\alpha_{\mathbf{A},d}$ ,  $\|\mathbf{g}\|_{0,\Omega}$ ,  $\mu_0$ ,  $L_c$  (cf. Lemma 4.2), and  $r$ , such that

$$\|\mathbf{S}_h(\mathbf{w}_h, \varphi_h) - \mathbf{S}_h(\mathbf{z}_h, \xi_h)\|_{0,4;\Omega} \leq L_{\mathbf{S},d} \left\{ \|\varphi_h - \xi_h\|_{0,4;\Omega} + \|\mathbf{S}_h(\mathbf{z}_h, \xi_h)\|_{0,4;\Omega} \|\mathbf{w}_h - \mathbf{z}_h\|_{0,4;\Omega} \right\}$$

for all  $(\mathbf{w}_h, \varphi_h), (\mathbf{z}_h, \xi_h) \in \mathbf{W}_h(r) \times (\mathbf{H}_h^\phi \times \mathbf{H}_h^\phi)$ .

**Lemma 5.5.** Let  $r \in (0, r_{0,d}]$ , with  $r_{0,d}$  as in (5.14). Then, there exists a positive constant  $L_{\tilde{\mathbf{S}},d}$ , depending only on  $\tilde{C}_j$  (cf. (4.28)),  $R_j$ , and  $\tilde{\alpha}_{j,d}$ ,  $j \in \{1, 2\}$ , such that

$$\begin{aligned} & \|\tilde{\mathbf{S}}_h(\mathbf{w}_h) - \tilde{\mathbf{S}}_h(\mathbf{z}_h)\|_{0,4;\Omega} \\ & \leq L_{\tilde{\mathbf{S}},d} \sum_{j=1}^2 \left\{ \|g_j\|_{0,4/3;\Omega} + (1 + \|\mathbf{w}_h\|_{0,4;\Omega}) \|\phi_{j,D}\|_{1/2,\Gamma} \right\} \|\mathbf{w}_h - \mathbf{z}_h\|_{0,4;\Omega} \end{aligned}$$

for all  $\mathbf{w}_h, \mathbf{z}_h \in \mathbf{W}_h(r)$ .

Thanks to Lemmas 5.4 and 5.5, the Lipschitz continuity of  $\mathbf{T}_h$  (cf. (5.9)) is stated as follows.

**Lemma 5.6.** Let  $r \in (0, r_{0,d}]$ , with  $r_{0,d}$  as in (5.14). Then, there exists a positive constant  $L_{\mathbf{T},d}$ , depending only on  $L_{\mathbf{S},d}$ ,  $L_{\tilde{\mathbf{S}},d}$ ,  $C_{\mathbf{S},d}$ , and  $C_{\tilde{\mathbf{S}},d}$ , such that

$$\begin{aligned} & \|\mathbf{T}_h(\mathbf{w}_h) - \mathbf{T}_h(\mathbf{z}_h)\|_{0,4;\Omega} \\ & \leq L_{\mathbf{T},d} \left\{ \|\mathbf{u}_D\|_{1/2,\Gamma} + \|\phi_r\|_{0,4;\Omega} + \sum_{j=1}^2 \left\{ \|g_j\|_{0,4/3;\Omega} + (1+r) \|\phi_{j,D}\|_{1/2,\Gamma} \right\} \right\} \|\mathbf{w}_h - \mathbf{z}_h\|_{0,4;\Omega} \end{aligned}$$

for all  $\mathbf{w}_h, \mathbf{z}_h \in \mathbf{W}_h(r)$ .

We end this section with the solvability result for (5.10) (and hence for (5.2)).

**Theorem 5.7.** Given  $r \in (0, r_{0,d}]$ , with  $r_{0,d}$  as in (5.14), assume that, in addition to the hypothesis of Lemma 5.3 (cf. (5.25)), the data satisfy

$$L_{\mathbf{T},d} \left\{ \|\mathbf{u}_D\|_{1/2,\Gamma} + \|\phi_r\|_{0,4;\Omega} + \sum_{j=1}^2 \left\{ \|g_j\|_{0,4/3;\Omega} + (1+r) \|\phi_{j,D}\|_{1/2,\Gamma} \right\} \right\} < 1. \quad (5.26)$$

Then, there exists a unique  $\mathbf{u}_h \in \mathbf{W}_h(r)$  (cf. (5.24)) fixed point of  $\mathbf{T}_h$  (cf. (5.10)). Equivalently, (5.2) has a unique solution  $(\boldsymbol{\sigma}_h, \vec{\mathbf{u}}_h) := (\boldsymbol{\sigma}_h, (\mathbf{u}_h, \boldsymbol{\gamma}_h)) \in \mathbf{H}_h \times \mathbf{Q}_h$  and  $(\vec{\phi}_{j,h}, \boldsymbol{\vartheta}_{j,h}) := ((\phi_{j,h}, \mathbf{t}_{j,h}), \boldsymbol{\vartheta}_{j,h}) \in \tilde{\mathbf{H}}_h \times \tilde{\mathbf{Q}}_h$ ,  $j \in \{1, 2\}$ , with  $\mathbf{u}_h \in \mathbf{W}_h(r)$ . Moreover, there exist positive constants  $\mathcal{C}_d$ ,  $\mathcal{C}_{1,d}$ , and  $\mathcal{C}_{2,d}$ , depending on  $\tilde{\alpha}_{1,d}$ ,  $\tilde{\alpha}_{2,d}$ ,  $\tilde{\beta}_d$ ,  $\|\mathbf{Q}_1\|_{0,\infty;\Omega}$ ,  $\|\mathbf{Q}_2\|_{0,\infty;\Omega}$ ,  $\|\mathbf{i}_4\|$ ,  $\mathbf{R}_1$ ,  $\mathbf{R}_2$ ,  $r$ ,  $\|\mathbf{i}_4\|$ ,  $\|\mathbf{g}\|_{0,\Omega}$ , and  $\alpha_{\mathbf{A},d}$ , such that there hold the following a priori bounds

$$\|(\boldsymbol{\sigma}_h, \vec{\mathbf{u}}_h)\|_{\mathbf{H} \times \mathbf{Q}} \leq \mathcal{C}_d \left\{ \|\mathbf{u}_D\|_{1/2,\Gamma} + \|\phi_r\|_{0,4;\Omega} + \sum_{j=1}^2 \left\{ \|g_j\|_{0,4/3;\Omega} + \|\phi_{j,D}\|_{1/2,\Gamma} \right\} \right\}, \quad (5.27)$$

and for each  $j \in \{1, 2\}$

$$\|(\vec{\phi}_{j,h}, \boldsymbol{\vartheta}_{j,h})\|_{\tilde{\mathbf{H}} \times \tilde{\mathbf{Q}}} \leq \mathcal{C}_{j,d} \left\{ \|g_j\|_{0,4/3;\Omega} + \|\phi_{j,D}\|_{1/2,\Gamma} \right\}. \quad (5.28)$$

*Proof.* It proceeds analogously to the proof of Theorem 4.6. Indeed, since  $\mathbf{T}_h$  is a contraction that maps the ball  $\mathbf{W}_h(r)$  into itself, which is consequence of Lemma 5.3 and assumption (5.26),

a direct application of the Banach fixed-point theorem confirms the solvabilities of (5.10) and (5.2). In turn, noting that  $\mathbf{u}_h = \mathbf{T}_h(\mathbf{u}_h) = \mathbf{S}_h(\mathbf{u}_h, \tilde{\mathbf{S}}_h(\mathbf{u}_h))$  and  $\boldsymbol{\phi}_h := (\phi_{1,h}, \phi_{2,h}) = \tilde{\mathbf{S}}_h(\mathbf{u}_h)$ , the a priori estimates (5.27) and (5.28) follow from (5.15), (5.22), and (5.23).  $\square$

## 5.4 A priori error analysis

In this section we consider arbitrary finite element subspaces satisfying the hypotheses **(H.1)** up to **(H.6)** introduced in Section 5.3, and derive the Céa estimate for the global error

$$\|(\boldsymbol{\sigma}, \bar{\mathbf{u}}) - (\boldsymbol{\sigma}_h, \bar{\mathbf{u}}_h)\|_{\mathbf{H} \times \mathbf{Q}} + \sum_{j=1}^2 \|(\vec{\phi}_j, \boldsymbol{\vartheta}_j) - (\vec{\phi}_{j,h}, \boldsymbol{\vartheta}_{j,h})\|_{\tilde{\mathbf{H}} \times \tilde{\mathbf{Q}}},$$

where  $((\boldsymbol{\sigma}, \bar{\mathbf{u}}), (\vec{\phi}_j, \boldsymbol{\vartheta}_j)) := ((\boldsymbol{\sigma}, (\mathbf{u}, \boldsymbol{\gamma})), ((\phi_j, \mathbf{t}_j), \boldsymbol{\vartheta}_j)) \in (\mathbf{H} \times \mathbf{Q}) \times (\tilde{\mathbf{H}} \times \tilde{\mathbf{Q}})$ ,  $j \in \{1, 2\}$ , with  $\mathbf{u} \in \mathbf{W}(r)$ , is the unique solution of (3.24), and  $((\boldsymbol{\sigma}_h, \bar{\mathbf{u}}_h), (\vec{\phi}_{j,h}, \boldsymbol{\vartheta}_{j,h})) := ((\boldsymbol{\sigma}_h, (\mathbf{u}_h, \boldsymbol{\gamma}_h)), ((\phi_{j,h}, \mathbf{t}_{j,h}), \boldsymbol{\vartheta}_{j,h})) \in (\mathbf{H}_h \times \mathbf{Q}_h) \times (\tilde{\mathbf{H}}_h \times \tilde{\mathbf{Q}}_h)$ ,  $j \in \{1, 2\}$ , with  $\mathbf{u}_h \in \mathbf{W}_h(r)$ , is the unique solution of (5.2). To this end, in what follows we apply known Strang-type estimates to the pairs of associated continuous and discrete schemes arising from (3.24) and (5.2), once they are split according to the two decoupled problems. Hereafter, given a subspace  $X_h$  of an arbitrary Banach space  $(X, \|\cdot\|_X)$ , we set

$$\text{dist}(x, X_h) := \inf_{x_h \in X_h} \|x - x_h\|_X.$$

We begin the analysis with the first two equations of (3.24) and (5.2), which can be rewritten as

$$\begin{aligned} \mathcal{A}((\boldsymbol{\sigma}, \bar{\mathbf{u}}), (\boldsymbol{\tau}, \vec{\mathbf{v}})) &= \mathcal{F}((\boldsymbol{\tau}, \vec{\mathbf{v}})) & \forall (\boldsymbol{\tau}, \vec{\mathbf{v}}) \in \mathbf{H} \times \mathbf{Q}, \quad \text{and} \\ \mathcal{A}_h((\boldsymbol{\sigma}_h, \bar{\mathbf{u}}_h), (\boldsymbol{\tau}_h, \vec{\mathbf{v}}_h)) &= \mathcal{F}_h((\boldsymbol{\tau}_h, \vec{\mathbf{v}}_h)) & \forall (\boldsymbol{\tau}_h, \vec{\mathbf{v}}_h) \in \mathbf{H}_h \times \mathbf{Q}_h, \end{aligned}$$

where

$$\begin{aligned}\mathcal{A}((\boldsymbol{\zeta}, \vec{\mathbf{w}}), (\boldsymbol{\tau}, \vec{\mathbf{v}})) &:= \mathbf{A}_{\mathbf{u}}((\boldsymbol{\zeta}, \vec{\mathbf{w}}), (\boldsymbol{\tau}, \vec{\mathbf{v}})) + \mathbf{b}(\mathbf{u}; \mathbf{w}, \boldsymbol{\tau}), \\ \mathcal{A}_h((\boldsymbol{\zeta}_h, \vec{\mathbf{w}}_h), (\boldsymbol{\tau}_h, \vec{\mathbf{v}}_h)) &:= \mathbf{A}_{\mathbf{u}_h}((\boldsymbol{\zeta}_h, \vec{\mathbf{w}}_h), (\boldsymbol{\tau}_h, \vec{\mathbf{v}}_h)) + \mathbf{b}(\mathbf{u}_h; \mathbf{w}_h, \boldsymbol{\tau}_h), \\ \mathcal{F}(\boldsymbol{\tau}, \vec{\mathbf{v}}) &:= \mathbf{G}(\boldsymbol{\tau}) + \mathbf{F}_{\phi}(\vec{\mathbf{v}}), \quad \text{and} \\ \mathcal{F}_h(\boldsymbol{\tau}_h, \vec{\mathbf{v}}_h) &:= \mathbf{G}(\boldsymbol{\tau}_h) + \mathbf{F}_{\phi_h}(\vec{\mathbf{v}}_h),\end{aligned}$$

