NONLINEAR TWOFOLD SADDLE POINT-BASED MIXED FINITE ELEMENT METHODS FOR GRANULAR FLOWS

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Abstract. We propose and analyze new mixed finite element methods for a regularized $\mu(I)$-rheology model of granular materials with an equivalent viscosity depending nonlinearly on the pressure and the euclidean norm of the symmetric part of the velocity gradient. To this end, and besides the velocity, the pressure and the aforementioned strain rate, we introduce the stress and the skew-symmetric vorticity as auxiliary tensor unknowns, thus yielding two related mixed variational formulations within a Banach spaces framework. The first one solves for the pressure through a iterative postprocess suggested by the incompressibility condition of the fluid, whereas the second one incorporates this variable as one more of the respective unknowns. Fixed-point strategies combined with a solvability result for a class of nonlinear twofold saddle point operator equations in Banach spaces, are employed to show, along with the classical Banach fixed-point theorem, the well-posedness of the continuous and discrete formulations. In particular, PEERS and AFW elements of order $\ell \geq 0$ for the stress, the velocity, and the skew-symmetric vorticity, and piecewise polynomials of degree $\leq \ell + n$ (resp. $\leq \ell + 1$) for the strain rate with PEERS (resp. with AFW), and of degree $\leq \ell$ for the pressure, yield stable Galerkin schemes. Optimal a priori error estimates are derived and associated rates of convergence are established. Finally, numerical results confirming the latter and illustrating the good performance of the methods, are reported.

Keywords: granular flows, nonlinear viscosity, twofold saddle point, mixed finite elements, fixed-point theory, a priori error analysis

Mathematics Subject Classifications (2020): 65N30, 65N12, 65N15, 47H10, 47J26, 76D05, 76T25,

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