WAVENUMBER-ROBUST DEEP RELU NEURAL NETWORK EMULATION IN ACOUSTIC WAVE SCATTERING

FERNANDO HENRÍQUEZ AND CHRISTOPH SCHWAB

ABSTRACT. We present wavenumber-robust error bounds using deep neural networks to emulate the solution to the time-harmonic, sound-soft acoustic scattering problem in the exterior of a smooth, convex, two-dimensional obstacle.

The starting point of our analysis is the reduction of the scattering problem in the unbounded exterior region to its (bounded) boundary by means of the wavenumber-robust Combined Field Integral Equation (CFIE), yielding a second-kind boundary integral equation posed on the smooth surface Γ of the scatterer. This BIE is well-posed in $L^2(\Gamma)$, with explicit bounds on the continuity and stability constants that depend explicitly on the (non-dimensional) wavenumber κ.

Utilizing well-known wavenumber-explicit asymptotics of the solution to this problem, as introduced in the work of Melrose and Taylor [1], we explore the numerical approximation of the BIE using fully connected, deep feed-forward neural networks (DNNs) with the Rectified Linear Unit (ReLU) as the chosen activation function [2]. It's worth noting that the results presented here can be straightforwardly extended to different activation functions such as the hyperbolic tangent or the Rectified Power Unit.

Through a constructive argument, we prove the existence of DNNs affording an ϵ-error in the $L^\infty(\Gamma)$-norm with a fixed and small width and a depth that increases *spectrally* with the accuracy ϵ and polynomially with respect to $\log(\kappa)$. By *spectral accuracy*, we mean that there exists $\alpha > 0$ such that for each $n \in \mathbb{N}$, there exists a constant $C_n > 0$, such that for a prescribed accuracy $\epsilon > 0$, the depth of the DNN is bounded by $C_n \epsilon^{-\frac{\alpha}{n}}$. The nature of this bound is not an artifact of the proof but a limitation determined by how the regularity of the problem as derived in [1].

The DNNs constructed in our proofs do not require any analytic information about the scatterer's shape or even the wavenumber κ and can efficiently approximate the various behaviors that make the numerical solution of this problem a challenge in computational mathematics [3, 4].

Keywords: High Wavenumber; Acoustic Scattering; Boundary Integral Equations; Deep Neural Networks.

Mathematics Subject Classifications (2010): 65N15; 65M32; 65N15; 68T07.

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CHAIR OF COMPUTATIONAL MATHEMATICS AND SIMULATION SCIENCE, ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE, 1015 LAUSANNE, SWITZERLAND.
Email address: fernando.henriquez@epfl.ch

SEMINAR FOR APPLIED MATHEMATICS, ETH ZÜRICH, 8092 ZÜRICH, SWITZERLAND.
Email address: schwab@math.ethz.ch