for all  $(\boldsymbol{\zeta}, \vec{\mathbf{w}}), (\boldsymbol{\tau}, \vec{\mathbf{v}}) \in \mathbf{H} \times \mathbf{Q}$ , for all  $(\boldsymbol{\zeta}_h, \vec{\mathbf{w}}_h), (\boldsymbol{\tau}_h, \vec{\mathbf{v}}_h) \in \mathbf{H}_h \times \mathbf{Q}_h$ . Then, applying the a priori error bound provided by [15, Lemma 5.1], and then suitably bounding the resulting consistency estimate, which is given by  $\|\mathcal{A}((\boldsymbol{\sigma}, \vec{\mathbf{u}}), (\cdot, \cdot)) - \mathcal{A}_h((\boldsymbol{\sigma}, \vec{\mathbf{u}}), (\cdot, \cdot))\|_{(\mathbf{H}_h \times \mathbf{Q}_h)'}$ , we deduce the existence of a positive constant  $\mathcal{C}_{\text{ST}}$ , depending only on  $\|\mathcal{A}\|$ ,  $\|\mathcal{A}_h\|$ , and  $\alpha_{\mathbf{A},d}$ , and thus, easily shown to be independent of  $h$ , such that

$$\begin{aligned}\|(\boldsymbol{\sigma}, \vec{\mathbf{u}}) - (\boldsymbol{\sigma}_h, \vec{\mathbf{u}}_h)\|_{\mathbf{H} \times \mathbf{Q}} &\leq \mathcal{C}_{\text{ST}} \left\{ \text{dist}(\boldsymbol{\sigma}, \mathbf{H}_h) + \text{dist}(\vec{\mathbf{u}}, \mathbf{Q}_h) + \|\mathbf{F}_{\phi} - \mathbf{F}_{\phi_h}\|_{\mathbf{Q}'_h} \right. \\ &\quad \left. + \|\mathbf{b}(\mathbf{u}; \mathbf{u}, \cdot) - \mathbf{b}(\mathbf{u}_h; \mathbf{u}, \cdot)\|_{\mathbf{H}'_h} + \|c(\mathbf{u}; \vec{\mathbf{u}}, \cdot) - c(\mathbf{u}_h; \vec{\mathbf{u}}, \cdot)\|_{\mathbf{Q}'_h} \right\}.\end{aligned}\tag{5.29}$$

Note, in particular, that  $\|\mathcal{A}\|$  depends proportionally on  $\|a\| = \frac{1}{\mu_0}$ ,  $\|b\| = 1$ ,  $\|\mathbf{b}(\mathbf{u}, \cdot, \cdot)\| = \frac{1}{\mu_0} \|\mathbf{u}\|_{0,4;\Omega}$ , and  $\|c(\mathbf{u}; \cdot, \cdot)\| = |\Omega|^{1/2} (D + F\|\mathbf{u}\|_{0,4;\Omega}^{\rho-2})$ , with  $\|\mathbf{u}\|_{0,4;\Omega}$  bounded by  $r$ . An analogue remark is valid for  $\|\mathcal{A}_h\|$ . Next, proceeding as for the derivation of (4.42), we readily obtain

$$\|\mathbf{F}_{\phi} - \mathbf{F}_{\phi_h}\|_{\mathbf{Q}'_h} \leq \|\mathbf{g}\|_{0,\Omega} \|\phi - \phi_h\|_{0,4;\Omega}.\tag{5.30}$$

In turn, bearing in mind the definition of  $\mathbf{b}$  (cf. (3.11)), we find that for each  $\boldsymbol{\tau}_h \in \mathbf{H}_h$  there holds

$$\mathbf{b}(\mathbf{u}; \mathbf{u}, \boldsymbol{\tau}_h) - \mathbf{b}(\mathbf{u}_h; \mathbf{u}, \boldsymbol{\tau}_h) = \mathbf{b}(\mathbf{u} - \mathbf{u}_h; \mathbf{u}, \boldsymbol{\tau}_h),$$

from which, employing the boundedness property of  $\mathbf{b}$  (cf. first row of (4.3)), we conclude that

$$\|\mathbf{b}(\mathbf{u}; \mathbf{u}, \cdot) - \mathbf{b}(\mathbf{u}_h; \mathbf{u}, \cdot)\|_{\mathbf{H}'_h} \leq \frac{1}{\mu_0} \|\mathbf{u}\|_{0,4;\Omega} \|\mathbf{u} - \mathbf{u}_h\|_{0,4;\Omega}.\tag{5.31}$$

Similarly, using the continuity property of  $c$  provided by (4.33) (cf. Lemma 4.2), we get

$$\begin{aligned} \|c(\mathbf{u}; \bar{\mathbf{u}}, \cdot) - c(\mathbf{u}_h; \bar{\mathbf{u}}, \cdot)\|_{\mathbf{Q}'_h} &\leq L_c \left\{ \|\mathbf{u}\|_{0,4;\Omega} + \|\mathbf{u}_h\|_{0,4;\Omega} \right\}^{\rho-3} \|\mathbf{u}\|_{0,4;\Omega} \|\mathbf{u} - \mathbf{u}_h\|_{0,4;\Omega} \\ &\leq L_c (2r)^{\rho-3} \|\mathbf{u}\|_{0,4;\Omega} \|\mathbf{u} - \mathbf{u}_h\|_{0,4;\Omega}. \end{aligned} \quad (5.32)$$

In this way, replacing the bounds given by (5.30), (5.31), and (5.32), back into (5.29), we arrive at

$$\begin{aligned} \|(\boldsymbol{\sigma}, \bar{\mathbf{u}}) - (\boldsymbol{\sigma}_h, \bar{\mathbf{u}}_h)\|_{\mathbf{H} \times \mathbf{Q}} &\leq \widehat{\mathcal{C}}_{\text{ST}} \left\{ \text{dist}(\boldsymbol{\sigma}, \mathbf{H}_h) + \text{dist}(\bar{\mathbf{u}}, \mathbf{Q}_h) \right. \\ &\quad \left. + \|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_{0,4;\Omega} + \|\mathbf{u}\|_{0,4;\Omega} \|\mathbf{u} - \mathbf{u}_h\|_{0,4;\Omega} \right\}, \end{aligned} \quad (5.33)$$

where  $\widehat{\mathcal{C}}_{\text{ST}}$  is a positive constant depending only on  $\mathcal{C}_{\text{ST}}$ ,  $\|\mathbf{g}\|_{0,\Omega}$ ,  $\mu_0$ ,  $L_c$ ,  $r$ , and  $\rho$ .

On the other hand, proceeding analogously with the third and fourth equations of (3.24) and (5.2), but using now the particular Strang-type estimate provided by [18, Lemma 6.1] (see also [5, Lemma 5.1] for a slightly more general result), we deduce, for each  $j \in \{1, 2\}$ , the existence of a positive constant  $\mathcal{C}_{j,\text{ST}}$  depending only on  $\tilde{\alpha}_{j,d}$ ,  $\tilde{\beta}_d$ ,  $\|\tilde{a}_j\| = \|\mathbf{Q}_j\|_{0,\infty;\Omega}$ ,  $\|\tilde{c}_j(\mathbf{u}; \cdot, \cdot)\| = R_j \|\mathbf{u}\|_{0,4;\Omega}$ ,  $\|\tilde{c}_j(\mathbf{u}_h; \cdot, \cdot)\| = R_j \|\mathbf{u}_h\|_{0,4;\Omega}$ , and  $\|\tilde{b}\| = 2$ , and hence, easily shown to be independent of  $h$ , such that

$$\begin{aligned} \|(\vec{\phi}_j, \boldsymbol{\vartheta}_j) - (\vec{\phi}_{j,h}, \boldsymbol{\vartheta}_{j,h})\|_{\tilde{\mathbf{H}} \times \tilde{\mathbf{Q}}} &\leq \mathcal{C}_{j,\text{ST}} \left\{ \text{dist}(\vec{\phi}_j, \tilde{\mathbf{H}}_h) + \text{dist}(\boldsymbol{\vartheta}_j, \tilde{\mathbf{Q}}_h) \right. \\ &\quad \left. + \|\tilde{c}_j(\mathbf{u}; \vec{\phi}_j, \cdot) - \tilde{c}_j(\mathbf{u}_h; \vec{\phi}_j, \cdot)\|_{\tilde{\mathbf{H}}'_h} \right\}. \end{aligned} \quad (5.34)$$

Now, bearing in mind the definition of  $\tilde{c}_j$  (cf. (3.23)), we obtain for each  $\vec{\psi}_{j,h} \in \tilde{\mathbf{H}}_h$

$$\tilde{c}_j(\mathbf{u}; \vec{\phi}_j, \vec{\psi}_{j,h}) - \tilde{c}_j(\mathbf{u}_h; \vec{\phi}_j, \vec{\psi}_{j,h}) = \tilde{c}_j(\mathbf{u} - \mathbf{u}_h; \vec{\phi}_j, \vec{\psi}_{j,h}),$$

from which, using the boundedness property of  $\tilde{c}_j$  (cf. third row of (4.3)), we deduce that

$$\|\tilde{c}_j(\mathbf{u}; \vec{\phi}_j, \cdot) - \tilde{c}_j(\mathbf{u}_h; \vec{\phi}_j, \cdot)\|_{\tilde{\mathbf{H}}'_h} \leq R_j \|\vec{\phi}_j\|_{\tilde{\mathbf{H}}} \|\mathbf{u} - \mathbf{u}_h\|_{0,4;\Omega},$$

so that (5.34) becomes

$$\|(\vec{\phi}_j, \boldsymbol{\vartheta}_j) - (\vec{\phi}_{j,h}, \boldsymbol{\vartheta}_{j,h})\|_{\tilde{\mathbf{H}} \times \tilde{\mathbf{Q}}} \leq \widehat{\mathcal{C}}_{j,\text{ST}} \left\{ \text{dist}(\vec{\phi}_j, \tilde{\mathbf{H}}_h) + \text{dist}(\boldsymbol{\vartheta}_j, \tilde{\mathbf{Q}}_h) + \|\vec{\phi}_j\|_{\tilde{\mathbf{H}}} \|\mathbf{u} - \mathbf{u}_h\|_{0,4;\Omega} \right\}, \quad (5.35)$$

where  $\widehat{\mathcal{C}}_{j,\text{ST}}$  is a positive constant depending only on  $\mathcal{C}_{j,\text{ST}}$  and  $R_j$ .

We now proceed to suitably combine (5.33) and (5.35) to derive the final Céa estimate. Indeed, multiplying (5.33) by  $\frac{1}{2\widehat{\mathcal{C}}_{\text{ST}}}$ , summing up in (5.35) over  $j \in \{1, 2\}$ , adding the resulting inequalities, bounding  $\|\bar{\mathbf{u}}\|_{0,4;\Omega}$  and  $\|\vec{\phi}_j\|_{\tilde{\mathbf{H}}}$  by the right hand sides of (4.52) and (4.53), respectively, and then performing some algebraic manipulations, we find that

$$\begin{aligned} & \frac{1}{2\widehat{\mathcal{C}}_{\text{ST}}} \|(\boldsymbol{\sigma}, \bar{\mathbf{u}}) - (\boldsymbol{\sigma}_h, \bar{\mathbf{u}}_h)\|_{\mathbf{H} \times \mathbf{Q}} + \sum_{j=1}^2 \|(\vec{\phi}_j, \boldsymbol{\vartheta}_j) - (\vec{\phi}_{j,h}, \boldsymbol{\vartheta}_{j,h})\|_{\tilde{\mathbf{H}} \times \tilde{\mathbf{Q}}} \leq \frac{1}{2} \|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_{0,4;\Omega} \\ & + \frac{1}{2} \left\{ \text{dist}(\boldsymbol{\sigma}, \mathbf{H}_h) + \text{dist}(\bar{\mathbf{u}}, \mathbf{Q}_h) \right\} + \sum_{j=1}^2 \widehat{\mathcal{C}}_{j,\text{ST}} \left\{ \text{dist}(\vec{\phi}_j, \tilde{\mathbf{H}}_h) + \text{dist}(\boldsymbol{\vartheta}_j, \tilde{\mathbf{Q}}_h) \right\} \\ & + \widehat{\mathcal{C}} \left\{ \|\mathbf{u}_D\|_{1/2,\Gamma} + \|\boldsymbol{\phi}_r\|_{0,4;\Omega} + \sum_{j=1}^2 \left\{ \|g_j\|_{0,4/3;\Omega} + \|\phi_{j,D}\|_{1/2,\Gamma} \right\} \right\} \|\mathbf{u} - \mathbf{u}_h\|_{0,4;\Omega}, \end{aligned} \quad (5.36)$$

where  $\widehat{\mathcal{C}}$  is a positive constant depending only on  $\mathcal{C}$  (cf. (4.52)),  $\mathcal{C}_j$  (cf. (4.53)), and  $\widehat{\mathcal{C}}_{j,\text{ST}}$ ,  $j \in \{1, 2\}$ .

