NEURAL CONTROL OF FINITE ELEMENT METHODS: QUASI-OPTIMAL CONVERGENCE OF QUASI-MINIMIZING NEURAL NETWORKS

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Abstract. There is tremendous potential in using neural networks to optimize numerical methods. In this talk, I will introduce and analyse a framework [1, 2, 3, 4] for the neural optimization of discrete weak formulations, suitable for finite element methods. The main idea of the framework is to include a neural-network function acting as a control variable in the weak form. Finding the neural control that (quasi-) minimizes a suitable cost (or loss) functional, then yields a numerical approximation with desirable attributes. In particular, the framework allows in a natural way the incorporation of known data of the exact solution, or the incorporation of stabilization mechanisms (e.g., to remove spurious oscillations).

I will present the analysis of well-posedness and convergence of the associated constrained-optimization problem, i.e., under certain conditions, discrete weak forms are stable, and quasi-minimizing neural controls exist, which converge quasi-optimally. The analysis results are specialized to Galerkin, least- squares and minimal-residual formulations, where the neural-network dependence appears in the form of suitable weights.

I will also consider the case of parametric PDEs, where we are motivated by the desire to find a method that is tailored to quantities of interest of solutions to the parametric PDE. The central component in our approach is an efficient neural-network-weighted Minimal-Residual formulation, which, after training, provides Galerkin-based approximations in standard discrete spaces that have accurate quantities of interest, regardless of the coarseness of the discrete space. Elementary numerical experiments support our findings and demonstrate the potential of the framework.

Keywords: Optimal neural control; Artificial neural networks; Data-driven discretization; Weighted finite element methods; Quasi-minimization; Quasi-optimal convergence

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