# OPTIMAL BLOCK PRECONDITIONERS FOR HIGH-ORDER DISCRETIZATIONS OF MULTIPHYSICS PROBLEMS IN THE DE RHAM COMPLEX

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ABSTRACT. The Riesz maps of the $L^2$ de Rham complex frequently arise as subproblems in the construction of fast preconditioners for multiphysics problems. In this work we present multigrid solvers for high-order finite element discretizations of these Riesz maps with the same time and space complexity as sum-factorized operator application, i.e., with optimal complexity in polynomial degree in the context of Krylov methods. The key idea of our approach is to build new finite elements for each space in the de Rham complex with orthogonality properties in both the $L^2$- and $H(\mathrm{d})$-inner products ($\mathrm{d} \in \{\mathrm{grad}, \mathrm{curl}, \mathrm{div}\}$) on the reference hexahedron. The resulting sparsity enables the fast solution of the patch problems arising in the Pavarino, Arnold–Falk–Winther, and Hiptmair space decompositions, in the separable case. In the non-separable case, the method can be applied to an auxiliary operator that is sparse by construction. With exact Cholesky factorizations of the sparse patch problems, the application complexity is optimal but the setup costs and storage are not. We overcome this with the finer Hiptmair space decomposition and the use of incomplete Cholesky factorizations imposing the sparsity pattern arising from static condensation, which applies whether static condensation is used for the solver or not. This yields multigrid relaxations with time and space complexity that are both optimal in the polynomial degree.

We illustrate our preconditioning approach by solving mixed formulations of a pseudostress-displacement formulation of linear elasticity and the vorticity-velocity-pressure formulation of Stokes flow, for which we observe robustness with respect to the mesh size, the polynomial degree, and the problem parameters.

**Keywords**: preconditioning, high-order, de Rham complex, additive Schwarz, multigrid

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