Having established (5.36), the announced Céa estimate is stated as follows.

**Theorem 5.1.** Assume that the data satisfy

$$\widehat{\mathcal{C}} \left\{ \|\mathbf{u}_D\|_{1/2,\Gamma} + \|\boldsymbol{\phi}_r\|_{0,4;\Omega} + \sum_{j=1}^2 \left\{ \|g_j\|_{0,4/3;\Omega} + \|\phi_{j,D}\|_{1/2,\Gamma} \right\} \right\} \leq \frac{1}{4\widehat{\mathcal{C}}_{\text{ST}}}. \quad (5.37)$$

Then, there exists a positive constant  $\tilde{\mathcal{C}}$ , depending only on  $\widehat{\mathcal{C}}_{\text{ST}}$  and  $\widehat{\mathcal{C}}_{j,\text{ST}}$ ,  $j \in \{1, 2\}$ , and hence, independent of  $h$ , such that

$$\begin{aligned} & \|(\boldsymbol{\sigma}, \bar{\mathbf{u}}) - (\boldsymbol{\sigma}_h, \bar{\mathbf{u}}_h)\|_{\mathbf{H} \times \mathbf{Q}} + \sum_{j=1}^2 \|(\vec{\phi}_j, \boldsymbol{\vartheta}_j) - (\vec{\phi}_{j,h}, \boldsymbol{\vartheta}_{j,h})\|_{\tilde{\mathbf{H}} \times \tilde{\mathbf{Q}}} \\ & \leq \tilde{\mathcal{C}} \left\{ \text{dist}(\boldsymbol{\sigma}, \mathbf{H}_h) + \text{dist}(\bar{\mathbf{u}}, \mathbf{Q}_h) + \sum_{j=1}^2 \left\{ \text{dist}(\vec{\phi}_j, \tilde{\mathbf{H}}_h) + \text{dist}(\boldsymbol{\vartheta}_j, \tilde{\mathbf{Q}}_h) \right\} \right\}. \end{aligned} \quad (5.38)$$

*Proof.* It follows directly from (5.36) after realizing that the first term on its right hand side can be subtracted from the second one on the left hand side, whereas, under (5.37), a similar procedure applies to the corresponding last and first terms.  $\square$

Furthermore, as suggested by (2.8), (2.14), and (3.7), we can approximate the pressure  $p$ , the velocity gradient  $\nabla \mathbf{u}$ , and the shear stress tensor  $\tilde{\boldsymbol{\sigma}}$ , by the following postprocessing formulae:

$$p_h := -\frac{1}{n} \operatorname{tr}(\boldsymbol{\sigma}_h + (\mathbf{u}_h \otimes \mathbf{u}_h)) + \frac{1}{n|\Omega|} \int_{\Omega} \operatorname{tr}(\mathbf{u}_h \otimes \mathbf{u}_h), \quad (5.39)$$

$$(\nabla \mathbf{u})_h := \frac{1}{\mu} \boldsymbol{\sigma}_h^d + \frac{1}{\mu} (\mathbf{u}_h \otimes \mathbf{u}_h)^d + \boldsymbol{\gamma}_h, \quad \text{and} \quad (5.40)$$

$$\tilde{\boldsymbol{\sigma}}_h := \boldsymbol{\sigma}_h - \left( \frac{1}{n|\Omega|} \int_{\Omega} \operatorname{tr}(\mathbf{u}_h \otimes \mathbf{u}_h) \right) \mathbb{I} + (\mathbf{u}_h \otimes \mathbf{u}_h). \quad (5.41)$$

Thus, it is not difficult to show that there exists a positive constant  $c$ , independent of  $h$ , though depending either on  $r$  or the data providing the a priori bounds for  $\|\mathbf{u}\|_{0,4;\Omega}$  and  $\|\mathbf{u}_h\|_{0,4;\Omega}$ , such that

$$\begin{aligned} & \|p - p_h\|_{0,\Omega} + \|\nabla \mathbf{u} - (\nabla \mathbf{u})_h\|_{0,\Omega} + \|\tilde{\boldsymbol{\sigma}} - \tilde{\boldsymbol{\sigma}}_h\|_{0,\Omega} \\ & \leq c \left\{ \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{H}} + \|\mathbf{u} - \mathbf{u}_h\|_{0,4;\Omega} + \|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_{0,\Omega} \right\}, \end{aligned} \quad (5.42)$$

which, certainly, is bounded by the right hand side of (5.38) as well.

---

## Specific finite element subspaces

---

In this chapter we resort to [27, Section 4.4] and [26, Section 4.5] to specify two examples of finite element subspaces  $\widetilde{\mathbb{H}}_h^\sigma$ ,  $\mathbf{H}_h^\mu$ ,  $\mathbb{H}_h^\gamma$ ,  $\mathbf{H}_h^\phi$ ,  $\mathbf{H}_h^\dagger$ , and  $\mathbf{H}_h^\varrho$  satisfying the hypotheses **(H.1)** up to **(H.6)** stated in Section 5.3, and then establish the associated rates of convergence for the Galerkin scheme (5.2). Although it will become clear below, we remark in advance that the two aforementioned examples are actually determined by two possible choices for the first three subspaces since the remaining three are kept the same in both cases.

### 6.1 Preliminaries

Given an integer  $k \geq 0$  and  $K \in \mathcal{T}_h$ , we let  $\mathbf{P}_k(K)$  be the space of polynomials of degree at most  $k$  defined on  $K$  with vector and tensorial counterparts  $\mathbf{P}_k(K) := [\mathbf{P}_k(K)]^n$  and  $\mathbb{P}_k(K) := [\mathbf{P}_k(K)]^{n \times n}$ , respectively. In addition, we let  $\mathbf{RT}_k(K) := \mathbf{P}_k(K) + \mathbf{P}_k(K)\mathbf{x}$  be the local Raviart–Thomas space of order  $k$  defined on  $K$ , where  $\mathbf{x}$  stands for a generic vector in  $\mathbb{R}^n$ . Furthermore, denoting by  $b_K$  the bubble function on  $K$ , which is given by the product of its  $n + 1$  barycentric

coordinates, we set the local bubble space of order  $k$  as

$$\begin{aligned} \mathbf{B}_k(K) &:= \text{curl}(b_K \mathbf{P}_k(K)) \quad \text{if } n = 2, \quad \text{and} \\ \mathbf{B}_k(K) &:= \text{curl}(b_K \mathbf{P}_k(K)) \quad \text{if } n = 3, \end{aligned} \tag{6.1}$$

where  $\text{curl}(v) := (\frac{\partial v}{\partial x_2}, -\frac{\partial v}{\partial x_1})$  if  $n = 2$  and  $v : K \rightarrow \mathbb{R}$ , and  $\text{curl}(\mathbf{v}) = \nabla \times \mathbf{v}$  if  $n = 3$  and  $\mathbf{v} : K \rightarrow \mathbb{R}^3$ . Next, we introduce the global spaces

$$\begin{aligned} \mathbf{P}_k(\Omega) &:= \left\{ v_h \in \mathbf{L}^2(\Omega) : v_h|_K \in \mathbf{P}_k(K), \quad \forall K \in \mathcal{T}_h \right\}, \\ \mathbf{P}_k(\Omega) &:= \left\{ \mathbf{v}_h \in \mathbf{L}^2(\Omega) : \mathbf{v}_h|_K \in \mathbf{P}_k(K), \quad \forall K \in \mathcal{T}_h \right\}, \\ \mathbb{P}_k(\Omega) &:= \left\{ \boldsymbol{\delta}_h \in \mathbb{L}^2(\Omega) : \boldsymbol{\delta}_h|_K \in \mathbb{P}_k(K), \quad \forall K \in \mathcal{T}_h \right\}, \\ \mathbf{RT}_k(\Omega) &:= \left\{ \boldsymbol{\eta}_h \in \mathbf{H}(\mathbf{div}; \Omega) : \boldsymbol{\eta}_h|_K \in \mathbf{RT}_k(K), \quad \forall K \in \mathcal{T}_h \right\}, \\ \mathbb{RT}_k(\Omega) &:= \left\{ \boldsymbol{\tau}_h \in \mathbb{H}(\mathbf{div}; \Omega) : \boldsymbol{\tau}_{h,i}|_K \in \mathbf{RT}_k(K), \quad \forall i \in \{1, \dots, n\}, \forall K \in \mathcal{T}_h \right\}, \\ \mathbb{B}_k(\Omega) &:= \left\{ \boldsymbol{\tau}_h \in \mathbb{H}(\mathbf{div}; \Omega) : \boldsymbol{\tau}_{h,i}|_K \in \mathbf{B}_k(K), \quad \forall i \in \{1, \dots, n\}, \forall K \in \mathcal{T}_h \right\}, \end{aligned}$$

where  $\boldsymbol{\tau}_{h,i}$  denotes the  $i$ th-row of  $\boldsymbol{\tau}_h$ . It is clear that  $\mathbf{P}_k(\Omega)$  and  $\mathbf{P}_k(\Omega)$  are also subspaces of  $\mathbf{L}^4(\Omega)$  and  $\mathbf{L}^4(\Omega)$ , respectively. In addition, being  $\mathbf{H}(\mathbf{div}; \Omega)$  and  $\mathbb{H}(\mathbf{div}; \Omega)$  contained in  $\mathbf{H}(\mathbf{div}_{4/3}; \Omega)$  and  $\mathbb{H}(\mathbf{div}_{4/3}; \Omega)$ , respectively, we notice that the spaces  $\mathbb{RT}_k(\Omega)$  and  $\mathbb{B}_k(\Omega)$  are both subspaces of  $\mathbb{H}(\mathbf{div}_{4/3}; \Omega)$  as well, whereas  $\mathbf{RT}_k(\Omega)$  is contained in  $\mathbf{H}(\mathbf{div}_{4/3}; \Omega)$ .

## 6.2 Two examples

To begin with, we proceed as in [25, Section 4.4] and [26, Section 4.5], and employ the stable triplets for linear elasticity derived in [27, Section 4.4], to define two examples of finite element subspaces  $\tilde{\mathbb{H}}_h^\sigma$ ,  $\mathbf{H}_h^\mu$ , and  $\mathbb{H}_h^\gamma$ , satisfying **(H.1)** up to **(H.3)**. In what follows,  $\ell$  is a non-negative integer.

