NEAR-OPTIMAL LEARNING OF BANACH-VALUED, HIGH-DIMENSIONAL FUNCTIONS VIA DEEP NEURAL NETWORKS FOR PARAMETRIC PDES

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Abstract. This work establishes practical existence theorems for approximating high-dimensional parametric functions from limited samples using deep neural networks motivated by, but not limited to, parametrized partial differential equations. Over the last decade, deep learning has become one of the most studied techniques in scientific computing. However, the mathematical principles of deep learning in terms of stability, accuracy and sample complexity for such functions are still in development. On the one hand, the ever-expanding literature on approximation theory using deep neural networks suggests that they can approximate functions from relevant function classes. On the other hand, deep learning is becoming an ever more practical tool for recovering parametric maps of physical systems modelled as differential equations. Moreover, while standard PDEs are naturally posed in a weak form in Hilbert spaces, there has been an increasing interest in solving problems whose solution belongs to more general Banach space. Despite the substantial recent work, the theory typically does not address what sample complexity suffices to construct such networks. Here, we observe a gap between theory and practice. This work is motivated by the desire to close this gap. We provide theoretical arguments combining tools from deep learning, best s-term polynomial approximation theory, compressed sensing and convex optimization to compute near-optimal approximations to Banach-valued functions. These approximations are also robust to all key sources of error in the problem, including sampling, optimization, approximation and physical discretization errors. Furthermore, our method is non-intrusive and establishes efficient approximation to infinite-dimensional Banach-valued functions from limited data. We also provide preliminary numerical results illustrating the practical performance for problems arising as solutions to parametric PDEs.

Keywords: deep neural networks, high-dimensional approximation, parametric PDEs, uncertainty quantification, Banach spaces

Mathematics Subject Classifications (2020): 65D40; 68T07; 68Q32

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