# VARIATIONAL PHYSICS INFORMED NEURAL NETWORKS: THE ROLE OF QUADRATURES AND TEST FUNCTIONS AND BOUNDARY CONDITIONS

STEFANO BERRONE, CLAUDIO CANUTO, MORENO PINTORE, AND N. SUKUMAR

Abstract. In this presentation, we investigate the impact of different quadrature rules' precision and varying degrees of piecewise polynomial test functions on the convergence rate of Variational Physics Informed Neural Networks (VPINNs) when addressing elliptic boundary-value problems through mesh refinement. Employing a Petrov-Galerkin framework, we derive an a priori error estimate in the energy norm. The proposed interpolation operator is crucial to obtain an inf-sup stable method and to remove all spurious modes from the output of the neural network. Our findings may seem counterintuitive, suggesting that, for smooth solutions, the optimal approach for achieving a rapid error decay rate is to opt for test functions with the lowest polynomial degree while employing precision-oriented quadrature formulas.

We proceed introducing an “a posteriori” error estimator, comprising a residual component, a loss function term, and data oscillation terms. Our analysis demonstrates the reliability and efficiency of this estimator in controlling the energy norm error between the exact solution and the VPINN-derived solution.

Furthermore, a comprehensive exploration of four distinct approaches for enforcing Dirichlet boundary conditions in PINNs and VPINNs is performed. Traditionally, these conditions are enforced by introducing penalization terms in the loss function and carefully selecting corresponding scaling coefficients, a process that often demands resource-intensive tuning. Through a series of numerical tests, we establish that modifying the neural network's output to precisely align with the specified values yields more efficient and accurate solvers. Our most promising results emerge from the exact enforcement of Dirichlet boundary conditions, achieved by employing an approximate distance function. We also demonstrate that variational imposition of these conditions using Nitsche's method yields suboptimal solvers.

**Keywords**: VPINN, rates of convergence, Dirichlet boundary conditions

**Mathematics Subject Classifications (2010)**: 35B45 , 35J20 , 65K10 , 65N20 , 68T07

## References

[1] S. Berrone, C. Canuto, and M. Pintore. Solving PDEs by variational physics-informed neural networks: an a posteriori error analysis. *Annali dell'Università di Ferrara*, 68:575–595, 2022.

[2] S. Berrone, C. Canuto, and M. Pintore. Variational physics informed neural networks: the role of quadratures and test functions. *Journal of Scientific Computing*, 92(3):1–27, 2022.

[3] S. Berrone, C. Canuto, M. Pintore, and N. Sukumar. Enforcing dirichlet boundary conditions in physics-informed neural networks and variational physics-informed neural networks. *Heliyon*, 9(8):e18820, 2023.

Politecnico di Torino
*Email address*: stefano.berrone@polito.it

Politecnico di Torino
*Email address*: claudio.canuto@polito.it

Politecnico di Torino
*Email address*: moreno.pintore@polito.it

University of California
*Email address*: nsukumar@ucdavis.edu