The first example for  $\tilde{\mathbb{H}}_h^\sigma$ ,  $\mathbf{H}_h^\mu$ , and  $\mathbb{H}_h^\gamma$ , is  $\text{PEERS}_\ell$ , the plane elasticity element with reduced symmetry of order  $\ell \geq 0$ , whose stability for the mixed finite element formulation of the linear

elasticity problem, within the classical Hilbertian framework, was originally established in [3] for  $\ell = 0$  and  $n = 2$ , and later on proved for  $\ell \geq 0$  and  $n \in \{2, 3\}$  in [32]. The corresponding subspaces are defined as follows:

$$\begin{aligned} \tilde{\mathbb{H}}_h^\sigma &:= \mathbb{RT}_\ell(\Omega) \oplus \mathbb{B}_\ell(\Omega), & \mathbf{H}_h^u &:= \mathbf{P}_\ell(\Omega), & \text{and} \\ \mathbb{H}_h^\gamma &:= [C(\bar{\Omega})]^{n \times n} \cap \mathbb{L}_{\text{skew}}^2(\Omega) \cap \mathbb{P}_{\ell+1}(\Omega). \end{aligned} \quad (6.2)$$

The second example for  $\tilde{\mathbb{H}}_h^\sigma$ ,  $\mathbf{H}_h^u$ , and  $\mathbb{H}_h^\gamma$ , is  $\text{AFW}_\ell$ , the Arnold-Falk-Winther element of order  $\ell \geq 0$ , whose corresponding aforementioned stability can be found in [4]. In this case, the subspaces are given by:

$$\tilde{\mathbb{H}}_h^\sigma := \mathbb{P}_{\ell+1}(\Omega) \cap \mathbb{H}(\mathbf{div}; \Omega), \quad \mathbf{H}_h^u := \mathbf{P}_\ell(\Omega), \quad \text{and} \quad \mathbb{H}_h^\gamma := \mathbb{L}_{\text{skew}}^2(\Omega) \cap \mathbb{P}_\ell(\Omega). \quad (6.3)$$

Regarding the verification of the hypotheses by the subspaces specified in (6.2) and (6.3), we first observe that **(H.1)** is clearly satisfied in both cases. The same holds with **(H.2)** since  $\mathbf{div}(\mathbb{RT}_\ell(\Omega))$  and  $\mathbf{div}(\mathbb{P}_{\ell+1}(\Omega))$  are contained in  $\mathbf{P}_\ell(\Omega)$ , which coincides with  $\mathbf{H}_h^u$  in the two examples, whereas, according to (6.1), the tensors in  $\mathbb{B}_\ell(\Omega)$  are divergence-free. In turn, we recall that the discrete inf-sup condition for  $b$  required in the assumption **(H.3)**, was proved in [27, Lemma 4.8] for  $\text{PEERS}_\ell$  as well as for  $\text{AFW}_\ell$ . We omit further details and refer to the analysis developed in [27, Section 4.4.2].

On the other hand, specific finite element subspaces  $\mathbf{H}_h^\phi$ ,  $\mathbf{H}_h^t$ , and  $\mathbf{H}_h^\vartheta$ , are set as follows:

$$\mathbf{H}_h^\phi := \mathbf{P}_\ell(\Omega), \quad \mathbf{H}_h^t := \mathbf{P}_\ell(\Omega), \quad \text{and} \quad \mathbf{H}_h^\vartheta := \mathbf{RT}_\ell(\Omega). \quad (6.4)$$

Similarly as a previous remark, the fact that  $\mathbf{div}(\mathbf{RT}_\ell(\Omega))$  is contained in  $\mathbf{P}_\ell(\Omega) = \mathbf{H}_h^\phi$ , guarantees that **(H.4)** is satisfied. In addition, knowing from (5.17) that, besides being contained in  $\mathbf{H}_h^\vartheta = \mathbf{RT}_\ell(\Omega)$ , the vector functions of  $Z_{0,h}$  are divergence-free, we deduce, from a particular argument provided in the proof of [23, Theorem 3.3], that  $Z_{0,h} \subseteq \mathbf{P}_\ell(\Omega)$ , which confirms **(H.5)**. Finally, the discrete inf-sup condition required by **(H.6)**, which coincides with [18, eq. (5.64)],

is basically proved in the last part of [18, Section 5.5] by realizing that it reduces to the vector version of [18, Lemma 5.5, eq. (5.45)].

### 6.3 The rates of convergence

In this section we first collect the approximation properties of the finite element spaces defined in Section 6.2, and then establish the associated rates of convergence of the Galerkin scheme (5.2).

We begin with the approximation properties of  $\text{PEERS}_\ell$  (cf. (6.2)) and  $\text{AFW}_\ell$  (cf. (6.3)), which basically follow from the analogue properties of the Raviart–Thomas and AFW interpolation operators, and of the orthogonal projectors  $\mathcal{P}_h^\ell : \mathbf{L}^1(\Omega) \rightarrow \mathbf{P}_\ell(\Omega)$  and  $\mathcal{P}_h^\ell : \mathbb{L}^1(\Omega) \rightarrow \mathbb{P}_\ell(\Omega)$  (cf. [21, Proposition 1.135]), along with the use of the commuting diagram properties and of the interpolation estimates of Sobolev spaces. They read as follows (cf. [27, Section 4.4.3], [25, Section 4.4.4]):

( $\mathbf{AP}_h^\sigma$ ) there exists a positive constant  $C$ , independent of  $h$ , such that for each  $s \in (0, \ell + 1]$ , and for each  $\boldsymbol{\tau} \in \mathbb{H}^s(\Omega) \cap \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega)$ , with  $\mathbf{div}(\boldsymbol{\tau}) \in \mathbf{W}^{s,4/3}(\Omega)$ , there holds

$$\text{dist}(\boldsymbol{\tau}, \mathbf{H}_h) \leq C h^s \left\{ \|\boldsymbol{\tau}\|_{s,\Omega} + \|\mathbf{div}(\boldsymbol{\tau})\|_{s,4/3;\Omega} \right\},$$

( $\mathbf{AP}_h^{\mathbf{u}}$ ) there exists a positive constant  $C$ , independent of  $h$ , such that for each  $s \in [0, \ell + 1]$ , and for each  $\mathbf{v} \in \mathbf{W}^{s,4}(\Omega)$ , there holds

$$\text{dist}(\mathbf{v}, \mathbf{H}_h^{\mathbf{u}}) \leq C h^s \|\mathbf{v}\|_{s,4;\Omega}, \quad \text{and}$$

( $\mathbf{AP}_h^\gamma$ ) there exists a positive constant  $C$ , independent of  $h$ , such that for each  $s \in [0, \ell + 1]$ , and for each  $\boldsymbol{\delta} \in \mathbb{H}^s(\Omega) \cap \mathbb{L}_{\text{skew}}^2(\Omega)$ , there holds

$$\text{dist}(\boldsymbol{\delta}, \mathbb{H}_h^\gamma) \leq C h^s \|\boldsymbol{\delta}\|_{s,\Omega}.$$

Furthermore, regarding the approximation properties of the subspaces defined in (6.4), they are given as indicated next:

( $\mathbf{AP}_h^\phi$ ) there exists a positive constant  $C$ , independent of  $h$ , such that for each  $s \in [0, \ell + 1]$ , and for each  $\psi \in W^{s,4}(\Omega)$ , there holds

$$\text{dist}(\psi, \mathbf{H}_h^\phi) \leq C h^s \|\psi\|_{s,4;\Omega},$$

( $\mathbf{AP}_h^t$ ) there exists a positive constant  $C$ , independent of  $h$ , such that for each  $s \in [0, \ell + 1]$ , and for each  $\mathbf{r} \in \mathbf{H}^s(\Omega)$ , there holds

$$\text{dist}(\mathbf{r}, \mathbf{H}_h^t) \leq C h^s \|\mathbf{r}\|_{s;\Omega}, \quad \text{and}$$

( $\mathbf{AP}_h^\vartheta$ ) there exists a positive constant  $C$ , independent of  $h$ , such that for each  $s \in (0, \ell + 1]$ , and for each  $\boldsymbol{\eta} \in \mathbf{H}^s(\Omega) \cap \mathbf{H}(\text{div}_{4/3}; \Omega)$ , with  $\text{div}(\boldsymbol{\eta}) \in W^{s,4/3}(\Omega)$ , there holds

$$\text{dist}(\boldsymbol{\eta}, \mathbf{H}_h^\vartheta) \leq C h^s \left\{ \|\boldsymbol{\eta}\|_{s,\Omega} + \|\text{div}(\boldsymbol{\eta})\|_{s,4/3;\Omega} \right\}.$$

In this way, as a consequence of Theorem 5.1, (5.42), and the approximation properties ( $\mathbf{AP}_h^\sigma$ ), ( $\mathbf{AP}_h^u$ ), ( $\mathbf{AP}_h^\gamma$ ), ( $\mathbf{AP}_h^\phi$ ), ( $\mathbf{AP}_h^t$ ), and ( $\mathbf{AP}_h^\vartheta$ ), we conclude the rates of convergence of the Galerkin Scheme (5.2) with the finite element subspaces defined in Section 6.2. More precisely, we have the following result.

**Theorem 6.1.** In addition to the hypotheses of Theorems 4.6, 5.7, and 5.1, assume that there exists  $s \in (0, \ell + 1]$  such that  $\boldsymbol{\sigma} \in \mathbb{H}^s(\Omega) \cap \mathbb{H}_0(\text{div}_{4/3}; \Omega)$ ,  $\text{div}(\boldsymbol{\sigma}) \in \mathbf{W}^{s,4/3}(\Omega)$ ,  $\mathbf{u} \in \mathbf{W}^{s,4}(\Omega)$ ,  $\boldsymbol{\gamma} \in \mathbb{H}^s(\Omega) \cap \mathbb{L}_{\text{skew}}^2(\Omega)$ ,  $\phi_j \in W^{s,4}(\Omega)$ ,  $\mathbf{t}_j \in \mathbf{H}^s(\Omega)$ ,  $\boldsymbol{\vartheta}_j \in \mathbf{H}^s(\Omega) \cap \mathbf{H}(\text{div}_{4/3}; \Omega)$ , and  $\text{div}(\boldsymbol{\vartheta}_j) \in$

$W^{s,4/3}(\Omega)$ ,  $j \in \{1, 2\}$ . Then, there exists a positive constant  $C$ , independent of  $h$  such that

$$\begin{aligned}
& \|(\boldsymbol{\sigma}, \vec{\mathbf{u}}) - (\boldsymbol{\sigma}_h, \vec{\mathbf{u}}_h)\|_{\mathbf{H} \times \mathbf{Q}} + \sum_{j=1}^2 \|(\vec{\phi}_j, \boldsymbol{\vartheta}_j) - (\vec{\phi}_{j,h}, \boldsymbol{\vartheta}_{j,h})\|_{\tilde{\mathbf{H}} \times \tilde{\mathbf{Q}}} \\
& + \|p - p_h\|_{0,\Omega} + \|\nabla \mathbf{u} - (\nabla \mathbf{u})_h\|_{0,\Omega} + \|\tilde{\boldsymbol{\sigma}} - \tilde{\boldsymbol{\sigma}}_h\|_{0,\Omega} \\
& \leq C h^s \left\{ \|\mathbf{u}\|_{s,4;\Omega} + \|\boldsymbol{\sigma}\|_{s,\Omega} + \|\mathbf{div}(\boldsymbol{\sigma})\|_{s,4/3;\Omega} + \|\boldsymbol{\gamma}\|_{s,\Omega} \right. \\
& \left. + \sum_{j=1}^2 \left( \|\phi_j\|_{s,4;\Omega} + \|\mathbf{t}_j\|_{s,\Omega} + \|\boldsymbol{\vartheta}_j\|_{s,\Omega} + \|\mathbf{div}(\boldsymbol{\vartheta}_j)\|_{s,4/3;\Omega} \right) \right\}.
\end{aligned} \tag{6.5}$$

---

## Numerical results

---

In this chapter we consider the two pairs of finite element subspaces detailed in Chapter 6 to present three examples illustrating the performance of the mixed finite element method (5.2) on a set of quasi-uniform triangulations of the respective domains. In what follows, we refer to the corresponding sets of finite element subspaces generated by  $\ell = \{0, 1\}$  as simply  $\text{PEERS}_\ell - \mathbf{P}_\ell - \mathbf{P}_\ell - \mathbf{RT}_\ell$  and  $\text{AFW}_\ell - \mathbf{P}_\ell - \mathbf{P}_\ell - \mathbf{RT}_\ell$ . The numerical methods have been implemented using open source finite element libraries: FEniCS [1] and FreeFem++ [29]. We have used FEniCS for Examples 1 and 2, and FreeFem++ for the Example 3. A Newton–Raphson algorithm with a fixed tolerance  $\text{tol} = 1\text{E}-06$  is used for the resolution of the nonlinear problem (5.2). As usual, the iterative method is finished when the relative error between two consecutive iterations of the complete coefficient vector, namely  $\mathbf{coeff}^m$  and  $\mathbf{coeff}^{m+1}$ , is sufficiently small, that is,

$$\frac{\|\mathbf{coeff}^{m+1} - \mathbf{coeff}^m\|_{\text{DOF}}}{\|\mathbf{coeff}^{m+1}\|_{\text{DOF}}} \leq \text{tol},$$

where  $\|\cdot\|_{\text{DOF}}$  stands for the usual Euclidean norm in  $\mathbf{R}^{\text{DOF}}$  with DOF denoting the total number of degrees of freedom defining the finite element subspaces  $\tilde{\mathbb{H}}_h^\sigma, \mathbf{H}_h^u, \mathbb{H}_h^\gamma, H_h^\phi, \mathbf{H}_h^t$ , and  $\mathbf{H}_h^\vartheta$  (cf.

