MULTI-LEVEL NEURAL NETWORKS FOR ACCURATE SOLUTIONS OF BOUNDARY-VALUE PROBLEMS

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ABSTRACT. The solution to partial differential equations using deep learning approaches, such as the physics-informed neural networks (PINNs) [1], the deep Ritz method [2], or the weak adversarial networks [3], have shown in recent years promising results for several classes of initial and boundary-value problems. However, their ability to surpass, particularly in terms of accuracy, classical discretization methods such as the finite element methods, remains a significant challenge. One of the main obstacles of deep learning approaches lies in their inability to consistently reduce the relative error in the computed solution. To better control the error, we present a new methodology for deep learning methods. The main idea consists in computing an initial approximation to the problem using a simple neural network and in estimating, in an iterative manner, a correction by solving the problem for the residual error with a new network of increasing complexity. A similar approach has been suggested in [4] to control the error in the case of symmetric problems. This sequential reduction of the residual associated with the partial differential equation allows one to decrease the solution error, which, in some cases, can be reduced to machine precision. The underlying explanation is that the method is able to capture at each level smaller scales of the solution using a new network. Numerical examples in 1D and 2D are presented to demonstrate the effectiveness of the proposed approach using PINNs. It is noted that the approach can also be extended to other neural network solvers based on weak or strong formulations of the residual.

Keywords: Neural networks, Partial differential equations, Physics-informed neural networks, Numerical error, Convergence, Frequency analysis

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