(6.2)–(6.4)).

We now introduce some additional notation. The individual errors are denoted by

$$\begin{aligned} \mathbf{e}(\boldsymbol{\sigma}) &:= \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\text{div}_{4/3};\Omega}, & \mathbf{e}(\mathbf{u}) &:= \|\mathbf{u} - \mathbf{u}_h\|_{0,4;\Omega}, & \mathbf{e}(\boldsymbol{\gamma}) &:= \|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_{0,\Omega}, \\ \mathbf{e}(p) &:= \|p - p_h\|_{0,\Omega}, & \mathbf{e}(\nabla \mathbf{u}) &:= \|\nabla \mathbf{u} - (\nabla \mathbf{u})_h\|_{0,\Omega}, \\ \mathbf{e}(\phi_j) &:= \|\phi_j - \phi_{j,h}\|_{0,4;\Omega}, & \mathbf{e}(\mathbf{t}_j) &:= \|\mathbf{t}_j - \mathbf{t}_{j,h}\|_{0,\Omega}, & \mathbf{e}(\boldsymbol{\vartheta}_j) &:= \|\boldsymbol{\vartheta}_j - \boldsymbol{\vartheta}_{j,h}\|_{\text{div}_{4/3};\Omega}, \quad j \in \{1, 2\}, \end{aligned}$$

where  $p_h$  and  $(\nabla \mathbf{u})_h$  stand for the post-processed pressure and velocity gradient suggested by (5.39) and (5.40), respectively. We stress here that we are also able to recover the shear stress tensor  $\tilde{\boldsymbol{\sigma}}$  by the post-processing formula (5.41). However, for the sake of simplicity, in the numerical essays below we will focus only on the pressure field and velocity gradient tensor. Moreover, for each  $\star \in \{\boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\gamma}, p, \nabla \mathbf{u}, \phi_j, \mathbf{t}_j, \boldsymbol{\vartheta}_j\}$  we let  $r(\star)$  be the experimental rate of convergence given by  $r(\star) := \log(\mathbf{e}(\star)/\hat{\mathbf{e}}(\star))/\log(h/\hat{h})$ , where  $h$  and  $\hat{h}$  denote two consecutive meshsizes with errors  $\mathbf{e}$  and  $\hat{\mathbf{e}}$ , respectively.

The examples to be considered in this section are described next. In all of them, we take  $\varrho = 1, R_1 = 1, R_2 = 1$ , and  $\boldsymbol{\phi}_r = (0, 0)$ . In turn, in the first two examples the tensors  $\mathbf{Q}_1$  and  $\mathbf{Q}_2$  are taken as the identity matrix  $\mathbb{I}$ , which satisfy (2.4). In addition, the null mean value of  $\text{tr}(\boldsymbol{\sigma}_h)$  over  $\Omega$  is fixed via a real Lagrange multiplier strategy.

## Example 1: Convergence against smooth exact solutions in a 2D domain

In this test we corroborate the rates of convergence in a two-dimensional domain. The domain is the square  $\Omega = (0, 1)^2$ . We consider the inertial power  $\rho = 3$ , the potential type gravitational acceleration  $\mathbf{g} = (0, -1)^t$ , the effective viscosity  $\mu(x_1, x_2) = \exp(-x_1 x_2)$ , and adjust the data

$\mathbf{f}$ ,  $g_1$ , and  $g_2$  in (2.13) such that the exact solution is given by

$$\mathbf{u}(x_1, x_2) = \begin{pmatrix} \sin(\pi x_1) \cos(\pi x_2) \\ -\cos(\pi x_1) \sin(\pi x_2) \end{pmatrix}, \quad p(x_1, x_2) = \cos(\pi x_1) \sin(0.5 \pi x_2),$$

$$\phi_1(x_1, x_2) = 0.5 + 0.5 \cos(x_1 x_2), \quad \text{and} \quad \phi_2(x_1, x_2) = 0.1 + 0.3 \exp(x_1 x_2).$$

The model problem is then complemented with the appropriate Dirichlet boundary conditions. Tables 7.1 and 7.2 show the convergence history for a sequence of quasi-uniform mesh refinements, including the number of Newton iterations when  $D = 1$  and  $F = 10$ . As already announced, we stress that we are able not only to approximate the original unknowns but also the pressure field and the velocity gradient through the formulae (5.39)–(5.40). The results confirm that the optimal rates of convergence  $\mathcal{O}(h^{\ell+1})$  predicted by Theorem 6.1 are attained for  $\ell = \{0, 1\}$  for both  $\text{PEERS}_\ell$  and  $\text{AFW}_\ell$  based schemes. The Newton method exhibits a behavior independent of the meshsize, converging in five iterations in almost all cases. In Figure 7.1 we display some solutions obtained with the mixed  $\text{PEERS}_1 - \text{P}_1 - \mathbf{P}_1 - \mathbf{RT}_1$  approximation with meshsize  $h = 0.013$  and 24,200 triangle elements (actually representing 1,260,602 DOF).

## Example 2: Convergence against smooth exact solutions in a 3D domain

In the second example we consider the cube domain  $\Omega = (0, 1)^3$ , the model parameter  $\rho = 3.5$ ,  $D = 1$ ,  $F = 10$ ,  $\mu(x_1, x_2, x_3) = \exp(-x_1 x_2 x_3)$ , and  $\mathbf{g} = (0, 0, -1)$ . The manufactured solution is given by

$$\mathbf{u}(x_1, x_2, x_3) = \begin{pmatrix} \sin(\pi x_1) \cos(\pi x_2) \cos(\pi x_3) \\ -2 \cos(\pi x_1) \sin(\pi x_2) \cos(\pi x_3) \\ \cos(\pi x_1) \cos(\pi x_2) \sin(\pi x_3) \end{pmatrix}, \quad p(x_1, x_2, x_3) = \cos(\pi x_1) \exp(x_2 + x_3),$$

$$\phi_1(x_1, x_2, x_3) = 0.5 + 0.5 \cos(x_1 x_2 x_3), \quad \text{and} \quad \phi_2(x_1, x_2, x_3) = 0.1 + 0.3 \exp(x_1 x_2 x_3).$$

Similarly to the first example, the data  $\mathbf{f}$ ,  $g_1$ ,  $g_2$  and  $\mathbf{u}_D$ ,  $\phi_{1,D}$ ,  $\phi_{2,D}$  are computed from (2.13) using the above solution. The convergence history for a set of quasi-uniform mesh refinements using  $\ell = 0$  is shown in Table 7.3. Again, the mixed finite element method converges optimally with order  $\mathcal{O}(h)$ , as it was proved by Theorem 6.1. In addition, some components of the numerical solution are displayed in Figure 7.2, which were built using the mixed PEERS<sub>0</sub>–P<sub>0</sub>–P<sub>0</sub>–RT<sub>0</sub> approximation with meshsize  $h = 0.087$  and 48,000 tetrahedral elements (actually representing 1,479,784 DOF).

### Example 3: Flow through a 2D porous media with fracture network.

Inspired by [16, Example 4, Section 6], we finally focus on a flow through a porous medium with a fracture network considering strong jump discontinuities of the parameters  $D$  and  $F$  across the two regions. We consider the square domain  $\Omega = (-1, 1)^2$  with an internal fracture network denoted as  $\Omega_f$  (see the first plot in Figure 7.3), and boundary  $\Gamma$ , whose left, right, upper and lower parts are given by  $\Gamma_{\text{left}} = \{-1\} \times (-1, 1)$ ,  $\Gamma_{\text{right}} = \{1\} \times (-1, 1)$ ,  $\Gamma_{\text{top}} = (-1, 1) \times \{1\}$ , and  $\Gamma_{\text{bottom}} = (-1, 1) \times \{-1\}$ , respectively. Note that the boundary of the internal fracture network is defined as a union of segments. The prescribed mesh file is available in [https://github.com/scaucaao/Fracture\\_network-mesh\\_CBF-DD](https://github.com/scaucaao/Fracture_network-mesh_CBF-DD). We consider the coupling of the convective Brinkman–Forchheimer and double-diffusion equations (2.13) in the whole domain  $\Omega$  with inertial power  $\rho = 4$ ,  $\mu = 1$ ,  $\mathbf{Q}_1 = 0.1 \mathbb{I}$  and  $\mathbf{Q}_2 = 0.2 \mathbb{I}$ , but with different values of the parameters  $D$  and  $F$  for the interior and the exterior of the fracture, namely

$$F = \begin{cases} 10 & \text{in } \Omega_f \\ 1 & \text{in } \bar{\Omega} \setminus \Omega_f \end{cases} \quad \text{and} \quad D = \begin{cases} 1 & \text{in } \Omega_f \\ 1000 & \text{in } \bar{\Omega} \setminus \Omega_f \end{cases}. \quad (7.1)$$

The parameter choice corresponds to increased inertial effect ( $F = 10$ ) in the fracture and a high permeability ( $D = 1$ ), compared to reduced inertial effect ( $F = 1$ ) in the porous medium and low permeability ( $D = 1000$ ). In addition,  $\mathbf{g} = (0, -1)$ , the source terms are  $g_1 = 0$  and

$g_2 = 0$ , and the boundaries conditions are

$$\begin{aligned} \boldsymbol{\sigma} \boldsymbol{\nu} &= \begin{cases} (-100(x_2 - 1), 0)^t & \text{on } \Gamma_{\text{left}}, \\ (0, -100(x_1 - 1))^t & \text{on } \Gamma_{\text{top}}, \end{cases} & \boldsymbol{\sigma} \boldsymbol{\nu} &= (0, 0)^t & \text{on } \Gamma_{\text{right}} \cup \Gamma_{\text{bottom}}, \\ \phi_1 &= 0.3 & \text{on } \Gamma_{\text{bottom}}, & \phi_1 &= 0 & \text{on } \Gamma_{\text{top}}, & \boldsymbol{\vartheta}_1 \cdot \boldsymbol{\nu} &= 0 & \text{on } \Gamma_{\text{left}} \cup \Gamma_{\text{right}}, \\ \phi_2 &= 0.2 & \text{on } \Gamma_{\text{bottom}}, & \phi_2 &= 0 & \text{on } \Gamma_{\text{top}}, & \boldsymbol{\vartheta}_2 \cdot \boldsymbol{\nu} &= 0 & \text{on } \Gamma_{\text{left}} \cup \Gamma_{\text{right}}, \end{aligned} \quad (7.2)$$

which drives the flow in a diagonal direction from the left-top corner to the right-bottom corner of the square domain  $\Omega$ . We remark that the analysis developed in the previous sections can be extended, with minor modifications, to the case of mixed boundary conditions considered in this example.

In Figure 7.3, we display the computed magnitude of the pseudostress tensor, velocity, velocity gradient, and gradients of the temperature and concentration, and the temperature and concentration fields, which were built using the fully-mixed  $\text{AFW}_0 - \text{P}_0 - \mathbf{P}_0 - \mathbf{RT}_0$  scheme on a mesh with  $h = 0.029$  and 31,932 triangle elements (actually representing 576,216 DOF). As we expected, the velocity in the fractures is higher than the velocity in the porous medium, due to smaller fractures thickness and the parameter setting (7.1). In addition, the velocity is higher in branches of the network where the fluid enters from the left-top corner and decreases toward the right-bottom corner of the domain. In turn, we observe a sharp velocity gradient across the interfaces between the fractures and the porous medium. The pseudostress is consistent with the boundary conditions (7.2) and it is more diffused since it includes the pressure field. In turn, the temperature and concentration are zero on the top of the domain and go increasing towards the bottom of it, which is consistent with the behavior observed in the magnitude of the temperature and concentration gradients. This example illustrates the ability of the method to provide accurate resolution and numerically stable results for heterogeneous inclusions with high aspect ratio and complex geometry, as presented in the network of thin fractures. We notice that the mesh used in this example was built by considering a quasi-uniform refinement. Nevertheless, this refinement can be improved and automatized by employing a suitable *a posteriori* error indicator, as in [14] and [11], that captures the aforementioned discontinuity of

the parameters and localize the refinement where it is needed. The corresponding *a posteriori* error analysis and numerical implementation will be addressed in a future work.

PEERS <sub>0</sub> - P <sub>0</sub> - P <sub>0</sub> - RT <sub>0</sub> approximation												
DOF	$h$	$e(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	$e(\mathbf{u})$	$r(\mathbf{u})$	$e(\boldsymbol{\gamma})$	$r(\boldsymbol{\gamma})$	$e(p)$	$r(p)$	$e(\nabla \mathbf{u})$	$r(\nabla \mathbf{u})$	
570	0.354	2.2E+00	–	2.3E-01	–	1.6E-01	–	2.8E-01	–	8.6E-01	–	
2194	0.177	1.1E+00	0.988	1.2E-01	1.003	5.1E-02	1.666	1.4E-01	0.948	4.5E-01	0.927	
8610	0.088	5.5E-01	1.003	5.8E-02	1.063	2.1E-02	1.304	6.9E-02	1.039	2.3E-01	0.973	
30002	0.047	2.9E-01	1.004	3.1E-02	1.057	9.2E-03	1.301	3.6E-02	1.037	1.2E-01	0.993	
119402	0.024	1.5E-01	1.002	1.6E-02	1.111	3.4E-03	1.422	1.8E-02	1.015	6.2E-02	1.000	
400402	0.013	8.0E-02	1.001	8.5E-03	1.059	1.4E-03	1.479	9.7E-03	1.005	3.4E-02	1.001	
$e(\phi_1)$	$r(\phi_1)$	$e(\mathbf{t}_1)$	$r(\mathbf{t}_1)$	$e(\boldsymbol{\vartheta}_1)$	$r(\boldsymbol{\vartheta}_1)$	$e(\phi_2)$	$r(\phi_2)$	$e(\mathbf{t}_2)$	$r(\mathbf{t}_2)$	$e(\boldsymbol{\vartheta}_2)$	$r(\boldsymbol{\vartheta}_2)$	it
2.2E-02	–	8.4E-02	–	1.6E-01	–	4.2E-02	–	7.6E-02	–	1.5E-01	–	6
1.1E-02	0.979	5.0E-02	0.736	8.9E-02	0.883	2.1E-02	0.981	4.1E-02	0.904	7.8E-02	0.946	6
5.5E-03	0.997	2.7E-02	0.916	4.6E-02	0.961	1.1E-02	0.995	2.1E-02	0.962	3.9E-02	0.980	5
2.9E-03	1.000	1.4E-02	0.973	2.5E-02	0.988	5.7E-03	0.999	1.1E-02	0.987	2.1E-02	0.993	5
1.5E-03	1.000	7.3E-03	0.991	1.2E-02	0.996	2.8E-03	1.000	5.6E-03	0.996	1.1E-02	0.998	5
8.0E-04	1.000	4.0E-03	0.997	6.7E-03	0.999	1.6E-03	1.000	3.1E-03	0.999	5.8E-03	0.999	5
AFW <sub>0</sub> - P <sub>0</sub> - P <sub>0</sub> - RT <sub>0</sub> approximation												
DOF	$h$	$e(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	$e(\mathbf{u})$	$r(\mathbf{u})$	$e(\boldsymbol{\gamma})$	$r(\boldsymbol{\gamma})$	$e(p)$	$r(p)$	$e(\nabla \mathbf{u})$	$r(\nabla \mathbf{u})$	
625	0.354	2.0E+00	–	2.3E-01	–	4.0E-01	–	1.5E-01	–	8.2E-01	–	
2401	0.177	9.8E-01	1.027	1.2E-01	0.965	2.1E-01	0.944	6.7E-02	1.164	4.2E-01	0.943	
9409	0.088	4.9E-01	1.010	5.8E-02	0.991	1.0E-01	0.986	3.2E-02	1.078	2.1E-01	0.987	
32761	0.047	2.6E-01	1.003	3.1E-02	0.998	5.6E-02	0.996	1.7E-02	1.024	1.1E-01	0.997	
130321	0.024	1.3E-01	1.001	1.6E-02	0.999	2.8E-02	0.999	8.2E-03	1.007	5.7E-02	0.999	
436921	0.013	7.0E-02	1.000	8.5E-03	1.000	1.5E-02	1.000	4.5E-03	1.002	3.1E-02	1.000	
$e(\phi_1)$	$r(\phi_1)$	$e(\mathbf{t}_1)$	$r(\mathbf{t}_1)$	$e(\boldsymbol{\vartheta}_1)$	$r(\boldsymbol{\vartheta}_1)$	$e(\phi_2)$	$r(\phi_2)$	$e(\mathbf{t}_2)$	$r(\mathbf{t}_2)$	$e(\boldsymbol{\vartheta}_2)$	$r(\boldsymbol{\vartheta}_2)$	it
2.2E-02	–	8.3E-02	–	1.6E-01	–	4.2E-02	–	7.6E-02	–	1.5E-01	–	5
1.1E-02	0.978	5.0E-02	0.731	8.9E-02	0.883	2.1E-02	0.981	4.1E-02	0.905	7.8E-02	0.947	5
5.5E-03	0.997	2.7E-02	0.915	4.6E-02	0.960	1.1E-02	0.996	2.1E-02	0.963	3.9E-02	0.980	5
2.9E-03	1.000	1.4E-02	0.973	2.5E-02	0.987	5.7E-03	0.999	1.1E-02	0.987	2.1E-02	0.993	5
1.5E-03	1.000	7.3E-03	0.991	1.2E-02	0.996	2.8E-03	1.000	5.6E-03	0.996	1.1E-02	0.998	5
8.0E-04	1.000	4.0E-03	0.997	6.7E-03	0.999	1.6E-03	1.000	3.1E-03	0.999	5.8E-03	0.999	5

Table 7.1: [Example 1,  $\ell = 0$ ] Number of degrees of freedom, meshsizes, errors, rates of convergence, and Newton iteration count for the fully-mixed approximations with  $\rho = 3$ ,  $D = 1$ , and  $F = 10$ .

PEERS <sub>1</sub> - P <sub>1</sub> - P <sub>1</sub> - RT <sub>1</sub> approximation												
DOF	$h$	$e(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	$e(\mathbf{u})$	$r(\mathbf{u})$	$e(\boldsymbol{\gamma})$	$r(\boldsymbol{\gamma})$	$e(p)$	$r(p)$	$e(\nabla \mathbf{u})$	$r(\nabla \mathbf{u})$	
1746	0.354	3.4E-01	–	4.1E-02	–	5.7E-02	–	3.4E-02	–	1.3E-01	–	
6818	0.177	8.8E-02	1.959	1.1E-02	1.969	1.7E-02	1.733	9.4E-03	1.836	3.6E-02	1.867	
26946	0.088	2.3E-02	1.964	2.6E-03	1.992	5.4E-03	1.674	2.6E-03	1.851	1.0E-02	1.861	
94202	0.047	6.5E-03	1.979	7.5E-04	1.998	1.7E-03	1.817	7.9E-04	1.912	3.0E-03	1.905	
375602	0.024	1.6E-03	1.991	1.9E-04	1.999	4.5E-04	1.926	2.0E-04	1.960	7.8E-04	1.954	
1260602	0.013	4.9E-04	1.996	5.6E-05	2.000	1.4E-04	1.973	6.1E-05	1.983	2.3E-04	1.980	
$e(\phi_1)$	$r(\phi_1)$	$e(\mathbf{t}_1)$	$r(\mathbf{t}_1)$	$e(\boldsymbol{\vartheta}_1)$	$r(\boldsymbol{\vartheta}_1)$	$e(\phi_2)$	$r(\phi_2)$	$e(\mathbf{t}_2)$	$r(\mathbf{t}_2)$	$e(\boldsymbol{\vartheta}_2)$	$r(\boldsymbol{\vartheta}_2)$	it
2.0E-03	–	1.3E-02	–	2.2E-02	–	3.4E-03	–	7.9E-03	–	1.7E-02	–	6
5.1E-04	2.006	3.7E-03	1.792	6.1E-03	1.878	8.9E-04	1.971	2.2E-03	1.827	4.5E-03	1.926	6
1.3E-04	2.002	9.8E-04	1.919	1.6E-03	1.950	2.2E-04	1.994	5.9E-04	1.929	1.2E-03	1.969	5
3.6E-05	2.001	2.9E-04	1.965	4.5E-04	1.978	6.1E-05	1.999	1.7E-04	1.967	3.3E-04	1.985	5
9.0E-06	2.000	7.2E-05	1.984	1.1E-04	1.990	1.5E-05	2.000	4.3E-05	1.984	8.3E-05	1.993	5
2.7E-06	2.000	2.2E-05	1.992	3.4E-05	1.995	4.5E-06	2.000	1.3E-05	1.992	2.5E-05	1.996	5
AFW <sub>1</sub> - P <sub>1</sub> - P <sub>1</sub> - RT <sub>1</sub> approximation												
DOF	$h$	$e(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	$e(\mathbf{u})$	$r(\mathbf{u})$	$e(\boldsymbol{\gamma})$	$r(\boldsymbol{\gamma})$	$e(p)$	$r(p)$	$e(\nabla \mathbf{u})$	$r(\nabla \mathbf{u})$	
1745	0.354	3.1E-01	–	4.1E-02	–	6.4E-02	–	1.8E-02	–	1.3E-01	–	
6817	0.177	7.6E-02	2.010	1.1E-02	1.959	1.7E-02	1.949	4.3E-03	2.042	3.3E-02	1.958	
26945	0.088	1.9E-02	2.006	2.6E-03	1.989	4.2E-03	1.980	1.1E-03	1.963	8.4E-03	1.986	
94201	0.047	5.3E-03	2.003	7.5E-04	1.997	1.2E-03	1.992	3.2E-04	1.972	2.4E-03	1.995	
375601	0.024	1.3E-03	2.001	1.9E-04	1.999	3.0E-04	1.997	8.1E-05	1.984	6.0E-04	1.998	
1260601	0.013	4.0E-04	2.001	5.6E-05	2.000	8.9E-03	1.998	2.4E-05	1.992	1.8E-04	1.999	
$e(\phi_1)$	$r(\phi_1)$	$e(\mathbf{t}_1)$	$r(\mathbf{t}_1)$	$e(\boldsymbol{\vartheta}_1)$	$r(\boldsymbol{\vartheta}_1)$	$e(\phi_2)$	$r(\phi_2)$	$e(\mathbf{t}_2)$	$r(\mathbf{t}_2)$	$e(\boldsymbol{\vartheta}_2)$	$r(\boldsymbol{\vartheta}_2)$	it
2.0E-03	–	1.3E-02	–	2.2E-02	–	3.4E-03	–	7.9E-03	–	1.7E-02	–	6
5.1E-04	2.006	3.7E-03	1.783	6.1E-03	1.877	8.6E-04	1.971	2.2E-03	1.820	4.5E-03	1.926	5
1.3E-04	2.002	9.8E-04	1.918	1.6E-03	1.950	2.2E-04	1.994	5.9E-04	1.928	1.2E-03	1.969	5
3.6E-05	2.001	2.9E-04	1.965	4.5E-04	1.978	6.1E-05	1.999	1.7E-04	1.968	3.3E-04	1.985	5
9.0E-06	2.000	7.2E-05	1.984	1.1E-04	1.990	1.5E-05	2.000	4.3E-05	1.984	8.3E-05	1.993	5
2.7E-06	2.000	2.2E-05	1.992	3.4E-05	1.995	4.5E-06	2.000	1.3E-05	1.992	2.5E-05	1.996	5

Table 7.2: [Example 1,  $\ell = 1$ ] Number of degrees of freedom, meshsizes, errors, rates of convergence, and Newton iteration count for the fully-mixed approximations with  $\rho = 3$ ,  $D = 1$ , and  $F = 10$ .

PEERS <sub>0</sub> - P <sub>0</sub> - P <sub>0</sub> - RT <sub>0</sub> approximation												
DOF	$h$	$e(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	$e(\mathbf{u})$	$r(\mathbf{u})$	$e(\boldsymbol{\gamma})$	$r(\boldsymbol{\gamma})$	$e(p)$	$r(p)$	$e(\nabla \mathbf{u})$	$r(\nabla \mathbf{u})$	
1642	0.866	9.2E+00	-	5.7E-01	-	1.2E+00	-	1.2E+00	-	2.4E+00	-	
12376	0.433	4.8E+00	0.953	3.1E-01	0.898	4.3E-01	1.477	6.3E-01	0.881	1.4E+00	0.721	
96268	0.217	2.4E+00	1.002	1.6E-01	0.971	1.2E-01	1.793	3.1E-01	1.054	7.6E-01	0.910	
509926	0.124	1.4E+00	1.015	8.9E-02	0.994	5.0E-02	1.604	1.6E-01	1.104	4.4E-01	0.968	
1479784	0.087	9.4E-01	1.012	6.3E-02	0.998	2.9E-02	1.577	1.1E-01	1.084	3.1E-01	0.987	
$e(\phi_1)$	$r(\phi_1)$	$e(\mathbf{t}_1)$	$r(\mathbf{t}_1)$	$e(\boldsymbol{\vartheta}_1)$	$r(\boldsymbol{\vartheta}_1)$	$e(\phi_2)$	$r(\phi_2)$	$e(\mathbf{t}_2)$	$r(\mathbf{t}_2)$	$e(\boldsymbol{\vartheta}_2)$	$r(\boldsymbol{\vartheta}_2)$	it
2.9E-02	-	1.0E-01	-	3.2E-01	-	6.0E-02	-	1.1E-01	-	2.3E-01	-	6
1.6E-02	0.868	8.2E-02	0.295	1.9E-01	0.760	3.1E-02	0.931	6.3E-02	0.849	1.3E-01	0.864	6
8.0E-03	0.971	4.8E-02	0.788	9.9E-02	0.926	1.6E-02	0.982	3.3E-02	0.918	6.6E-02	0.957	6
4.6E-03	0.994	2.8E-02	0.925	5.7E-02	0.975	9.1E-03	0.995	1.9E-02	0.966	3.8E-02	0.986	6
3.2E-03	0.998	2.0E-02	0.965	4.0E-02	0.989	6.4E-03	0.998	1.4E-02	0.983	2.7E-02	0.994	6
AFW <sub>0</sub> - P <sub>0</sub> - P <sub>0</sub> - RT <sub>0</sub> approximation												
DOF	$h$	$e(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	$e(\mathbf{u})$	$r(\mathbf{u})$	$e(\boldsymbol{\gamma})$	$r(\boldsymbol{\gamma})$	$e(p)$	$r(p)$	$e(\nabla \mathbf{u})$	$r(\nabla \mathbf{u})$	
1993	0.866	8.7E+00	-	5.6E-01	-	1.1E+00	-	1.0E+00	-	2.1E+00	-	
14881	0.433	4.3E+00	1.032	3.0E-01	0.903	6.4E-01	0.727	4.9E-01	1.093	1.3E+00	0.712	
114817	0.217	2.1E+00	1.045	1.6E-01	0.959	3.4E-01	0.928	2.2E-01	1.119	6.8E-01	0.922	
605641	0.124	1.2E+00	1.019	8.9E-02	0.988	1.9E-01	0.980	1.3E-01	1.050	3.9E-01	0.978	
1754401	0.087	8.1E-01	1.008	6.2E-02	0.995	1.4E-01	0.992	8.7E-02	1.021	2.7E-01	0.992	
$e(\phi_1)$	$r(\phi_1)$	$e(\mathbf{t}_1)$	$r(\mathbf{t}_1)$	$e(\boldsymbol{\vartheta}_1)$	$r(\boldsymbol{\vartheta}_1)$	$e(\phi_2)$	$r(\phi_2)$	$e(\mathbf{t}_2)$	$r(\mathbf{t}_2)$	$e(\boldsymbol{\vartheta}_2)$	$r(\boldsymbol{\vartheta}_2)$	it
2.9E-02	-	9.9E-02	-	3.1E-01	-	6.0E-02	-	1.1E-01	-	2.3E-01	-	5
1.6E-02	0.869	8.1E-02	0.275	1.9E-01	0.747	3.1E-02	0.934	6.3E-02	0.846	1.3E-01	0.863	5
8.0E-03	0.972	4.8E-02	0.775	9.8E-02	0.917	1.6E-02	0.984	3.3E-02	0.915	6.6E-02	0.954	5
4.6E-03	0.995	2.8E-02	0.922	5.7E-02	0.970	9.1E-03	0.996	1.9E-02	0.965	3.8E-02	0.983	5
3.2E-03	0.998	2.0E-02	0.964	4.0E-02	0.986	6.4E-03	0.998	1.4E-02	0.983	2.7E-02	0.992	5

Table 7.3: [Example 2] Number of degrees of freedom, meshsizes, errors, rates of convergence, and Newton iteration count for the fully-mixed approximations with  $\rho = 3.5$ ,  $D = 1$ , and  $F = 10$ .

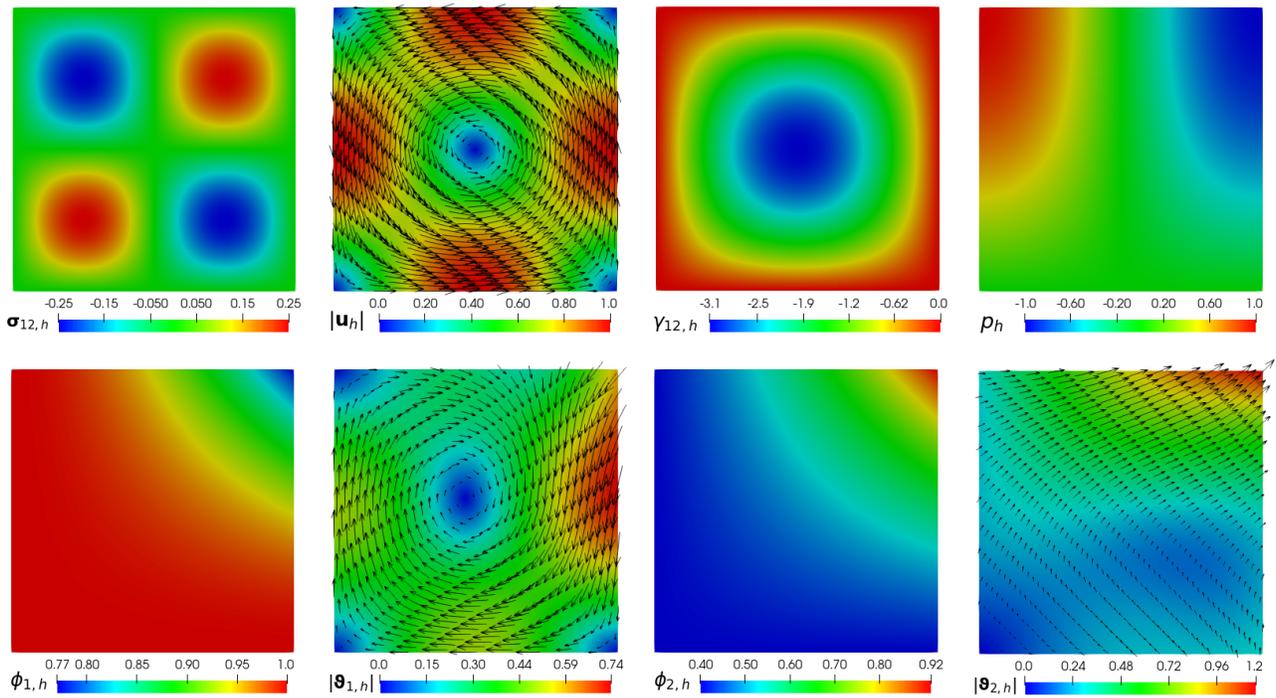


Figure 7.1: [Example 1,  $\ell = 1$ ] Computed pseudostress tensor component, magnitude of the velocity, vorticity component, and pressure field (top plots); temperature field, magnitude of the pseudoheat vector, concentration field, and magnitude of the pseudodiffusion vector (bottom plots).

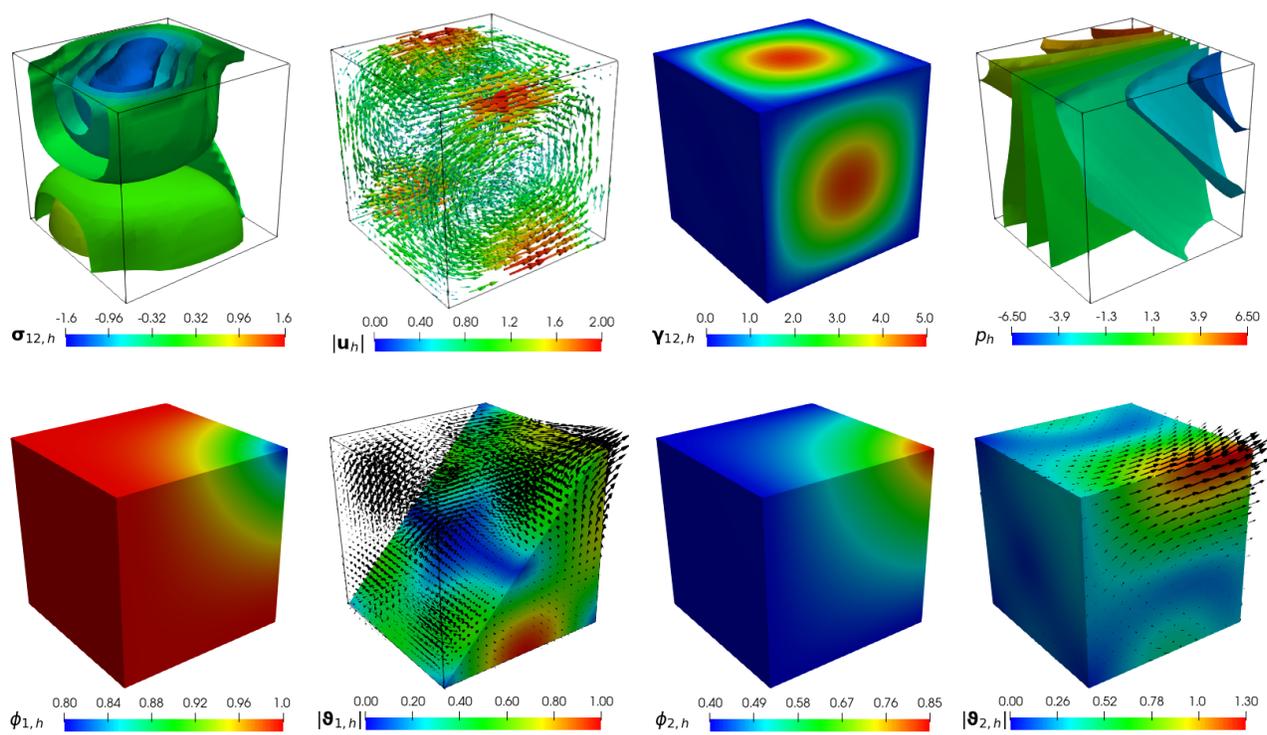


Figure 7.2: [Example 2] Computed pseudostress tensor component, magnitude of the velocity, vorticity component, and pressure field (top plots); temperature field, magnitude of the pseudoheat vector, concentration field, and magnitude of the pseudodiffusion vector (bottom plots).

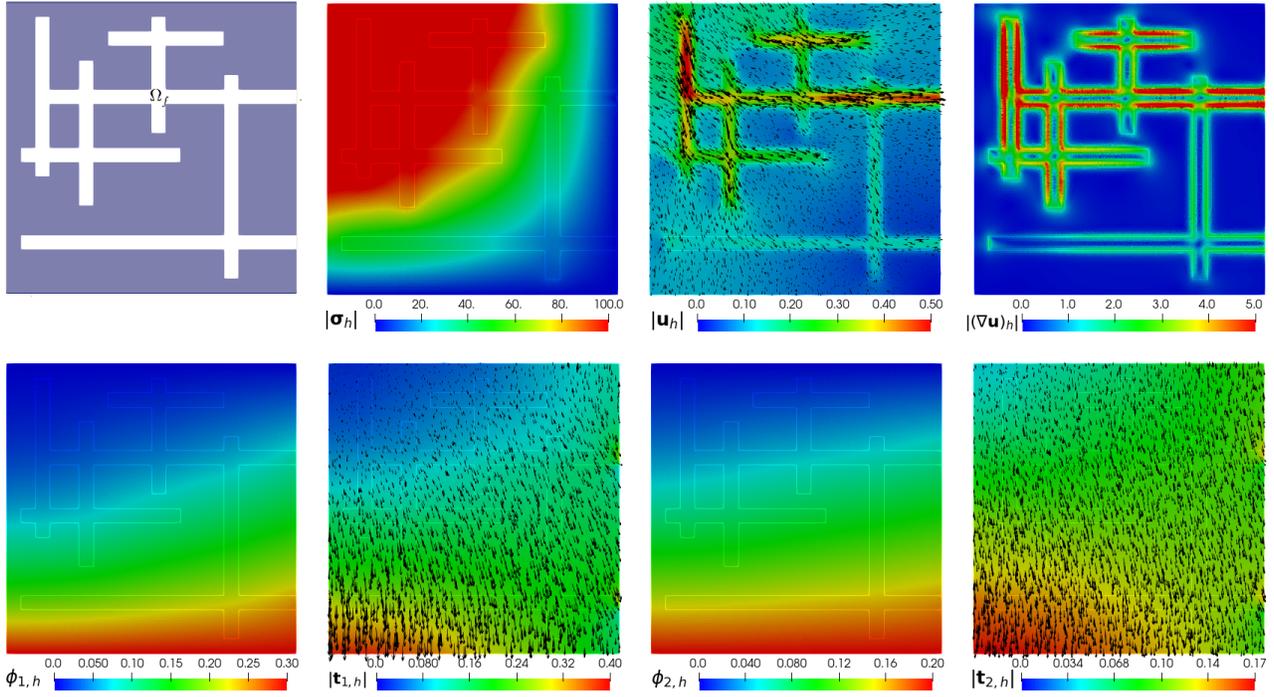


Figure 7.3: [Example 3] Domain configuration, computed magnitude of the pseudostress tensor, velocity, and velocity gradient tensor (top plots); concentration field, magnitude of the temperature gradient, concentration field, and magnitude of the concentration gradient (bottom plots).

---

## Conclusions and Future Works

---

In this chapter we summarize the main contributions of this work and give a brief description of eventual future works.

### 8.1 Conclusions

Upon the results presented in this thesis, we can extract the following conclusions:

- We developed a new mixed formulation for the convective Brinkman-Forchheimer equations, whose analysis made use of diverse tools and abstract results in Banach spaces.
- We showed that the pressure field, the velocity gradient and the shear stress tensor can be obtained by using a post-processing formula based on the computed variables.
- We proved that both configurations of finite element spaces proposed for fluid variables achieve optimal convergence, which is confirmed by numerical examples.

## 8.2 Future Works

The methods developed and the results obtained here have motivated some possibilities of future work, which are described below:

- To extend the analysis of the coupled convective Brinkman-Forchheimer and double-diffusion equations to the case in which the viscosity  $\mu$  depends on the module of the strain tensor  $\mathbf{e}(\mathbf{u})$ , namely

$$\begin{aligned}
 -\operatorname{div}(\mu(|\mathbf{e}(\mathbf{u})|)\mathbf{e}(\mathbf{u})) + (\nabla\mathbf{u})\mathbf{u} + \mathbf{D}\mathbf{u} + \mathbf{F}|\mathbf{u}|^{\rho-2}\mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } \Omega, \\
 \operatorname{div}(\mathbf{u}) = 0 \quad \text{in } \Omega, \quad \mathbf{u} = \mathbf{u}_D \quad \text{on } \Gamma, \quad \int_{\Omega} p &= 0.
 \end{aligned}$$

- To couple the convective Brinkman-Forchheimer equations with other models of interest, such as Darcy's equation, and analyze it in a Banach spaces framework.
- To develop the corresponding a posteriori error analyses of some of the above models.
- To extend the analysis and results to the unsteady state case of some of the above models.

---

## Bibliography

---

- [1] M.S. ALNAES, J. BLECHTA, J. HAKE, A. JOHANSSON, B. KEHLET, A. LOGG, C. RICHARDSON, J. RING, M.E. ROGNES, AND G.N. WELLS, *The FEniCS project version 1.5*. Arch. Numer. Softw. 3 (2015) 9–23.
- [2] A.K. ALZHRANI, *Importance of Darcy–Forchheimer porous medium in 3D convective flow of carbon nanotubes*. Phys. Lett. A 382 (2018), no. 40, 2938–2943.
- [3] D.N. ARNOLD, F. BREZZI AND J. DOUGLAS, PEERS: *A new mixed finite element method for plane elasticity*. Jpn. J. Appl. Math. 1 (1984), 347–367.
- [4] D.N. ARNOLD, R.S. FALK AND R. WINTHER, *Mixed finite element methods for linear elasticity with weakly imposed symmetry*. Math. Comp. 76 (2007), no. 260, 1699–1723.
- [5] G.A. BENAVIDES, S. CAUCAO, G.N. GATICA AND A.A. HOPPER, *A new non-augmented and momentum-conserving fully-mixed finite element method for a coupled flow-transport problem*. Calcolo 59 (2022), no. 1, Paper No. 6, 44 pp.
- [6] M.M. BHATTI, A. ZEESHAN, R. ELLAHI, AND G.C. SHIT, *Mathematical modeling of heat and mass transfer effects on MHD peristaltic propulsion of two-phase flow through*

- a Darcy–Brinkman–Forchheimer porous medium*. Adv. Powder Techn. 29 (2018), no. 5, 1189–1197.
- [7] F. BREZZI AND M. FORTIN, *Mixed and Hybrid Finite Element Methods*. Springer Series in Computational Mathematics, 15. Springer-Verlag, New York, 1991.
- [8] J. CAMAÑO, C. GARCÍA, AND R. OYARZÚA, *Analysis of a momentum conservative mixed-FEM for the stationary Navier-Stokes problem*. Numer. Methods Partial Differential Equations 37 (2021), no. 5, 2895–2923.
- [9] J. CAMAÑO, C. MUÑOZ, AND R. OYARZÚA, *Numerical analysis of a dual-mixed problem in non-standard Banach spaces*. Electron. Trans. Numer. Anal. 48 (2018), 114–130.
- [10] S. CAUCAO, E. COLMENARES, G.N. GATICA AND C. INZUNZA, *A Banach spaces-based fully-mixed finite element method for the stationary chemotaxis-Navier-Stokes problem*. Comp. Math. Appl. 145 (2023), 65–89.
- [11] S. CAUCAO AND J. ESPARZA, *An augmented mixed FEM for the convective Brinkman-Forchheimer problem: a priori and a posteriori error analysis*. arXiv:2209.02894 [math.NA], (2023).
- [12] S. CAUCAO, G.N. GATICA, AND L.F. GATICA, *A Banach spaces-based mixed finite element method for the stationary convective Brinkman-Forchheimer problem*. Preprint 2023-10, Centro de Investigación en Ingeniería Matemática (CI<sup>2</sup>MA), Universidad de Concepción, Concepción, Chile, (2022).
- [13] S. CAUCAO, G.N. GATICA, AND J.P. ORTEGA, *A fully-mixed formulation in Banach spaces for the coupling of the steady Brinkman-Forchheimer and double-diffusion equations*. ESAIM Math. Model. Numer. Anal. 55 (2021), no. 6, 2725–2758.
- [14] S. CAUCAO, G.N. GATICA, AND J.P. ORTEGA, *A posteriori error analysis of a Banach spaces-based fully mixed FEM for double-diffusive convection in a fluid-saturated porous medium*. Comput. Geosci. 27 (2023), no. 2, 289–316.

- [15] S. CAUCAO, G.N. GATICA, R. OYARZÚA, AND N. SÁNCHEZ, *A fully-mixed formulation for the steady double-diffusive convection system based upon Brinkman-Forchheimer equations*. J. Sci. Comput. 85 (2020), no. 2, Paper No. 44, 37 pp.
- [16] S. CAUCAO, R. OYARZÚA, S. VILLA-FUENTES AND I. YOTOV, *A three-field Banach spaces-based mixed formulation for the unsteady Brinkman-Forchheimer equations*. Comput. Methods Appl. Mech. Engrg. 394 (2022), Paper No. 114895, 32 pp.
- [17] A.O. CELEBI, V.K. KALANTAROV AND D. UGURLU, *Continuous dependence for the convective Brinkman-Forchheimer equations*. Appl. Anal. 84 (2005), no. 9, 877–888.
- [18] E. COLMENARES, G.N. GATICA, AND S. MORAGA, *A Banach spaces-based analysis of a new fully-mixed finite element method for the Boussinesq problem*. ESAIM Math. Model. Numer. Anal. 54 (2020), no. 5, 1525–1568.
- [19] C.I. CORREA AND G.N. GATICA, *On the continuous and discrete well-posedness of perturbed saddle-point formulations in Banach spaces*. Comput. Math. Appl. 117 (2022), 14–23.
- [20] C.I. CORREA, G.N. GATICA AND R. RUIZ-BAIER, *New mixed finite element methods for the coupled Stokes and Poisson-Nernst-Planck equations in Banach spaces*. ESAIM Math. Model. Numer. Anal. 57 (2023), no. 3, 1511–1551.
- [21] A. ERN AND J.-L. GUERMOND, *Theory and Practice of Finite Elements*. Applied Mathematical Sciences, 159. Springer-Verlag, New York, 2004.
- [22] J. FAULKNER, B.X. HU, S. KISH, AND F. HUA, *Laboratory analog and numerical study of ground water flow and solute transport in a karst aquifer with conduit and matrix domains*. J. Contaminant Hydrology 110 (2009), no. 1-2, 34–44.
- [23] G.N. GATICA, *A Simple Introduction to the Mixed Finite Element Method. Theory and Applications*. SpringerBriefs in Mathematics. Springer, Cham, 2014.

- [24] G.N. GATICA AND C. INZUNZA, *On the well-posedness of Banach spaces-based mixed formulations for the nearly incompressible Navier-Lamé and Stokes equations*. *Comp. Math. Appl.* 102 (2021), 87–94.
- [25] G.N. GATICA, N. NÚÑEZ AND R. RUIZ-BAIER, *New non-augmented mixed finite element methods for the Navier-Stokes-Brinkman equations using Banach spaces*. *J. Numer. Math.*, <https://doi.org/10.1515/jnma-2022-0073>.
- [26] G.N. GATICA, N. NÚÑEZ AND R. RUIZ-BAIER, *Mixed-primal methods for natural convection driven phase change with Navier-Stokes-Brinkman equations*. *J. Sci. Comput.* 95 (2023), no. 3, Paper No. 79.
- [27] G.N. GATICA, R. OYARZÚA, R. RUIZ-BAIER, AND Y. SOBRAL, *Banach spaces-based analysis of a fully-mixed finite element method for the steady-state model of fluidized beds*. *Comp. Math. Appl.* 84 (2021), 244–276
- [28] R. GLOWINSKI AND A. MARROCCO, *Sur l'approximation, par éléments finis d'ordre un, et la résolution, par pénalisations-dualité d'une classe de problèmes de Dirichlet non lineaires*. *R.A.I.R.O.* 9 (1975), no. 2, 41–76.
- [29] F. HECHT, *New development in FreeFem++*. *J. Numer. Math.* 20 (2012), 251–265.
- [30] P.N. KALONI AND J. GUO, *Steady nonlinear double-diffusive convection in a porous medium based upon the Brinkman-Forchheimer model*. *J. Math. Anal. Appl.* 204 (1996), no. 1, 138–155.
- [31] D. LIU AND K. LI, *Mixed finite element for two-dimensional incompressible convective Brinkman-Forchheimer equations*. *Appl. Math. Mech. (English Ed.)* 40 (2019), no. 6, 889–910.
- [32] M. LONSING AND R. VERFUÜRTH, *On the stability of BDMS and PEERS elements*. *Numer. Math.* 99 (2004), no. 1, 131–140.
- [33] M. ÔTANI AND S. UCHIDA, *Global solvability of some double-diffusive convection system coupled with Brinkman-Forchheimer equations*. *Lib. Math. (N.S.)* 33 (2013), no. 1, 79–107.

- [34] S. SAFI AND S. BENISSAAD, *Double-diffusive convection in an anisotropic porous layer using the Darcy–Brinkman–Forchheimer formulation*. Arch. Mech. (Arch. Mech. Stos.) 70 (2018), no. 1, 89–102.
- [35] C. ZHAO AND Y. YOU, *Approximation of the incompressible convective Brinkman–Forchheimer equations*. J. Evol. Equ. 12 (2012), no. 4, 767–788.
- [36] Y.J. ZHUANG, H.Z. YU, AND Q.Y. ZHU, *A thermal non-equilibrium model for 3D double diffusive convection of power-law fluids with chemical reaction in the porous medium*. Int. J. Heat Mass Transf. 115-B (2017), 670–694.